HIROSHIMA MATH. J.
4 (1974), 9–28

On the Group of Self-Equivalences of a Mapping Cone

Shichirô OKA, Norichika SAWASHITA and Masahiro SUGAWARA

(Received August 3, 1973)

Introduction

The set $\mathscr{E}(X)$ of homotopy classes of self-(homotopy-)equivalences of a based space X forms a group by the composition of maps, and this group is studied by several authors.

The purpose of this note is to study the group $\mathscr{E}(C_f)$ for a mapping cone $C_f = B \cup_f CA$ of $f: A \to B$ with certain conditions, by the dual considerations of J. W. Rutter [11] using the homotopy exact sequences of cofiberings.

In §1, after preparing some results on $\mathscr{E}(C_f)$, we represent the group $\mathscr{E}(B \vee SA)$, which is the case that f is the constant map, as the split extension of a certain group H by $\mathscr{E}(B) \times \mathscr{E}(SA)$ (Theorem 1.13). In the case that A is the $(m-1)$-sphere S^{m-1}, the above group H is equal to the homotopy group $\pi_m(B)$.

In §2, we have the exact sequence

$$0 \longrightarrow H \longrightarrow \mathscr{E}(B \cup_f e^m) \longrightarrow G \longrightarrow 1$$

for $A = S^{m-1}$, where H is the factor group of $\pi_m(B)$ and G is the subgroup of $\mathscr{E}(B) \times \mathscr{E}(S^m) = \mathscr{E}(B) \times Z_2$. This result is essentially the theorem of W. D. Barcus and M. G. Barratt [1, Th. 6.1].

Furthermore, we study in §3 some cases that the above sequence is split. For the case $2f=0$, we see in Theorem 3.9 that G is the direct product $G_1 \times G_2$ and the subgroup $G_2 = 1 \times Z_2$ is split. By these results, we obtain in Theorem 3.13 the split extension

$$0 \longrightarrow H \longrightarrow \mathscr{E}(S^n \cup_f e^m) \longrightarrow G \longrightarrow 1$$

for a two-cell complex $S^n \cup_f e^m$ $(2 \leqq n \leqq m-2)$ whose attaching map $f \in \pi_{m-1}(S^n)$ is a suspension Sf' and the orders of f and f' are equal. Here

$$H = \pi_m(S^n)/(f_* \pi_m(S^{m-1}) + (Sf)^* \pi_{n+1}(S^n)),$$

$$G = Z_2 \times Z_2 \quad \text{if} \quad 2f=0, \qquad = Z_2 \quad \text{if} \quad 2f \neq 0,$$

and the action of G on H is given by

$$(\tau, \rho)\cdot a = \tau a \rho \quad \text{for} \quad a \in \pi_m(S^n),\ (\tau, \rho) \in Z_2 \times Z_2, \quad \text{if} \quad 2f=0,$$

$$\rho \cdot a = a\rho \quad \text{for} \quad a \in \pi_m(S^n),\ \rho \in Z_2, \quad \text{if} \quad 2f \neq 0.$$

This is a slight improvement of the recent result [8, Th. 3.3].

In § 4, we give some examples of $\mathscr{E}(X)$ for cell complexes X with two or three cells.

In § 5, we are concerned with the product space $B \times \Omega A$ of B and a loop space ΩA, under the dual considerations of § 1, and obtain the dual result of Theorem 1.13 in Theorem 5.8, whose corollary is a slight improvement of the result of Y. Nomura [7, Th. 2.10].

§1. $\mathscr{E}(C_f)$ of a mapping cone C_f

Throughout this note, all (topological) spaces are arcwise connected spaces with base point $*$ and have homotopy types of CW-complexes, and all (continuous) maps and homotopies preserve the base point. For given spaces X and Y, we denote by $[X, Y]$ the set of (based) homotopy classes of maps of X to Y, and by the same letter f a map $f: X \to Y$ and its homotopy class $f \in [X, Y]$. Also, we denote usually by

$$g_*: [X, Y] \longrightarrow [X, Z], \qquad g^*: [Z, X] \longrightarrow [Y, X]$$

the induced maps of a given map $g: Y \to Z$.

For any space X, we denote by $1: X \to X$ the identity map. Then, the set $[X, X]$ is a semi-group with respect to the composition of maps having unit 1, and the group

$$\mathscr{E}(X) \ (\subset [X, X])$$

of self-equivalences of X is the group of invertible elements of $[X, X]$.

In §§ 1–4, we consider the group $\mathscr{E}(C_f)$ of a mapping cone

$$(1.1) \qquad C_f = B \cup_f CA \quad \text{of} \quad f: A \longrightarrow B,$$

under the condition (1.9) below. Let

$$A \xrightarrow{f} B \xrightarrow{i} C_f \xrightarrow{p} SA \xrightarrow{Sf} SB$$

be the sequence of the induced cofiberings, where i is the inclusion, S is the suspension functor and p is the projection.

The co-multiplication

$$l: SA \longrightarrow SA \vee SA,$$

collapsing $A \times \{1/2\}$ of SA to $*$, defines the usual group multiplication $+$ of any homotopy set $[SA, X]$ with unit 0, the class of the constant map $*$. Also, the co-action

$$l: C_f \longrightarrow C_f \vee SA$$

of SA is defined by collapsing $A \times \{1/2\}$ of CA to $*$, and this defines the map

$$\lambda\colon [SA, C_f] \longrightarrow [C_f, C_f], \tag{1.2}$$

$$\lambda(\alpha) = \nabla(1 \vee \alpha) l : C_f \longrightarrow C_f \vee SA \longrightarrow C_f \vee C_f \longrightarrow C_f$$

for $\alpha\colon SA \to C_f$, where ∇ is the folding map. Then, the second multiplication

$$\oplus : [SA, C_f] \times [SA, C_f] \longrightarrow [SA, C_f] \tag{1.3}$$

is defined by

$$\alpha \oplus \beta = \alpha + \lambda(\alpha)\beta \qquad \text{for} \quad \alpha, \beta \in [SA, C_f],$$

and we have the following lemma, which is the dual of [10, Lemmas 3.7–8].

LEMMA 1.4. (i) *The multiplication* $\oplus$ *of* (1.3) *defines a semi-group structure on* $[SA, C_f]$ *with unit* 0, *and the map* λ *of* (1.2) *is a homomorphism of this semi-group to the semi-group* $[C_f, C_f]$.

(ii) $\alpha \oplus \beta = \alpha + \beta$ *if* β *belongs to the image of* i_*: $[SA, B] \to [SA, C_f]$.

PROOF. (i) The equality $\lambda(\alpha \oplus \beta) = \lambda(\alpha)\lambda(\beta)$ follows immediately from the following commutative diagram:

$$\begin{array}{ccccc}
C_f \vee SA & \xrightarrow{1 \vee l} & C_f \vee SA \vee SA & \xrightarrow{1 \vee \alpha \vee \lambda(\alpha)\beta} & C_f \vee C_f \vee C_f \\
\uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l \vee 1} & & \downarrow{\scriptstyle \nabla \vee 1} \\
C_f & \xrightarrow{l} & C_f \vee SA & \xrightarrow{\lambda(\alpha) \vee \lambda(\alpha)\beta} & C_f \vee C_f \\
 & & \downarrow{\scriptstyle 1 \vee \beta} & & \downarrow{\scriptstyle \nabla} \\
 & & C_f \vee C_f \xrightarrow{\nabla} C_f & \xrightarrow{\lambda(\alpha)} & C_f .
\end{array}$$

The associativity of $\oplus$ is proved as follows:

$$\begin{aligned}
(\alpha \oplus \beta) \oplus \gamma &= (\alpha + \lambda(\alpha)\beta) + \lambda(\alpha \oplus \beta)\gamma \\
&= \alpha + \lambda(\alpha)\beta + \lambda(\alpha)\lambda(\beta)\gamma = \alpha \oplus (\beta \oplus \gamma).
\end{aligned}$$

(ii) The desired result follows immediately from the definitions. *q.e.d.*

Consider the map

$$\pi\colon [SA, C_f] \longrightarrow [SA, SA], \tag{1.5}$$

defined by $\pi(\alpha) = 1 + p\alpha$ for $\alpha \in [SA, C_f]$, and the diagram

$$\begin{array}{ccccc}
[SA, C_f] & \xrightarrow{\lambda} & [C_f, C_f] & \xrightarrow{i^*} & [B, C_f] \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_*} & & \uparrow{\scriptstyle i_*} \\
[SA, SA] & \xrightarrow{p^*} & [C_f, SA] & & [B, B]
\end{array} \tag{1.6}$$

where $p: C_f \to SA$ is the projection and $i: B \to C_f$ is the inclusion.

