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On the l-Number of 1-Cycles on an Abelian Variety

Tsutomu SEKIGUCHI

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Introduction

For any divisor D on an abelian variety X of dimension r, we denote by ϕ_D the homomorphism from X to the Picard variety $\hat{X}$ of X associated with the divisorial correspondence $m^*\mathcal{O}_X(D)\otimes p_1^*\mathcal{O}_X(-D)\otimes p_2^*\mathcal{O}_X(-D)$ on $X\times X$, where $m: X\times X\rightarrow X$ is the addition morphism of X and p_i: $X\times X\rightarrow X$ means the projection to the i-th component for $i=1, 2$. By *Riemann-Roch theorem*, the number of *Euler-Poincaré* $\chi(\mathcal{O}_X(D))$ is related with the self-intersection number (D^r) and with deg ϕ_D as follows;

$$\chi(\mathcal{O}_X(D))=\frac{(D^r)}{r!} \quad \text{and} \quad \deg\phi_D=\{\chi(\mathcal{O}_X(D))\}^2.$$

In particular, if D is ample, then $\chi(\mathcal{O}_X(D))=\dim \mathrm{H}^\circ(X, \mathcal{O}_X(D))=l(D)$, where $l(D)$ is the dimension of the linear system $L(D)$, and $\deg\phi_D$ becomes $l(D)^2$.

On the other hand, for 1-cycles on an abelian variety, an analogy to the relation $\deg\phi_D=1(D)^2$ was given by K. Toki [6], in the following way; that is, *for a positive* 1-*cycle* C *on an r-dimensional abelian variety* X *over the complex number fiield* $\boldsymbol{C}$, *the l-number* $l(C)$, *defined by* $l(C)=\deg(\overset{r}{*}C)/r!$ *where* $\overset{r}{*}C$ *means the r-times Pontrjagin product of* C, *satisfies the equality*

$$\deg\phi_C=l(C)^2.$$

Here ϕ_C *is the homomorphism from Picard variety* $\hat{X}$ *of* X *to* X *defined by* $\phi_C(\hat{x})=S(C\cdot(P|_{\{\hat{x}\}\times X}))$ *for any* $\hat{x}\in\hat{X}$, *where* P *is the Poincaré divisor on* $\hat{X}\times X$ *and* S *means the sum on* X. He proved this equality replacing the intersection and the Pontrjagin products by cohomological languages.

The purpose in this paper is to prove the same equality with no restriction on the characteristic of the ground field, by means of Jacobian varieties of curves.

In n°1, we shall review some results on abelian varieties and curves as preliminaries for our main theorem which will be proved in n°2.

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1. Let k be an algebraically closed field of any characteristic, which will be fixed throughout the paper. Let K, X and Y be three abelian varieties over k. If these abelian varieties form an exact sequence $0\longrightarrow K\xrightarrow{\iota} X\xrightarrow{\pi} Y\longrightarrow 0$ in the

category of commutative group schemes of finite type over the field k, then the dual varieties of them also make the exact sequence $0 \longrightarrow \hat{Y} \xrightarrow{\hat{\pi}} \hat{X} \xrightarrow{\hat{\iota}} \hat{K} \longrightarrow 0$.

For an invertible sheaf L on an abelian variety X, as stated in the introduction we denote by ϕ_L the homomorphism associated with the divisorial correspondence $m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$, where $m: X \times X \to X$ is the addition morphism of X and $p_i: X \times X \to X$ means the projection to the i-th component for $i=1, 2$. We denote by $K(L)$ the scheme theoretic kernel of ϕ_L. Under this situation we obtain

PROPOSITION 1. *Let K, X, Y and ι, π be as above. Let L be a non-degenerate invertible sheaf on X with $|\chi(L)|=1$. Then $K(\iota^*L) \xrightarrow{\sim} K(\pi^*L)$, in particular* $\deg(\phi_{\iota^*L}) = \deg(\phi_{\pi^*L})$.

In fact, let P_X and P_Y be the Poincaré invertible sheaves of $\hat{X} \times X$ and $\hat{Y} \times Y$ respectively. We identify X and $\hat{\hat{X}}$ (resp. Y and $\hat{\hat{Y}}$) by the divisorial correspondence P_X (resp. P_Y). Moreover we identify X and $\hat{X}$ by the isomorphism ϕ_L.

