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On the KO-Ring of S^{4n+3}/H_m

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§1. Introduction

The purpose of this note is to study the KO-ring $KO(N^n(m))$ of real vector bundles over the $(4n+3)$-dimensional quotient manifold

$$N^n(m) = S^{4n+3}/H_m \qquad (m \geqq 2),$$

whose K-ring $K(N^n(m))$ of complex vector bundles is studied in the previous note [3]. Here, H_m is the generalized quaternion group generated by two elements x and y with the two relations

$$x^{2^{m-1}} = y^2 \quad \text{and} \quad xyx = y,$$

that is, H_m is the subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$ generated by the two elements

$$x = \exp(\pi i/2^{m-1}) \quad \text{and} \quad y = j,$$

and the action of H_m on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space $\boldsymbol{H}^{n+1}$ is given by the diagonal action.

Consider the real line bundles

$$\alpha'_0, \quad \beta'_0 \in KO(N^n(m)),$$

whose first Stiefel-Whitney classes generate the cohomology group $H^1(N^n(m); Z_2) = Z_2 \oplus Z_2$, and the real restriction

$$\delta'_0 = r\pi^!\lambda \in KO(N^n(m))$$

of the induced bundle $\pi^!\lambda$, where λ is the canonical complex plane bundle over the quaternion projective space $HP^n = S^{4n+3}/S^3$ and $\pi: N^n(m) \to HP^n$ is the natural projection. Also, it is proved by B. J. Sanderson [7] that the complexification $c: KO(HP^n) \to K(HP^n)$ is monomorphic and $(\lambda-2)^2 \in cKO(HP^n)$, and so we can consider the element

$$x_0 = \pi^! c^{-1}((\lambda-2)^2) \in KO(N^n(m)).$$

Then we have the following

THEOREM 1.1. *The reduced KO-ring* $\widetilde{KO}(N^n(m))$ $(m\geqq 2)$ *is generated multiplicatively by the four elements*

$$\alpha_0=\alpha'_0-1,\quad \beta_0=\beta'_0-1,\quad \delta_0=\delta'_0-4 \quad and \quad x_0.$$

This theorem shows that the natural ring homomorphism

$$\xi\colon \widetilde{RO}(H_m)\longrightarrow \widetilde{KO}(N^n(m))$$

is an epimorphism, where $\widetilde{RO}(H_m)$ is the reduced orthogonal representation ring of H_m. Since the kernel of this homomorphism ξ is determined by D. Pitt [6, Th. 2.5], we have the following

COROLLARY 1.2. *The above* ξ *induces the ring isomorphism*

$$\widetilde{KO}(N^n(m))\cong\begin{cases}\widetilde{RO}(H_m)/c^{-1}((\chi_4-2)^{n+1})RO(H_m) & \text{if } n \text{ is odd},\\ \widetilde{RO}(H_m)/c^{-1}((\chi_4-2)^{n+1}c'RSp(H_m)) & \text{if } n \text{ is even}.\end{cases}$$

Here, $\chi_4\in R(H_m)$ is the complexification of the symplectic representation given by the inclusion $H_m\subset S^3=Sp(1)$, and the monomorphisms $c\colon RO(H_m)\to R(H_m)$, $c'\colon RSp(H_m)\to R(H_m)$ are the complexifications, where $R(H_m)$ is the (unitary) representation ring and $RSp(H_m)$ is the symplectic representation group of H_m.

For the case $m=2$, $H_2=\{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group and we have

THEOREM 1.3. *As an abelian group,*

$$\widetilde{KO}(N^n(2))=\begin{cases}Z_{2^{n+1}}\oplus Z_{2^{n+1}}\oplus Z_{2^{2n+1}}\oplus Z_{2^{n-1}} & \text{if } n \text{ is odd},\\ Z_{2^{n+2}}\oplus Z_{2^{n+2}}\oplus Z_{2^{2n}}\oplus Z_{2^n} & \text{if } n \text{ is even}.\end{cases}$$

If n is odd, the direct summands are generated by

$$\alpha_0,\quad \beta_0,\quad \delta_0,\quad and \quad x_0+(2+2^n)\delta_0,$$

respectively, and the last summand does not appear in the case $n=1$.
If n is even, the direct summands are generated by

$$\alpha_0,\quad \beta_0,\quad \delta_0,\quad and \quad x_0+2\delta_0,$$

respectively, and the last two summands do not appear in the case $n=0$.
The multiplicative structure of $\widetilde{KO}(N^n(2))$ *is given by*

$$\alpha_0^2=-2\alpha_0,\quad \beta_0^2=-2\beta_0,\quad \delta_0^2=4x_0,\quad \alpha_0\delta_0=-4\alpha_0,\quad \beta_0\delta_0=-4\beta_0,$$

$$\alpha_0\beta_0 = -2\alpha_0 - 2\beta_0 + x_0 + 2\delta_0, \quad \alpha_0 x_0 = 4\alpha_0, \quad \beta_0 x_0 = 4\beta_0,$$

$$x_0^{m+1} = 0 \quad if\ n = 2m+1, \quad \delta_0 x_0^m = x_0^{m+1} = 0 \quad if\ n = 2m.$$