LEMMA 1.7. (i) *The square in* (1.6) *is commutative.*

(ii) *π is a homomorphism of the semi-group $[SA, C_f]$ with $\oplus$ to $[SA, SA]$ with the composition.*

(iii) (cf. [9, Cor. 3.2.2]) *The upper sequence in* (1.6) *is exact, i.e.,* $\mathrm{Im}\lambda = i^{*-1}(i)$.

PROOF. (i) is clear, and we have (ii) since

$$\begin{aligned} \pi(\alpha \oplus \beta) &= 1 + p(\alpha + \lambda(\alpha)\beta) = \pi(\alpha) + p\lambda(\alpha)\beta \\ &= \pi(\alpha) + \pi(\alpha)p\beta = \pi(\alpha)\pi(\beta). \end{aligned}$$

(iii) is proved easily by definition. *q.e.d.*

LEMMA 1.8. *Assume that C_f is 1-connected. Then*

(i) $\lambda(\pi^{-1}(1)) \subset \mathscr{E}(C_f)$.

(ii) *$\pi^{-1}(1)$ is the group with the multiplication $\oplus$ of* (1.3).

PROOF. (i) If $\alpha \in \pi^{-1}(1)$, then (i) and (iii) of the above lemma show that the diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \xrightarrow{i} & C_f & \xrightarrow{p} & SA & \longrightarrow & SB \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \lambda(\alpha)} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ A & \longrightarrow & B & \xrightarrow{i} & C_f & \xrightarrow{p} & SA & \longrightarrow & SB \end{array}$$

of the cofiberings is homotopy commutative. Therefore, we have the induced commutative diagram of the exact sequences of homology groups, and so we see that $\lambda(\alpha)$ induces isomorphisms of homology groups by 5-Lemma. Hence, $\lambda(\alpha) \in \mathscr{E}(C_f)$ by the theorem of J. H. C. Whitehead.

(ii) Consider the element

$$\alpha' = -\lambda(\alpha)^{-1}\alpha \qquad \text{for} \quad \alpha \in \pi^{-1}(1),$$

where $\lambda(\alpha)^{-1}$ is a homotopy inverse of $\lambda(\alpha)$. Then

$$\alpha \oplus \alpha' = \alpha - \lambda(\alpha)\lambda(\alpha)^{-1}\alpha = 0.$$

Lemma 1.7 (ii) and this equality show that $\alpha \oplus \beta$, $\alpha' \in \pi^{-1}(1)$ if α, $\beta \in \pi^{-1}(1)$. *q.e.d.*

From now on, we assume that a mapping cone C_f of (1.1) satisfies the following condition:

(1.9) *The two maps i_* and p^* in* (1.6) *are bijective.*

Then, the two maps

$$(1.10)\qquad \begin{aligned}\varphi &= i_*^{-1} i^*\colon [C_f, C_f] \longrightarrow [B, B],\\ \psi &= p^{*-1} p_*\colon [C_f, C_f] \longrightarrow [SA, SA]\end{aligned}$$

are defined, and it is clear that $\varphi(h)$ and $\psi(h)$ are determined uniquely by the following homotopy commutative diagram:

$$\begin{array}{ccccc} B & \xrightarrow{i} & C_f & \xrightarrow{p} & SA \\ \downarrow{\scriptstyle \varphi(h)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \psi(h)} \\ B & \xrightarrow{i} & C_f & \xrightarrow{p} & SA. \end{array}$$

Therefore, φ and ψ preserve the composition and the images of $\mathscr{E}(C_f)$ by these maps are contained in $\mathscr{E}(B)$ and $\mathscr{E}(SA)$, respectively.

Hence, we have the following proposition by the above lemmas.

PROPOSITION 1.11. *If the mapping cone C_f of* (1.1) *is* 1*-connected and satisfies the condition* (1.9), *then the sequence*

$$\pi^{-1}(1) \xrightarrow{\lambda} \mathscr{E}(C_f) \xrightarrow{\varphi\times\psi} \mathscr{E}(B) \times \mathscr{E}(SA)$$

is exact, where $\pi^{-1}(1)$ is the group of Lemma 1.8 (ii), *the homomorphism λ is the restriction of* (1.2) *and $\varphi\times\psi$ is the homomorphism of* (1.10).

As a sufficient condition for (1.9), we have

LEMMA 1.12. *Assume that A is $(m-2)$-connected and*

$$\dim B \leqq m-2 \quad \text{if } f \neq 0, \qquad \dim B \leqq m-1 \quad \text{if } f=0.$$

Then, the condition (1.9) *holds.*

PROOF. Since A is $(m-2)$-connected, $C_f = B \cup SA$ is considered as a space obtained from B by attaching cells of dimension greater than $m-1$. Therefore,

$$i_*\colon [B, B] \longrightarrow [B, C_f]$$

is bijective if $\dim B \leqq m-2$ and surjective if $\dim B \leqq m-1$, by the cellular approximation theorem. i_* is also injective if $f=0$, since $C_0 = B \vee SA$.

Consider the homotopy exact sequence

$$[SB, SA] \longrightarrow [SA, SA] \xrightarrow{p^*} [C_f, SA] \longrightarrow [B, SA]$$

of cofiberings. Since SA is $(m-1)$-connected, the first set is 0 if $\dim B \leqq m-2$ and the last set is 0 if $\dim B \leqq m-1$. It is clear that p^* is injective if $f=0$, and so we have the lemma. *q.e.d.*

Here, we notice the following theorem for the case $f=0$.

THEOREM 1.13. *Assume that A is $(m-2)$-connected, B is* 1*-connected and*

$\dim B \leqq m-1$ ($m \geqq 3$). *Then, the exact sequence of Proposition* 1.11 *for* $B \vee SA = C_0$ *is the following split exact sequence*:

$$0 \longrightarrow \pi^{-1}(1) \xrightarrow{\lambda} \mathscr{E}(B \vee SA) \xrightarrow{\varphi \times \psi} \mathscr{E}(B) \times \mathscr{E}(SA) \longrightarrow 1.$$

PROOF. It is clear that $\varphi \times \psi$ has a right inverse in this case. By the definition (1.2), we have easily $\lambda(\alpha)i_2 = i_2 + \alpha$ for any $\alpha: SA \to B \vee SA$ and the inclusion $i_2: SA \to B \vee SA$. Therefore, we see that $\alpha = 0$ if $\lambda(\alpha) = 1$, and so we have the desired results by Proposition 1.11 and Lemma 1.12. *q.e.d.*

REMARK. In the above theorem, the group $\pi^{-1}(1)$ has the multiplication $\oplus$ of (1.3), and $\pi^{-1}(1) = p_*^{-1}(0)$ is also a group with $+$ where $p_*: [SA, B \vee SA] \to [SA, SA]$. For the latter group, we have the exact sequence

$$0 \longrightarrow [SA, F] \longrightarrow p_*^{-1}(0) \longrightarrow [SA, B] \longrightarrow 0,$$

where $F = \Omega(B \times SA; B \vee SA, *)$, the space of paths in $B \times SA$ from $B \vee SA$ to $*$.