Then under these identifications, $\phi_L^{-1} = \phi_{(\phi_L)_* L}$ and we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & K(\iota^*L) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & K & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{\iota^*L}} & & \downarrow{\scriptstyle \phi_L}\wr & & \uparrow{\scriptstyle \phi_{\hat{\pi}^*L}} & & \\
0 & \longleftarrow & \hat{K} & \xleftarrow{\hat{\iota}} & \hat{X} & \xleftarrow{\hat{\pi}} & \hat{Y} & \longleftarrow & 0 \\
 & & \downarrow & & & & \uparrow & & \\
 & & 0 & & & & K(\hat{\pi}^*L) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

Then obviously our assertion follows from this diagram.

Moreover we quote a result of T. Matsusaka [2], which will paly an essential part in the proof of our main theorem in next section.

Let J^g be the Jacobian variety of a complete non-singular curve C, φ be the canonical inclusion map of C into J, and Θ be the corresponding canonical divisor on J. For any integer r such that $1 \leq r \leq g$, we put

$\varphi^{(r)} = \varphi \circ p_1 + \cdots + \varphi \circ p_r$: $C^r = \overbrace{C \times \cdots \times C}^{r} \to J$, and W^r = the set theoretic image of $\varphi^{(r)}$, where p_i: $\overbrace{C \times \cdots \times C}^{r} \to C$ means the projection to the i-th component for

$i=1, \ldots, n$.

PROPOSITION 2. $\deg(\varphi^{(r)}: C^r \to W^r) = r!$.
(cf. [1], p. 37)

PROPOSITION 3. (T. Matsusaka) *Let $u_1, \ldots, u_{g-r}$ be $g-r$ independent generic points of J. Then $\Theta_{u_1} \ldots \Theta_{u_{g-r}}$ is numerically equivalent to $(g-r)!W^r$.*
(cf. [2], appendix.)

2. Let X be an abelian variety of dimension r over the field k. For an irreducible curve C on X, and its inclusion morphism f, we define a morphism $f^{(s)}$ by

$$f^{(s)} = f \circ p_1 + \cdots + f \circ p_s \colon C^s = \overbrace{C \times \cdots \times C}^{s} \to X$$

for any positive integer s, where p_i means as above. In particular for $s=0$, we mean by C° a point and by $f^{(\circ)}$ a constant map $C^\circ \hookrightarrow X$ with value origin O.

For a given positive integer n, let $C_i (i=1, \ldots, n)$ be irreducible curves on X and f_i be the inclusion morphism $C_i \hookrightarrow X$ for each i. Let $r_1, \ldots, r_n$ be n non-negative integers such that $r_1 + \cdots + r_n = r$. We denote by $F^{(r_1, \ldots, r_n)}$ the morphism

$$f_1^{(r_1)} \circ p_1^{(r_1)} + \cdots + f_n^{(r_n)} \circ p_n^{(r_n)} \colon C_1^{r_1} \times \cdots \times C_n^{r_n} \longrightarrow X,$$

and we put $d(C_1^{r_1}, \ldots, C_n^{r_n}) = \deg F^{(r_1, \ldots, r_n)}$, where $p_i^{(r_i)} \colon C_1^{r_1} \times \cdots \times C_n^{r_n} \to C_i^{r_i}$ means the projection to the component $C_i^{r_i}$ for each i. Obviously, if $r_i >$ genus of C_i for some i, then $d(C_1^{r_1}, \cdots, C_n^{r_n}) = 0$.

For any 1-cycle $C = \sum_{i=1}^{n} m_i C_i \in \mathscr{Z}^1(X)$, we define the Pontrjagin product $\overset{r}{*}C$ and its degree by

$$\overset{r}{*}C = \sum_{\substack{r_1+\cdots+r_n=r \\ r_i \geq 0}} \frac{r!}{r_1! \cdots r_n!} m_1^{r_1} \cdots m_n^{r_n} d(C_1^{r_1}, \ldots, C_n^{r_n}) \cdot X$$

and

$$\deg(\overset{r}{*}C) = \sum_{\substack{r_1+\cdots+r_n=r \\ r_i \geq 0}} \frac{r!}{r_1! \cdots r_n!} m_1^{r_1} \ldots m_n^{r_n} d(C_1^{r_1}, \ldots, C_n^{r_n}), \text{ respectively.}$$

Obviously these definitions do not depend on the expression $\sum_{i=1}^{n} m_i C_i$ of C. Then from the above notice, we obtain

$$\deg(\overset{r}{*}C) = \sum_{\substack{r_1+\cdots+r_n=r \\ \text{genus of } c_i \geq r_i \geq 0}} \frac{r!}{r_1! \ldots r_n!} m_1^{r_1} \ldots m_n^{r_n} d(C_1^{r_1}, \ldots, C_n^{r_n}). \tag{1}$$

According to K. Toki, we define the *l-number* of a positive 1-cycle C by

$l(C)=\deg(\overset{r}{*}C)/r!$. Moreover for a 1-cycle C on X, we define the homomorphism ϕ_C from $\hat{X}$ to X by $\phi_C(\hat{x})=S(C\cdot(P|_{\{\hat{x}\}\times X}))$ for any $\hat{x}\in\hat{X}$, where P is the Poincaré divisor on $\hat{X}\times X$, and S means the sum on X.