In §2, we recall the cell structure and the cohomology groups of $N^n(m)$. In §3, we consider the orthogonal representation ring $RO(H_m)$, which is determined by D. Pitt [6], and represent the elements α_0, β_0, δ_0 and x_0 in Theorem 1.1 as the ξ-images. Also, we study some relations between these elements and the known elements of $KO(L^{2n+1}(Z_4))$ of [5], where $L^{2n+1}(Z_4) = S^{4n+3}/Z_4$ is the lens space. Using these results we prove Theorem 1.1 in §4 by the induction on the skeletons on $N^n(m)$. Finally, Theorem 1.3 is proved in §5 by using Corollary 1.2.

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§2. Cohomology grougs of $N^n(m)$

The generalized quaternion group H_m ($m \geqq 2$) is the subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$, generated by the two elements

$$x = \exp(\pi i/2^{m-1}) \quad \text{and} \quad y = j.$$

In this note, we consider the diagonal action of H_m on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space $\boldsymbol{H}^{n+1}$, given by

$$q(q_1, \dots, q_{n+1}) = (qq_1, \dots, qq_{n+1}),$$

for $q \in H_m$ and $(q_1, \dots, q_{n+1}) \in S^{4n+3}$, and the quotient $(4n+3)$-manifold

$$N^n(m) = S^{4n+3}/H_m.$$

This manifold has the CW-decomposition $\{e^{4k+s}, e_1^{4k+t}, e_2^{4k+t};\ 0 \leqq k \leqq n,\ s=0, 3,\ t=1, 2\}$ with the boundary formulas:

$$\partial e^{4k} = 2^{m+1} e_1^{4k-1}, \quad \partial e_1^{4k+1} = \partial e_2^{4k+1} = 0,$$

$$\partial e_1^{4k+2} = 2^{m-1} e_1^{4k+1} - 2e_2^{4k+1}, \quad \partial e_2^{4k+2} = 2e_1^{4k+1}, \quad \partial e^{4k+3} = 0.$$

(cf. [3, Lemma 2.1]). Also, the cohomology groups of $N^n(m)$ are given by

$$H^k(N^n(m); Z) = \begin{cases} Z & \text{for } k=0,\ 4n+3, \\ Z_{2^{m+1}} & \text{for } k \equiv 0(4),\ 0<k<4n+3, \\ Z_2 \oplus Z_2 & \text{for } k \equiv 2(4),\ 0<k<4n+4, \\ 0 & \text{otherwise,} \end{cases}$$

$$H^k(N^n(m); Z_2) = \begin{cases} Z_2 \oplus Z_2 & \text{for} \quad k \equiv 1,\ 2(4),\ 0 < k < 4n+3, \\ Z_2 & \text{for} \quad k \equiv 0,\ 3(4),\ 0 \leqq k \leqq 4n+3, \\ 0 & \text{otherwise}, \end{cases}$$

(cf. [3, Prop. 2.2]).

Let $0 \longrightarrow Z \xrightarrow{\times 2} Z \xrightarrow{j} Z_2 \longrightarrow 0$ be the exact coefficient sequence, and $H^1(N^n(m); Z_2) \xrightarrow{\Delta} H^2(N^n(m); Z) \xrightarrow{\times 2} H^2(N^n(m); Z) \xrightarrow{j_*} H^2(N^n(m); Z_2)$ be the associated exact sequence. Then we have easily the following

LEMMA 2.1. *Δ and j_* are isomorphic.*

Now, let a and b be generators of

$$H^1(N^n(m); Z_2) = Z_2 \oplus Z_2,$$

and let α'_0 and β'_0 (resp. α' and β') be the real (resp. complex) line bundles over $N^n(m)$, whose first Stiefel-Whitney (resp. Chern) classes are given by

$$(2.2) \qquad \begin{aligned} w_1(\alpha'_0) &= a, \quad w_1(\beta'_0) = b, \\ c_1(\alpha') &= \Delta a, \quad c_1(\beta') = \Delta b. \end{aligned}$$

Denote their stable classes by

$$(2.3) \qquad \begin{aligned} \alpha_0 &= \alpha'_0 - 1, \quad \beta_0 = \beta'_0 - 1 \in \widetilde{KO}(N^n(m)), \\ \alpha &= \alpha' - 1, \quad \beta = \beta' - 1 \in \widetilde{K}(N^n(m)). \end{aligned}$$

The K- and KO-rings of the quaternion projective space HP^n are known as follows.