For the case that $A = S^{m-1}$, the $(m-1)$-sphere, we have $\mathscr{E}(S^m) = Z_2 = \{1, -1\}$ and

COROLLARY 1.14. *If B is 1-connected and* $\dim B \leqq n-1$ ($n \geqq 3$), *we have the split extension*

$$0 \longrightarrow \pi_m(B) \longrightarrow \mathscr{E}(B \vee S^m) \longrightarrow \mathscr{E}(B) \times Z_2 \longrightarrow 0,$$

where $\mathscr{E}(B) \times Z_2$ operates on the homotopy group $\pi_m(B)$ by

$$(h, \varepsilon)\cdot a = ha\varepsilon, \quad \textit{for} \quad (h, \varepsilon) \in \mathscr{E}(B) \times Z_2 \quad \textit{and} \quad a \in \pi_m(B).$$

PROOF. By the cellular approximation theorem, $j_*: \pi_m(B \vee S^m) \to \pi_m(B \times S^m)$ is isomorphic, where j is the inclusion. Therefore, we see that $p_*^{-1}(0)$ is isomorphic to $\pi_m(B)$ by $i_*: \pi_m(B) \to \pi_m(B \vee S^m)$ and the multiplication $\oplus$ of (1.3) is equal to $+$ by Lemma 1.4 (ii). *q.e.d.*

§2. The case $A = S^{m-1}$ and the theorem of Barcus-Barratt

In this section, we study a mapping cone

(2.1) $$C_f = B \cup_f e^m \quad \text{of} \quad f: S^{m-1} \longrightarrow B$$

for the case $A = S^{m-1}$, the $(m-1)$-sphere, under the condition (1.9).

It is clear that $\mathscr{E}(S^m) = \mathscr{E}(S^{m-1}) = Z_2 = \{1, -1\}$.

LEMMA 2.2. *The image G of the homomorphism*

$$\varphi \times \psi: \mathscr{E}(B \cup_f e^m) \longrightarrow \mathscr{E}(B) \times \mathscr{E}(S^m)$$

in Proposition 1.11 *for* $A=S^{m-1}$ *is given by*

$$G=\{(h,\varepsilon)\mid h\in\mathscr{E}(B),\ \varepsilon=\pm 1,\ hf=f\varepsilon \text{ in } \pi_{m-1}(B)\}.$$

PROOF. As is noticed ahead of Proposition 1.11, $\operatorname{Im}(\varphi\times\psi)$ is the set of (h,ε) $\in\mathscr{E}(B)\times\mathscr{E}(S^m)$ such that the middle square is commutative and the right one is homotopy commutative for some $h_1\in\mathscr{E}(B\cup e^m)$ in the following diagram:

$$(*)\qquad \begin{array}{ccccccc} S^{m-1} & \xrightarrow{f} & B & \xrightarrow{i} & B\cup e^m & \xrightarrow{p} & S^m \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle\varepsilon} \\ S^{m-1} & \xrightarrow{f} & B & \xrightarrow{i} & B\cup e^m & \xrightarrow{p} & S^m. \end{array}$$

Consider the commutative diagram of homotopy groups

$$\begin{array}{ccc} \pi_m(B\cup e^m, B) & \xrightarrow{\partial} & \pi_{m-1}(B) \\ \downarrow{\scriptstyle p_*} & & \uparrow{\scriptstyle f_*} \\ \pi_m(S^m) & \underset{\cong}{\xleftarrow{S}} & \pi_{m-1}(S^{m-1}). \end{array}$$

Let $\bar{f}\colon (CS^{m-1}, S^{m-1})\to(B\cup e^m, B)$ be the characteristic map of the cell e^m. Then the element $hf\in\pi_{m-1}(B)$ is equal to $\partial(h_1\bar{f})$ by definition, and so

$$hf=(f_*S^{-1}p_*)(h_1\bar{f})=(f_*S^{-1})(\varepsilon p\bar{f})=f\varepsilon.$$

Therefore the left square in $(*)$ is homotopy commutative, and we have the lemma. *q.e.d.*

Now, consider the homomorphism

$$\lambda\colon \pi^{-1}(1)\longrightarrow\mathscr{E}(C_f)=\mathscr{E}(B\cup_f e^m)$$

in Proposition 1.11, which is the restriction of (1.2).

LEMMA 2.3. *If* $A=S^{m-1}$ *and* B *is* 1-*connected, then the multiplication* $\oplus$ *of* (1.3) *coincides with the usual multiplication* $+$ *on* $\pi^{-1}(1)=p_*^{-1}(0)$, *where* $p_*\colon \pi_m(C_f)\to\pi_m(S^m)$ *is the induced homomorphism of the projection* p.

PROOF. It is clear that $\pi^{-1}(1)=p_*^{-1}(0)$ by (1.5).

Consider the product space $C_f\times S^m$ and the inclusions $j\colon C_f\vee S^m\to C_f\times S^m$, $j_1\colon C_f\to C_f\vee S^m$, $j_2\colon S^m\to C_f\vee S^m$, and the projections $p_1\colon C_f\times S^m\to C_f$, $p_2\colon C_f\times S^m\to S^m$. By the cellular approximation theorem, we see easily that

$$j_*\colon [S^m, C_f\vee S^m]\longrightarrow[S^m, C_f\times S^m]$$

is isomorphic since C_f is 1-connected. Therefore we have

$$g=j_1p_1jg+j_2p_2jg \qquad \text{for any} \qquad g\in[S^m, C_f\vee S^m].$$

This and the definition of $\oplus$ of (1.3) imply

$$\begin{aligned} \alpha\oplus\beta &= \alpha+\lambda(\beta)\alpha = \alpha+\nabla(1\vee\alpha)l\beta \\ &= \alpha+\nabla(1\vee\alpha)j_1p_1jl\beta+\nabla(1\vee\alpha)j_2p_2jl\beta, \end{aligned}$$

for α, $\beta\in[S^m, C_f]$. It is easy to see that the last is equal to $\alpha+\beta+\alpha p\beta$. Therefore we have $\alpha\oplus\beta=\alpha+\beta$ if $\beta\in\pi^{-1}(1)=p_*^{-1}(0)$. *q.e.d.*

To study the image of λ of the above, we consider the map λ of (1.2) and the map Γ of Barcus-Barratt [1, §§2–4], defined as follows. Assume that

(2.4) *A and B are homotopy associative co-H-spaces with homotopy inverses.*

Then, the mapping spaces X^A and X^B are naturally homotopy associative H-spaces with homotopy inverses. Furthermore, for any given maps $f\colon A\to B$ and $u\colon B\to X$, the homomorphism

$$\Gamma(u, f)\colon [SB, X]\longrightarrow[SA, X] \tag{2.5}$$

is defined to be the composition of

$$\begin{aligned} &[SB, X]=\pi_1(X^B, *)\xrightarrow{a_*}\pi_1(X^B, u) \\ &\xrightarrow{b_*}\pi_1(X^A, uf)\xrightarrow{c_*}\pi_1(X^A, *)=[SA, X], \end{aligned}$$

where $a\colon X^B\to X^B$ and $c\colon X^A\to X^A$ are the left translations by $u\in X^B$ and $(uf)^{-1}\in X^A$, respectively, and $b\colon X^B\to X^A$ is the map defined by the composition of f.

Now, for a mapping cone $C_f=B\cup_f e^m$ of (2.1) with the assumption (2.4) for B, we consider the following diagram:

$$\begin{array}{ccccccc} & & [SB, B] & \xrightarrow{i_*} & [SB, C_f] & & \pi_m(S^m) \\ & & \downarrow{\scriptstyle \Gamma(1,f)} & & \downarrow{\scriptstyle \Gamma(i,f)} & \nearrow{\scriptstyle p_*} & \uparrow{\scriptstyle p_*} \\ \pi_{m+1}(C_f, B) & \xrightarrow{\partial} & [S^m, B] & \xrightarrow{i_*} & [S^m, C_f] & \xrightarrow{j_*} & \pi_m(C_f, B) \\ \downarrow{\scriptstyle p_*} & & \uparrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \lambda} & & \\ \pi_{m+1}(S^m) & \underset{\cong}{\xleftarrow{S}} & \pi_m(S^{m-1}) & & [C_f, C_f] & & \end{array} \tag{2.6}$$

where λ is the map of (1.2), and the middle horizontal sequence is the homotopy exact sequence of (C_f, B). The left square and the triangle are commutative, and so is the middle square by the definition of (2.5).