From now on we denote by C a positive 1-cycle on X, which we write $\sum_{i=1}^{n}C_i$ (C_i's may not be distinct.), and we assume that C_i contains the origin, for each i. Let C_i^* be a non-singular model of C_i and J_i the Jacobian variety of it. Let φ_i denote the canonical inclusion map of C_i^* into J_i, and we may assume that for each i, $\varphi_i(C_i^*)$ contains the origin. Then there exists a canonical homomorphism h_i such that

$$\begin{tikzcd} C_i^* \arrow[rr, "f"] \arrow[dr, "\varphi_i"'] & & X \\ & J_i \arrow[ur, "h_i"'] & \end{tikzcd}$$ commutes, for each i.

Therefore we obtain the commutative diagram:

(2)
$$\begin{tikzcd} C_1^*\times\cdots\times C_n^* \arrow[d, "\varphi_1\times\cdots\times\varphi_n"'] \arrow[dr, "f_1\times\cdots\times f_n"] & \\ J_1\times\cdots\times J_n \arrow[r, "h_1\times\cdots h_n"] \arrow[dr, "H"'] & X\times\cdots\times X \arrow[d, "m^{(n)}"] \\ & X \end{tikzcd}$$

where H and $m^{(n)}$ mean the morphism $h_1\circ p_1+\cdots+h_n\circ p_n$ and the addition morphism respectively. Moreover let Θ_i be a canonical divisor on J_i corresponding to $\varphi_i(C_i^*)$ for each $i=1,\ldots,n$. Then we obtain

PROPOSITION 4. $\phi_C=H\circ(\phi_{\Theta_1}^{-1}\times\cdots\times\phi_{\Theta_n}^{-1})\circ\hat{H}$, i.e., $\phi_C=\phi_{H^*\Theta}$, where $\Theta=\sum_{i=1}^{n}J_1\times\cdots\times J_{i-1}\times\Theta_i\times J_{i+1}\times\cdots\times J_n$.

PROOF. Obviously $\widehat{m^{(n)}}=\Delta_{\hat{X}}: \hat{X}\hookrightarrow\overbrace{\hat{X}\times\cdots\times\hat{X}}^{n}$; the diagnoal map. Therefore by the diagram (2),

$$H=m^{(n)}\circ(h_1\times\cdots\times h_n)$$

and

$$\begin{aligned}\hat{H}&=(\hat{h}_1\times\cdots\times\hat{h}_n)\circ\widehat{m^{(n)}}\\ &=(\hat{\varphi}_1^{-1}\times\cdots\times\hat{\varphi}_n^{-1})\circ(\hat{f}_1\times\cdots\times\hat{f}_n)\circ\Delta_{\hat{X}}\\ &=(\phi_{\Theta_1}\times\cdots\times\phi_{\Theta_n})\circ(\hat{f}_1\times\cdots\times\hat{f}_n)\circ\Delta_{\hat{X}}.\end{aligned}$$

Hence

$$H\circ(\phi_{\Theta_1}^{-1}\times\cdots\times\phi_{\Theta_n}^{-1})\circ\hat{H}=H\circ(\hat{f}_1\times\cdots\times\hat{f}_n)\circ\Delta_{\hat{X}}=\phi_C.$$

q.e.d.

MAIN THEOREM. Let X be an abelian variety of dimension r. Then for any positive 1-cycle $C=\sum_{i=1}^{n}C_i$ on X, $\deg(\phi_C)=l(C)^2$.