The following lemma is proved by several authors.

LEMMA 2.7. (cf. [9, Cor. 3.2.2]) *The right vertical sequence in* (2.6) *is exact, i.e.,* $\operatorname{Im}\Gamma(i, f)=\lambda^{-1}(1)$.

LEMMA 2.8. *If* $\dim B\leqq m-2$, *then the upper* i_* *in* (2.6) *is surjective.*

Proof. This is clear by the cellular approximation theorem. *q.e.d.*

Lemma 2.9. *If B is 1-connected, then the right p_* is isomorphic and the left p_* is surjective, in* (2.6).

Proof. The desired results follow immediately from the theorem of Blakers-Massey [2, Th. II], since (C_f, B) is $(m-1)$-connected and B is 1-connected. *q.e.d.*

By the commutative diagram (2.6) and Lemmas 2.3, 2.7–9, we see that the image of the homomorphism

$$\lambda: \pi^{-1}(1) = p_*^{-1}(0) \longrightarrow \mathscr{E}(C_f)$$

in Proposition 1.11 is isomorphic to the group

$$H = \pi_m(B)/(\operatorname{Im} f_* + \operatorname{Im} \Gamma(1, f)). \tag{2.10}$$

Therefore, we have the following result, which is essentially the theorem of Barcus-Barratt [1, Th. 6.1], by Proposition 1.11 and Lemmas 1.12, 2.2.

Theorem 2.11. *Let B be a simply-connected CW-complex, $f: S^{m-1} \to B$ $(m \geqq 3)$ be a given map and $C_f = B \cup_f e^m$ be its mapping cone. Assume that $f \not\simeq 0$, $\dim B \leqq m-2$ and B is a homotopy associative co-H-space with a homotopy inverse. Then, the following sequence is exact:*

$$0 \longrightarrow H \xrightarrow{\lambda i_*} \mathscr{E}(C_f) \xrightarrow{\varphi \times \psi} G \longrightarrow 1,$$

where G is the group in Lemma 2.2 and H is the group of (2.10).

The homomorphism $\Gamma(u, f)$ of (2.5) is also defined if X is a homotopy associative H-space with a homotopy inverse, and it is easy to see that $\Gamma(u, f) = (Sf)^*$ (cf. [9, Th. 3.3.3]). Therefore, we have the following theorem by the diagram (2.6) in which $\Gamma(1, f)$ is replaced by $(Sf)^*$.

Theorem 2.12. *Theorem 2.11 in which $\Gamma(1, f)$ is replaced by $(Sf)^*$ also holds, under the assumption that C_f is a homotopy associative H-space with a homotopy inverse, instead of the assumption that B is a co-H-space. Also, H is isomorphic to*

$$\operatorname{Im}(i_*: \pi_m(B) \to \pi_m(C_f)) \,/\, \operatorname{Im}((Sf)^*: [SB, C_f] \to \pi_m(C_f)).$$

§3. Group extension in Theorem 2.11 and complexes with two cells

In Corollary 1.14, the group $\mathscr{E}(B \vee S^m)$ is determined as the split extension. In this section, we study some cases that the group extension in Theorem 2.11 is split.

Let $h: C_f \to C_f$ $(C_f = B \cup_f e^m)$ be a map such that

(3.1) $$h|B = i \text{ (the inclusion)},$$
$$h(S^{m-1} \times [0,3/4]) \subset B, \quad h(x, t) = (x, 4t-3) \text{ for } 3/4 \leqq t \leqq 1,$$

where $(x, t) \in C_f = B \cup_f CS^{m-1}$ is the image of $(x, t) \in S^{m-1} \times I$. Then, h defines the element $\alpha(h) \in \pi_m(B)$ by the composition

(3.2) $$\alpha(h) = \beta q: S^m \xrightarrow{q} S^{m-1} \times S^1 / * \times S^1 \xrightarrow{\beta} B,$$

where q is the map identifying $* \in S^m$ and its antipodal point, and $\beta(x, e^{2\pi i t}) = h(x, 3t/4)$ $(0 \leqq t \leqq 1)$.

We have the following lemma, for the composition

$$\lambda i_*: \pi_m(B) \xrightarrow{i_*} [S^m, C_f] \xrightarrow{\lambda} [C_f, C_f],$$

which induces the homomorphism $\lambda i_*: H \to \mathscr{E}(C_f)$ of Theorem 2.11.

LEMMA 3.3. *$\lambda i_* \alpha(h) = h$ for any $h \in [C_f, C_f]$ satisfying* (3.1).

PROOF. Let $h': S^m = SS^{m-1} \to C_f$ be the map defined by $h'(x, t) = (x, 1-2t)$ for $0 \leqq t \leqq 1/2$, $= h(x, 2t-1)$ for $1/2 \leqq t \leqq 1$. Then, we see easily $\lambda(h') = h$ by the definition (1.2) of λ, since $h|B = i$. Also, let

$$q': S^{m-1} \times S^1 / * \times S^1 \longrightarrow S^{m-1} \times S^1 / (* \times S^1 \cup S^{m-1} \times *) = S^m$$

be the quotient map. Then, $h'q'$ is homotopic to $i\beta$, since

$$h'q'(x, e^{2\pi i t}) = \begin{cases} h'q'(x, e^{2\pi i(1-t/4)}) & \text{for } 0 \leqq t \leqq 1/2, \\ \beta(x, e^{2\pi i(8t-4)/3}) & \text{for } 1/2 \leqq t \leqq 7/8, \end{cases}$$

where β is the map of (3.2). Therefore $i\alpha(h) = i\beta q$ is homotopic to $h'q'q$, and so to h' because $q'q: S^m \to S^m$ is a map of degree 1. These show the lemma. *q.e.d.*

Now, we consider the case that

(3.4) $$f \in \pi_{m-1}(B) \quad \textit{satisfies} \quad 2f = 0.$$

In this case, it is easy to see by the definition in Lemma 2.2 that the group G in Theorem 2.11 is the direct product:

(3.5) $$G = G_1 \times G_2, \quad G_1 = \{(h, 1) \mid h \in \mathscr{E}(B), hf = f\}, \quad G_2 = \{(1, 1), (1, \rho)\},$$

where $\rho = -1$, since $f = -f = f\rho$.

Take $\rho = -1: S^{m-1} \to S^{m-1}$ to be the reflexion. For a homotopy

(3.6) $$f_t: S^{m-1} \longrightarrow B, \quad f_0 = f, \quad f_1 = f\rho,$$

the map $\sigma: C_f \to C_f$ is defined by

(3.7) $$\sigma|B = i,\quad \sigma(x, t) = \begin{cases} f_{2t}(x) & \text{for } 0 \leqq t \leqq 1/2, \\ (\rho(x), 2t-1) & \text{for } 1/2 \leqq t \leqq 1, \end{cases}$$

and it is easy to see by (1.10) that

$$(\varphi \times \psi)\sigma = (1, \rho) \in G_2 \quad \text{and} \quad \sigma \in \mathscr{E}(C_f).$$

Also, $h = \sigma\sigma$ satisfies (3.1) and we have

LEMMA 3.8. *The element $\alpha(\sigma\sigma) \in \pi_m(B)$ of* (3.2) *satisfies*

$$\alpha(\sigma\sigma) \in \mathrm{Im}\,(f_*\colon \pi_m(S^{m-1}) \to \pi_m(B)).$$

If this is valid, we have $\sigma\sigma = 1$ by Lemma 3.3 because $\alpha(\sigma\sigma) = 0$ in H of (2.10). Therefore, we have a homomorphism $\sigma\colon G_2 \to \mathscr{E}(C_f)$ such that $(\varphi \times \psi)\sigma = 1$, and we obtain the following

THEOREM 3.9. *Assume that $f \in \pi_{m-1}(B)$ satisfies $2f = 0$. Then, in Theorem* 2.11, *the group G is the direct product $G_1 \times G_2$ of* (3.5), *and the subgroup $G_2 = Z_2$ is split. Therefore, we have the following exact sequence*:

$$1 \longrightarrow D(H) \longrightarrow \mathscr{E}(C_f) \longrightarrow G_1 \longrightarrow 1,$$

where H is the group of Theorem 2.11 *and $D(H)$ is the split extension*

(3.10) $$0 \longrightarrow H \longrightarrow D(H) \longrightarrow Z_2 \longrightarrow 1$$

acting $Z_2 = \{1, -1\}$ on H by $(-1)\cdot a = -a$ for $a \in H$.

Now, we prove Lemma 3.8. By (3.7) and (3.2), the element $\alpha(\sigma\sigma)$ is represented by the composition

$$\alpha(f_t) = \beta(f_t)q\colon S^m \longrightarrow S^{m-1} \times S^1 / * \times S^1 \longrightarrow B,$$

where q is the map identifying $*$ and its antipodal point, and

(3.11) $$\beta(f_t)(x, e^{2\pi i t}) = \begin{cases} f_{2t}(x) & \text{for } 0 \leqq t \leqq 1/2, \\ f_{2t-1}(\rho x) & \text{for } 1/2 \leqq t \leqq 1. \end{cases}$$

Then Lemma 3.8 is proved by the following

LEMMA 3.12. *For any homotopy f_t of* (3.6), *the element $\alpha(f_t) \in \pi_m(B)$ defined above belongs to $f_*(\pi_m(S^{m-1}))$.*

PROOF. Let K be the complex obtained from $S^{m-1} \times I$ by identifying $(x, 0) \sim (\rho x, 1)$ and shrinking $* \times I$ to $*$, and $p\colon S^{m-1} \times I \to K$ be the identification map. Then, a given homotopy f_t of (3.6) defines the map

$$F: K \longrightarrow B \quad \text{by} \quad Fp(x, t) = f_t(x).$$

Also, because p is a homotopy of the inclusion $i: S^{m-1} = S^{m-1} \times 0 \subset K$ to $i\rho$, the map $\beta(p): S^{m-1} \times S^1 / * \times S^1 \to K$ is defined in the same way as (3.11), and we have

$$\alpha(f_t) = \beta(f_t)q = F\beta(p)q.$$

On the other hand, K is the mapping cone $S^{m-1} \cup_2 e^m$ of the map $S^{m-1} \to S^{m-1}$ of degree 2, and so we have the exact sequence

$$\pi_m(S^{m-1}) \xrightarrow{\times 2} \pi_m(S^{m-1}) \xrightarrow{i_*} \pi_m(K) \longrightarrow \pi_m(S^m) \xrightarrow{\times 2} \pi_{m-1}(S^{m-1}),$$

which is obtained from the homotopy exact sequence of (K, S^{m-1}) using Lemma 2.9. Therefore, we see that i_* is an isomorphism, and so the element $\beta(p)q \in \pi_m(K)$ of above is contained in the image of i_*. Since $Fi = f$, these show that $\alpha(f_t) = F\beta(p)q \in F_* i_*(\pi_m(S^{m-1})) = f_*(\pi_m(S^{m-1}))$, as desired. *q.e.d.*

As an application of Theorems 2.11 and 3.9, we have the following theorem for suspended two-cell complexes, which is an improvement of [8, Th. 3.3].

THEOREM 3.13. *Let $S^n \cup_f e^m$ be a two-cell complex with an attaching map $f \in \pi_{m-1}(S^n)$ which is a suspension Sf', where $2 \leqq n \leqq m-2$.*

(i) *If $2f \neq 0$, then the group $\mathscr{E}(S^n \cup_f e^m)$ is the split extension*

$$0 \longrightarrow H \longrightarrow \mathscr{E}(S^n \cup_f e^m) \longrightarrow Z_2 \longrightarrow 1,$$

where $$H = \pi_m(S^n) / (f_* \pi_m(S^{m-1}) + (Sf)^* \pi_{n+1}(S^n))$$

and the action of Z_2 on H is given by $(-1)\cdot a = -(-1)a$ for $a \in \pi_m(S^n)$.

(ii) *If $2f = 0$, then we have the exact sequence*

$$1 \longrightarrow D(H) \longrightarrow \mathscr{E}(S^n \cup_f e^m) \longrightarrow Z_2 \longrightarrow 1$$

where $D(H)$ is the split extension (3.10) *of the above group H by Z_2.*

Furthermore, if $2f' = 0$, then the above sequence is split and the action of Z_2 on $D(H)$ is given by $(-1)\cdot(a\varepsilon) = -(-1)a\varepsilon$ for $a\varepsilon \in D(H)$.

PROOF. Since $f = Sf'$, we have $(-1)f = -f = f(-1)$, and so the group G in Theorem 2.11 for this case is given by

$$G = \mathscr{E}(S^n) \times \mathscr{E}(S^m) = Z_2 \times Z_2 \text{ if } 2f = 0, \quad = \{(1, 1), (-1, -1)\} \text{ if } 2f \neq 0.$$

Also, we see easily by the definition of (2.5) that the homomorphism $\Gamma(1, f): \pi_{n+1}(S^n) \to \pi_m(S^n)$ is equal to $(Sf)^*$ since $f = Sf'$, and so we have the desired exact sequence by Theorems 2.11 and 3.9.

We consider $\rho = -1$ as the reflexion on S^n (or S^{m-1}) fixing S^{n-1} (or S^{m-2}). Then ρf is equal to $f\rho$ since $f = Sf'$, and so the homeomorphism $R: S^n \cup e^m \to S^n \cup e^m$ is defined by $R|S^n = \rho$ and $R(x, t) = (\rho x, t)$. The element $R \in \mathscr{E}(S^n \cup e^m)$

satisfies

$$(\varphi\times\psi)R=(\rho,\ \rho)\in G\quad\text{and}\quad RR=1.$$

Hence, we have a splitting homomorphism $R\colon G\to\mathscr{E}(S^n\cup e^m)$ if $2f\neq 0$.

If $2f'=0$, we can choose such a homotopy f_t of (3.6) for $B=S^n$ that ρf_t is equal to $f_t\rho$, using a homotopy of f' to $-f'$. Therefore, we see that $\sigma R=R\sigma$ for the element $\sigma\in\mathscr{E}(S^n\cup e^m)$ of (3.7), and so we have a right inverse of $\varphi\times\psi$ in Theorem 2.11 by sending $(1,\ -1)$ and $(-1,\ -1)$ of $G=Z_2\times Z_2$ to σ and R, respectively. *q.e.d.*

The extension is not known to us, for the case $2f=0$ and $2f'\neq 0$ in the above theorem. Also for the case that f is not a suspension, we have only the following partial results. Let

$$\gamma(f)\colon \pi_{n+1}(S^n)\longrightarrow\pi_m(S^n) \tag{3.14}$$

be the homomorphism defined by

$$\gamma(f)\eta=\eta Sf+[\iota_n,\ \eta]Sh(f)\qquad\text{for }\eta\in\pi_{n+1}(S^n),$$

where $[\iota_n,\ \eta]\in\pi_{2n}(S^n)$ is the Whitehead product of $\iota_n=1\in\pi_n(S^n)$ and η, and $h(f)\in\pi_{m-1}(S^{2n-1})$ is the generalized Hopf invariant of f due to P. J. Hilton [3]. Also, set

$$H=\pi_m(S^n)/(f_*\pi_m(S^{m-1})+\gamma(f)\pi_{n+1}(S^n)).$$

THEOREM 3.15. *For a two-cell complex* $S^n\cup_f e^m(2\leqq n\leqq m-2)$, *we have the exact sequence*

$$1\longrightarrow H_1\longrightarrow\mathscr{E}(S^n\cup_f e^m)\longrightarrow G_1\longrightarrow 0,$$

$$H_1=H\quad\text{if}\quad 2f\neq 0,\quad =D(H)\quad\text{if}\quad 2f=0,$$

$$G_1=\begin{cases}Z_2 & \text{if } 2f=a(f),\text{ or } 2f\neq 0 \text{ and } a(f)=0,\\ 1 & \text{otherwise,}\end{cases}$$

where $D(H)$ *is the split extension of* (3.10) *and*

$$a(f)=f+(-1)f=[\iota_n,\ \iota_n]h(f). \tag{3.16}$$

PROOF. The last equality is proved in [3, Th. 6.7, Th. 6.9].