PROOF. Under the same notations as above, we have only to consider the case where H is surjective. Let K be the connected component at 0 of $(\ker H)_{red}$., and we put Y=the quotient variety $J_1\times\cdots\times J_n/K$. Let π be the canonical surjection from $J_1\times\cdots\times J_n$ to Y. Then there exists an isogeny $\bar{H}$ satisfying the commutative diagram

$$\begin{tikzcd} J_1\times\cdots\times J_n \arrow[r, "H"] \arrow[d, "\pi"'] & X \\ Y \arrow[ur, "\bar{H}"'] & \end{tikzcd}$$

and dualizing this diagram, we have

$$\begin{tikzcd} \hat{J}_1\times\cdots\times\hat{J}_n & \hat{X} \arrow[l, "\hat{H}"'] \arrow[dl, "\hat{\bar{H}}"] \\ \hat{Y} \arrow[u, "\hat{\pi}"] & \end{tikzcd}$$

Therefore from these diagrams and Proposition 4,

$$\phi_C=\bar{H}\circ\pi\circ(\phi_{\Theta_1}^{-1}\times\cdots\times\phi_{\Theta_n}^{-1})\circ\hat{\pi}\circ\hat{\bar{H}}=\bar{H}\circ\phi_{\hat{\pi}^*\Theta}\circ\hat{\bar{H}}. \tag{3}$$

While, by Proposition 1, $\deg(\phi_{\Theta|\hat{Y}})=\deg(\phi_{\Theta|K})$. From the Riemann-Roch theorem on K, we have $\deg(\phi_{\Theta|K})=\{K\cdot\Theta^{g_1+\cdots+g_n-r})/(g_1+\cdots+g_n-r)!\}^2$, where g_i is the genus of C_i for each i. On the other hand, by Proposition 3,

$$\begin{aligned}(K\cdot\Theta^{g_1+\cdots+g_n-r})&=\sum_{\substack{s_1+\cdots+s_n=g_1+\cdots+g_n-r\\ 0\le s_i\le g_i}}\frac{(g_1+\cdots+g_n-r)!}{s_1!\cdots s_n!}(K\cdot\Theta_1^{s_1}\times\cdots\times\Theta_n^{s_n})\\ &=\sum_{\substack{s_1+\cdots+s_n=g_1+\cdots+g_n-r\\ 0\le s_i\le g_i}}(g_1+\cdots+g_n-r)!(K\cdot W_1^{g_1-s_1}\times\cdots\times W_n^{g_n-s_n}),\\ &=(g_1+\cdots+g_n-r)!\sum_{\substack{r_1+\cdots+r_n=r\\ 0\le r_i\le g_i}}(K\cdot W_1^{r_1}\times\cdots\times W_n^{r_n}),\end{aligned}$$

where $r_i=g_i-s_i$. Therefore

$$\deg(\phi_{\Theta|\hat{Y}})=\{\sum_{\substack{r_1+\cdots+r_n=r\\ 0\le r_i\le g_i}}(K\cdot W_1^{r_1}\times\cdots\times W_n^{r_n})\}^2,$$

and from the equality (3),

$$(4) \qquad \deg \phi_C = (\deg \bar{H})^2 \cdot \deg (\phi_{\Theta|\hat{Y}})$$
$$= \{ \sum_{\substack{r_1+\cdots+r_n=r \\ 0\leq r_i\leq g_i}} \deg \bar{H} \cdot (K \cdot W_1^{r_1} \times \cdots \times W_n^{r_n}) \}^2.$$

On the other hand, we have the commutative diagram

$$\begin{tikzcd} C_1^{*r_1} \times \cdots \times C_n^{*r_n} \arrow[d, "\varphi_1^{(r_1)} \times \cdots \times \varphi_n^{(r_n)}"'] \arrow[dr, "F^{(r_1,\dots,r_n)}"] & \\ J_1 \times \cdots \times J_n \arrow[r, "H"'] & X. \end{tikzcd}$$

Therefore

$$d(C_1^{r_1},\dots, C_n^{r_n}) = \deg F^{(r_1,\dots,r_n)}$$
$$= \deg \bar{H} \cdot \deg (\varphi_1^{(r_1)} \times \cdots \times \varphi_n^{(r_n)} : C_1^{r_1} \times \cdots \times C_n^{r_n}$$
$$\longrightarrow W_1^{r_1} \times \cdots \times W_n^{r_n}) \cdot \deg(\pi|_{W_1^{r_1} \times \cdots \times W_n^{r_n}}).$$

Applying Proposition 2, we obtain

$$d(C_1^{r_1},\dots, C_n^{r_n}) = \deg \bar{H} \cdot (r_1! \dots r_n!)(K \cdot W_1 \times \cdots \times W_n).$$

Therefore from (4) and (1), we obtain

$$\deg \phi_C = \left\{ \frac{\deg(*_r C)}{r!} \right\}^2 = l(C)^2.$$

q.e.d.

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Department of Mathematics,
Faculty of Science and Engineering,
Chuo University