By Theorems 2.11 and 3.9, it is sufficient to show that the homomorphism $\Gamma(1,f)\colon\pi_{n+1}(S^n)\to\pi_m(S^n)$ in (2.6) for $B=S^n$ is equal to $\gamma(f)$ of (3.14). We notice that the map $\phi_v v_\#^{-1}$ in [1, Th. (4.6)] coincides by definition with $\Gamma(v,\ \phi)$ of (2.5). Hence, by applying [1, Th. (4.6)] to $\Gamma(1,f)$, we have

$$\Gamma(1,f)(\eta)=\eta Sf+[\iota_n,\ \eta]Sh(f)+A,$$

where A is the sum of compositions of some elements and the iterated Whitehead products $[\iota_n, [\iota_n, [\dots, \eta]]]$. Since $3[\iota_n, [\iota_n, \iota_n]]=0$ (cf. e.g. [3, Th. 6.10]), these Whitehead products are zero, and so we have the desired results. *q.e.d.*

§4. Some examples

In this section, we give some examples of $\mathscr{E}(X)$ for complexes X with two or three cells. For the calculations, we use several results on the homotopy groups of spheres which are referred mainly to Toda's book [12].

For a two-cell complex, we denote by $\mathscr{E}(f)=\mathscr{E}(S^n \cup_f e^m)$ for $f \in \pi_{m-1}(S^n)$. Also, we denote by $S: \mathscr{E}(f) \to \mathscr{E}(Sf)$ the homomorphism induced by the suspension.

EXAMPLE 4.1. *For the Hopf maps η_2, ν_4, σ_8 and their suspensions*

$$\eta_n \in \pi_{n+1}(S^n)\ (n \geqq 2), \quad \nu_n \in \pi_{n+3}(S^n)\ (n \geqq 4), \quad \sigma_n \in \pi_{n+7}(S^n)\ (n \geqq 8),$$

we have

$$\mathscr{E}(\eta_2)=Z_2, \qquad \mathscr{E}(\eta_n)=Z_2 \oplus Z_2 \quad \text{for } n \geqq 3,$$

$$\mathscr{E}(\nu_4)=Z_2, \qquad \mathscr{E}(\nu_n)=Z_2 \quad \text{for } n \geqq 5,$$

$$\mathscr{E}(\sigma_8)=Z_2 \oplus Z_2, \qquad \mathscr{E}(\sigma_n)=Z_2 \oplus Z_2 \quad \text{for } n \geqq 9.$$

$S: \mathscr{E}(\eta_n) \to \mathscr{E}(\eta_{n+1})$ is injective if $n=2$ and bijective if $n \geqq 3$, $S: \mathscr{E}(\nu_n) \to \mathscr{E}(\nu_{n+1})$ is trivial if $n=4$ and bijective if $n \geqq 5$, and $S: \mathscr{E}(\sigma_n) \to \mathscr{E}(_{n+1})$ maps onto Z_2 if $n=8$ and is bijective if $n \geqq 9$.

PROOF. Theorem 3.15 shows the desired results for $\mathscr{E}(\eta_n)$, since $a(\eta_2)= [\iota_2, \iota_2]=2\eta_2$ and $\pi_{n+2}(S^n)$ is generated by $\eta_n\eta_{n+1}$.

We have $G_1=1$ for $\mathscr{E}(\nu_4)$, since $a(\nu_4)=[\iota_4, \iota_4]=2\nu_4-S\omega$ (ω is the generator of $\pi_6(S^3)=Z_{12}$). Also, $\pi_8(S^4)=Z_2 \oplus Z_2$ is generated by $\nu_4\eta_7$ and $(S\omega)\eta_7$, and $\gamma(\nu_4)\eta_4=\eta_4\nu_5+[\iota_4, \iota_4]\eta_7=(S\omega)\eta_7+2\nu_4\eta_7-(S\omega)\eta_7=0$, and so we have $\mathscr{E}(\nu_4)=Z_2$ by Theorem 3.15. Also $\mathscr{E}(\nu_n)=Z_2$ ($n \geqq 5$), since $\pi_9(S^5)=\nu_{5*}\pi_9(S^8)$ and $\pi_{n+4}(S^n) =0$ for $n \geqq 6$. (Cf. [12, pp. 43–44]).

By the same way, $G_1=1$ for $\mathscr{E}(\sigma_8)$. Also, $\pi_{16}(S^8)=Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ is generated by $\sigma_8\eta_{15}$, $(S\sigma')\eta_{15}$, $\bar{\nu}_8$ and ε_8, and $\gamma(\sigma_8)\eta_8=\eta_8\sigma_9+[\iota_8, \iota_8]\eta_{15}=((S\sigma')\eta_{15} +\bar{\nu}_8+\varepsilon_8)+2\sigma_8\eta_{15}-(S\sigma')\eta_{15}=\bar{\nu}_8+\varepsilon_8$. Hence we have $\mathscr{E}(\sigma_8)=Z_2 \oplus Z_2$ by Theorem 3.15. Theorem 3.13 (i) shows that $\mathscr{E}(\sigma_n)=Z_2 \oplus Z_2$ for $n \geqq 9$ by the following results: $\pi_{17}(S^9)=Z_2 \oplus Z_2 \oplus Z_2$ is generated by $\sigma_9\eta_{16}$, $\bar{\nu}_9$ and ε_9; $\pi_{n+8}(S^n)=Z_2 \oplus Z_2$ is generated $\bar{\nu}_n$ and ε_n for $n \geqq 10$; $\sigma_{n+1}^*\eta_n=\eta_n\sigma_{n+1}=\bar{\nu}_n+\varepsilon_n$ for $n \geqq 9$. (Cf. [12, pp. 61, 64]).

EXAMPLE 4.2. *For the generator $\omega_3 \in \pi_6(S^3)=Z_{12}$ and its suspension*

$\omega_n \in \pi_{n+3}(S^n)$ $(n \geqq 3)$, *we have*

$$\mathcal{E}(\omega_3)=Z_2, \quad \mathcal{E}(\omega_4)=\mathcal{E}(\omega_5)=Z_2 \oplus Z_2, \quad \mathcal{E}(\omega_n)=Z_2 \ (n \geqq 6).$$

The suspension $S: \mathcal{E}(\omega_n) \to \mathcal{E}(\omega_{n+1})$ *is injective if* $n=3$, *surjective if* $n=5$, *and bijective if* $n=4$ *or* $n \geqq 6$.

PROOF. We have $a(\omega_3)=0$ since S^3 is an H-space, and so $\mathcal{E}(\omega_3)=Z_2$ by Theorem 3.15 since $\pi_7(S^3)=Z_2$ is generated by $\omega_3\eta_6$. Theorem 3.13 (i) shows the desired results for $\mathcal{E}(\omega_4)$ and $\mathcal{E}(\omega_5)$, since $\pi_8(S^4)=Z_2 \oplus Z_2$ is generated by $\nu_4\eta_7$ and $\omega_4\eta_7$, $\pi_9(S^5)=Z_2$ is gnerated by $\nu_5\eta_8$, and $\eta_4\omega_5=\eta_5\omega_6=\omega_5\eta_8=0$. $\mathcal{E}(\omega_n)=Z_2$ $(n \geqq 6)$ follows immediately since $\pi_{n+4}(S^n)=0$. (Cf. [12, p. 43]).

EXAMPLE 4.3. (i) $\mathcal{E}(\omega_4\eta_7)=Z_2 \oplus Z_2 \oplus Z_2$.

(ii) $\mathcal{E}(\omega_4\eta_7\eta_8)$ *is the split extension*

$$0 \longrightarrow Z_{24} \oplus Z_3 \longrightarrow \mathcal{E}(\omega_4\eta_7\eta_8) \longrightarrow Z_2 \times Z_2 \longrightarrow 1,$$

where the action of $Z_2 \times Z_2$ *on* $Z_{24} \oplus Z_3$ *is given by*

$$(1, -1)\cdot(a+b) = -a-b, \quad (-1, -1)\cdot(a+b) = -a+b \quad \text{for} \quad a \in Z_{24},\ b \in Z_2.$$

PROOF. These are the consequences of Theorem 3.13 (ii).

(i) By [12, p. 44], $\pi_9(S^4)=Z_2 \oplus Z_2$ is generated by $\nu_4\eta_7\eta_8$ and $\omega_4\eta_7\eta_8$, and $\eta_4\omega_5\eta_8=0$, and so we have $H=Z_2$ for $f=\omega_4\eta_7$ in Theorem 3.13 (ii).

(ii) By [12, pp. 46, 186], $\pi_{10}(S^4)=Z_{24} \oplus Z_3$ is generated by $\nu_4\nu_7$ and $S\alpha$ $(\alpha \in \pi_9(S^3)=Z_3)$. Since $(\omega_4\eta_7\eta_8)\eta_9=12\omega_4\nu_7=0$ and $\eta_4(\omega_5\eta_8\eta_9)=0$, we have $H=\pi_{10}(S^4)$ for $f=\omega_4\eta_7\eta_8$. The action of the split extension is determined by

$$(-1)\nu_4\nu_7 = (-\nu_4+[\iota_4, \iota_4])\nu_7 = (\nu_4-\omega_4)\nu_7 = \nu_4\nu_7. \qquad q.e.d.$$

EXAMPLE 4.4. *Let* $(S^n)_2$ *be the reduced product of* S^n *due to I. M. James. Then*

$$\mathcal{E}((S^n)_2)=\mathcal{E}([\iota_n, \iota_n])=\begin{cases} D(\pi_{2n}(S^n)/Y) \times Z_2 & \text{for odd } n, \\ D(\pi_{2n}(S^n)/Y) & \text{for even } n. \end{cases}$$

Here, D is the split extension of (3.10), *and the subgroup Y of* $\pi_{2n}(S^n)$ *is generated by* $[\iota_n, \iota_n]\eta_{2n-1}$ *and is given by*

$$Y=0 \quad \text{if } n \equiv -1 \ (4) \quad \text{or } n=2, 6, \quad =Z_2 \text{ otherwise}.$$

PROOF. $(S^n)_2$ is obtained from $S^n \times S^n$ by identifying $(x, *) \sim (*, x)$ for $x \in S^n$, and so it is the mapping cone of $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ by definition.

The group $\mathcal{E}([\iota_n, \iota_n])$ is determined without using Theorem 3.13. The exact sequence

$$0 \longrightarrow H \longrightarrow \mathscr{E}([\iota_n, \iota_n]) \xrightarrow{\varphi \times \psi} G \longrightarrow 1$$

of Theorem 2.11 for $f=[\iota_n, \iota_n]$ is given as follows:

$$G=\mathscr{E}(S^n)\times\mathscr{E}(S^{2n})=Z_2\times Z_2 \text{ if } n \text{ is odd,} \quad =Z_2\times 1 \text{ if } n \text{ is even,}$$

since $(-\iota_n)[\iota_n, \iota_n]=[\iota_n, \iota_n]$ for any n and $2[\iota_n, \iota_n]=0$ iff n is odd;

$$H=\pi_{2n-1}(S^n)/Y, \quad Y=[\iota_n, \iota_n]_*\pi_{2n}(S^{2n-1}),$$

which is given as above by [4, p. 232] and [5, Lemma 5.1].

Let σ, T: $(S^n)_2\to(S^n)_2$ be the homeomorphism induced by $(-\iota_n)\times(-\iota_n)$, T: $S^n\times S^n\to S^n\times S^n$, respectively, where $-\iota_n$ means the reflexion and T is the switching map. Then we have $(\varphi\times\psi)\sigma=(-1, 1)$, and $(\varphi\times\psi)T=(1, -1)$ if n is odd and $T=1$ if n is even. Since $\sigma\sigma=TT=1$ and $\sigma T=T\sigma$, we see that the above sequence is split. Since $(-1)a=-a+[\iota_n, \iota_n]h(a)\equiv -a$ in H for $a\in\pi_{2n}(S^n)$, the action of G on H is given by $(-1, 1)\cdot a=(1, -1)\cdot a=-a$ for $a\in H$, and we have the desired results. *q.e.d.*

Finally, we give examples for complexes with three cells.

EXAMPLE 4.5. *For the special unitary group $SU(3)$ and the symplectic group $Sp(2)$, we have*

$$\mathscr{E}(SU(3))=D(Z_{12})\times Z_2, \qquad \mathscr{E}(Sp(2))=D(Z_{120}).$$

PROOF. It is well known that $SU(3)=B\cup_f e^8$, $B=S^3\cup_{\eta_3}e^5$. By the homotopy exact sequence of the fibering $S^3\longrightarrow SU(3)\xrightarrow{p} S^5$, we see that

$$\pi_4(SU(3))=0, \qquad \pi_7(SU(3))=0, \qquad \pi_8(SU(3))=Z_{12},$$

and $p_*: \pi_8(SU(3))\to\pi_8(S^5)=Z_{24}$ is injective. By the exact sequence of $(SU(3), B)$, $\pi_7(B)$ is isomorphic to $\pi_8(SU(3), B)=Z$, and so $\pi_7(B)=Z$ is generated by f. Therefore, the group G in Theorem 2.11 is isomorphic to $\mathscr{E}(B)$, which is $Z_2\oplus Z_2$ by Example 4.1.

Since $i_*: \pi_8(B)\to\pi_8(SU(3))$ is surjective, the group H in Theorem 2.11 (or 2.12) is isomorphic to $\pi_8(SU(3))/\mathrm{Im}\,(Sf)^*$, where $(Sf)^*: [SB, SU(3)]\to\pi_8(SU(3))$. On the other hand, $Sf\in j_*\pi_8(S^4)$ by [6, (3.1)], where j: $S^4\to SB$ is the inclusion. Hence $(Sf)^*=0$ since $\pi_4(SU(3))=0$, and we have the exact sequence

$$0\longrightarrow Z_{12}\xrightarrow{\lambda}\mathscr{E}(SU(3))\longrightarrow Z_2\oplus Z_2\longrightarrow 0.$$

Let c, $v\in\mathscr{E}(SU(3))$ be the elements given by $c(x)=\bar{x}$, $v(x)=x^{-1}$ for $x\in SU(3)$, where $\bar{x}$ is the conjugate of x. Then, it is easy to see that the splitting homomorphism

$$\sigma: Z_2 \oplus Z_2 = \mathscr{E}(S^3) \oplus \mathscr{E}(S^5) \longrightarrow \mathscr{E}(SU(3))$$

in the above exact sequence is given by $\sigma(-1, -1)=v$, $\sigma(1, -1)=c$. Also, it is easy to see that $c\lambda(a)=\lambda(a)c$ and $v\lambda(a)=\lambda(-a)v$ in $\mathscr{E}(SU(3))$ for $a \in \pi_8(SU(3))$. Therefore, we have the desired result for $SU(3)$.

Since $Sp(2)=(S^3 \cup_{\omega_3} e^7) \cup e^{10}$, we can prove similarly the result for $Sp(2)$ by using Theorem 2.12 and Example 4.2. *q.e.d.*

§5. The group $\mathscr{E}(B \times \Omega A)$

In this section, we consider the dual situations of § 1, and study the product space $B \times \Omega A$ of B and a loop space ΩA of A.

For a given map $f: B \to A$, let

(5.1) $$E_f = \{(b, l) | b \in B, \quad l: [0, 1] \to A, \quad f(b)=l(0), \quad l(1)=*\}$$

be the mapping track of f, and let

$$\Omega B \xrightarrow{\Omega f} \Omega A \xrightarrow{i} E_f \xrightarrow{p} B \xrightarrow{f} A$$

be the sequence of the induced fiberings, where p is the projection, Ω is the loop functor and i is the inclusion.

As the dual of λ of (1.2), we define the map

(5.2) $$\kappa: [E_f, \Omega A] \longrightarrow [E_f, E_f]$$

by

$$\kappa(\alpha)=k(1 \times \alpha)\Delta: E_f \longrightarrow E_f \times E_f \longrightarrow E_f \times \Omega A \longrightarrow E_f$$

for $\alpha \in [E_f, \Omega A]$, where Δ is the diagonal map and k is the usual action of ΩA.

The usual multiplication on $[E_f, \Omega A]$ is denoted by $+$, and the second multiplication $\oplus$ is defined, dually to (1.3), by

(5.3) $$\alpha \oplus \beta = \alpha + \beta\kappa(\alpha) \qquad \text{for} \qquad \alpha, \beta \in [E_f, \Omega A].$$

Then, the dual of Lemma 1.4 is the following

LEMMA 5.4. (i) [10, Lemmas 3.7–8] $\oplus$ *defines a semi-group structure on* $[E_f, \Omega A]$ *with unit* 0, *and* κ *of* (5.2) *is a homomorphism of this semi-group to the semi-group* $[E_f, E_f]$, *i.e.,* $\kappa(\alpha \oplus \beta)=\kappa(\beta)\kappa(\alpha)$.

(ii) $\alpha \oplus \beta = \alpha + \beta$ *if* β *belongs to the image of* $p^*: [B, \Omega A] \to [E_f, \Omega A]$.

Now, we assume that

(5.5) *the two induced maps*

$$i_*: [\Omega A, \Omega A] \cong [\Omega A, E_f], \qquad p^*: [B, B] \cong [E_f, B],$$

of the inclusion i and the projection p, are bijective.

Then, $\varphi(h)$ and $\psi(h)$ are determined uniquely for $h \in [E_f, E_f]$ by the following homotopy commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i} & E_f & \xrightarrow{p} & B \\ \downarrow{\scriptstyle \psi(h)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi(h)} \\ A & \xrightarrow{i} & E_f & \xrightarrow{p} & B, \end{array}$$

and we have the following proposition, as the dual of Proposition 1.11.

PROPOSITION 5.6. *If the mapping track E_f of $f: B \to A$ satisfies* (5.5), *then the sequence*

$$i^{*-1}(0) \xrightarrow{\kappa} \mathscr{E}(E_f) \xrightarrow{\varphi \times \psi} \mathscr{E}(B) \times \mathscr{E}(\Omega A)$$

is exact, i.e., $\operatorname{Im} \kappa = (\varphi \times \psi)^{-1}(1, 1)$, *where* $i^*: [E_f, \Omega A] \to [\Omega A, \Omega A]$, *and* $i^{*-1}(0)$ *is the group with the multiplication* $\oplus$ *of* (5.3).

PROOF. We notice only that the assumption dual to the 1-connectedness of C_f in Lemma 1.8 is not necessary in this proposition, because we can prove $\kappa(i^{*-1}(0)) \subset \mathscr{E}(E_f)$ by the dual proof of Lemma 1.8(i), using homotopy groups instead of homology groups. *q.e.d.*

Now, we consider the special case $f=0: B \to A$, *i.e.*, the product space $E_0 = B \times \Omega A$.

LEMMA 5.7. *Assume that B is simple, and*

$$\pi_r(B) = 0 \quad \text{for } r \geqq n, \quad \pi_s(A) = 0 \quad \text{for } s \leqq n,$$

for some $n \geqq 2$. Then, the condition (5.5) *for $f=0$ is satisfied.*

PROOF. We have $[\Omega A, B]=0$, by the assumptions and the obstruction theory. Therefore, i_* in (5.5) is bijective, since $[\Omega A, B \times \Omega A] = [\Omega A, B] \times [\Omega A, \Omega A]$.

Also, $j^*: [B \times \Omega A, B] \to [B \vee \Omega A, B]$ is bijective by the assumptions and the obstruction theory, where j is the inclusion. Therefore, p^* in (5.5) is bijective since $[B \vee \Omega A, B] = [B, B] \times [\Omega A, B] = [B, B]$. *q.e.d.*

The following results are proved dually to Theorem 1.13 and Corollary 1.14, by Proposition 5.6 and Lemma 5.7.

THEOREM 5.8. *Assume that A is n-connected, and B is simple and $\pi_r(B)=0$ for $r \geqq n$ ($n \geqq 2$). Then $\mathscr{E}(B \times \Omega A)$ is the split extension*

$$0 \longrightarrow i^{*-1}(0) \longrightarrow \mathscr{E}(B \times \Omega A) \longrightarrow \mathscr{E}(B) \times \mathscr{E}(\Omega A) \longrightarrow 1,$$

where $i^*: [B \times \Omega A, \Omega A] \to [\Omega A, \Omega A]$ and $i^{*-1}(0)$ is the group with the multi-

plication $\oplus$ *of* (5.3).

COROLLARY 5.9. (cf. [7, Th. 2.10]) *If* B *is simple and* $\pi_r(B)=0$ *for* $r<m$ *or* $r\geqq n$, $\pi_s(A)=0$ *for* $s\leqq n$ *or* $s>n+m$, $(n\geqq m\geqq 1)$, *then we have the split extension*

$$0\longrightarrow[B,\ \Omega A]\longrightarrow\mathscr{E}(B\times\Omega A)\longrightarrow\mathscr{E}(B)\times\mathscr{E}(\Omega A)\longrightarrow 1,$$

where $[B,\ \Omega A]$ *is the group with usual multiplication* $+$.

REMARK. $i^{*-1}(0)$ in Theorem 5.8 is also the group by $+$, and the latter group is an extension

$$0\longrightarrow[B\wedge SA,\ \Omega A]\longrightarrow i^{*-1}(0)\longrightarrow[B,\ \Omega A]\longrightarrow 0.$$

EXAMPLE 5.10. *If* A *is 2-connected, then* $\mathscr{E}(S^1\times\Omega A)$ *is the split extension*

$$0\longrightarrow[S^1\wedge\Omega A,\ \Omega A]\longrightarrow\mathscr{E}(S^1\times\Omega A)\longrightarrow Z_2\times\mathscr{E}(\Omega A)\longrightarrow 1,$$

where the multiplication of the first group is induced by $\oplus$ *of* (5.3).

References

[1] W. D. Barcus and M. G. Barratt: *On the homotopy classification of a fixed map*, Trans. Amer. Math. Soc. **88** (1958), 57–74.

[2] A. L. Blakers and W. S. Massey: *The homotopy groups of a triad II*, Ann. of Math. **55** (1952), 192–201.

[3] P. J. Hilton: *On the homotopy groups of the union of spheres*, J. London Math. Soc. **30** (1955), 154–172.

[4] ————: *A note on the P-homomorphism in homotopy groups of spheres*, Proc. Cambridge Philos. Soc. **51** (1955), 230–233.

[5] W. C. Hsiang, J. Levine and R. H. Szczarba: *On the normal bundle of a homotopy sphere embedded in Euclidean space*, Topology **3** (1965), 173–181.

[6] I. M. James: *On the sphere-bundles over spheres*, Comment. Math. Helv. **35** (1961), 126–135.

[7] Y. Nomura: *Homotopy equivalences in a principal fiber space*, Math. Zeit. **92** (1966), 380–388.

[8] S. Oka: *Groups of self-equivalences of certain complexes*, Hiroshima Math. J. **2** (1972), 285–298.

[9] J. W. Rutter: *A homotopy classification of maps into an induced fibre space*, Topology **6** (1967), 379–403.

[10] ————: *Self-equivalences and principal morphisms*, Proc. London Math. Soc. (3) **20** (1970), 644–658.

[11] ————: *Groups of self homotopy equivalences of induced spaces*, Comment. Math. Helv. **45** (1970), 236–255.

[12] H. Toda: *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N. J., 1962.

Department of Mathematics,
Faculty of Science,
Hiroshima University,
Department of Mathematics,
Faculty of Engineering,
Tokushima University
and
Department of Mathematics,
Faculty of Science,
Hiroshima University