(2.4) (B. J. Sanderson [7, Th. 3.11, 3.12])

$$K(HP^n) = Z[z]/<z^{n+1}>,$$

where $z = \lambda - 2$ is the stable class of the canonical complex plane bundle λ over HP^n. Also, the complexification

$$c\colon KO(HP^n) \longrightarrow K(HP^n)$$

is monomorphic, and the ring $KO(HP^n)$ is generated by the two elements

$$z_0 = rz = c^{-1}(2z) \quad and \quad x = c^{-1}(z^2),$$

where r is the real restriction.

Using these results and the induced homomorphisms of the natural projection

(2.5) $$\pi\colon N^n(m)=S^{4n+3}/H_m \longrightarrow S^{4n+3}/S^3=HP^n,$$

we consider the following elements:

(2.6) $$\delta=\pi^! z\in \tilde{K}(N^n(m)),$$
$$\delta_0=r\delta=\pi^! z_0,\quad x_0=\pi^! x\in \widetilde{KO}(N^n(m)).$$

LEMMA 2.7. *For the complexification* $c\colon \widetilde{KO}(N^n(m))\to\tilde{K}(N^n(m))$,

$$c(\alpha_0)=\alpha,\ c(\beta_0)=\beta,\ c(\delta_0)=2\delta,\ c(x_0)=\delta^2.$$

PROOF. The total Stiefel-Whitney class of α'_0 is $w(\alpha'_0)=1+a$, by definition. Therefore,

$$w(rc\alpha'_0)=w(2\alpha'_0)=(w(\alpha'_0))^2=1+a^2=1+Sq^1a=1+j_*\Delta a=1+j_*c_1(\alpha').$$

On the other hand, it is well known that $w_2(rc\alpha'_0)=j_*c_1(c\alpha'_0)$, and we have $c_1(\alpha')=c_1(c\alpha'_0)$ by Lemma 2.1, and so $\alpha'=c\alpha'_0$. In the same way, we have the second equality. The last two equalities follow immediately by definition. q.e.d.

§3. Representation rings

We denote the unitary (resp. orthogonal) representation ring of the group G by $R(G)$ (resp. $RO(G)$), and the symplectic representation group by $RSp(G)$. By the natural inclusions $O(n)\subset U(n)$, $U(n)\subset O(2n)$, $Sp(n)\subset U(2n)$ and $U(n)\subset Sp(n)$, the following group homomorphisms are defined:

$$RO(G)\underset{c}{\overset{r}{\rightleftarrows}} R(G)\underset{h}{\overset{c'}{\rightleftarrows}} RSp(G).$$

The following facts (3.1) and (3.2) are well known (cf., e.g. [2]).

(3.1) These representation groups are free, and c is a ring homomorphism. Also

$$rc=2,\quad hc'=2,\quad cr=1+t=c'h,$$

(t denotes the conjugation), and c and c' are monomorphic.

(3.2) We have the commutative diagrams

$$\begin{array}{ccc} RO(G)\otimes_Z RSp(G) & \longrightarrow & RSp(G) \\ \downarrow{\scriptstyle c\otimes c'} & & \downarrow{\scriptstyle c'} \\ R(G)\otimes_Z R(G) & \longrightarrow & R(G) \end{array}\qquad \begin{array}{ccc} RSp(G)\otimes_Z RSp(G) & \longrightarrow & RO(G), \\ \downarrow{\scriptstyle c'\otimes c'} & & \downarrow{\scriptstyle c} \\ R(G)\otimes_Z R(G) & \longrightarrow & R(G), \end{array}$$

where the horizontal pairings are defined by tensoring over $\boldsymbol{R}$ or $\boldsymbol{H}$.

For the later purposes, we use the following facts for the representation rings or groups of H_m, S^3 and Z_4.

The generalized quaternion group H_m has three non-trivial representations of degree 1:

$$\begin{cases} \chi_1(x) = 1 \\ \chi_1(y) = -1, \end{cases} \quad \begin{cases} \chi_2(x) = -1 \\ \chi_2(y) = 1, \end{cases} \quad \begin{cases} \chi_3(x) = -1 \\ \chi_3(y) = -1, \end{cases}$$

and $2^{m-1}-1$ representations of degree 2:

$$\chi_{i+3}(x) = \begin{pmatrix} x^i & 0 \\ 0 & x^{-i} \end{pmatrix}, \qquad \chi_{i+3}(y) = \begin{pmatrix} 0 & (-1)^i \\ 1 & 0 \end{pmatrix},$$

for $i = 1, 2, \ldots, 2^{m-1}-1$.

LEMMA 3.3. (cf. [3, Prop. 3.1, 3.3]) *$R(H_m)$ is generated by χ_j ($j = 0, 1, \ldots, 2^{m-1}+2$) ($\chi_0 = 1$) as a free Z-module, and by 1, χ_1, χ_2 and χ_4 as a ring. The multiplicative structure is given by*

$$\chi_i\chi_j = \chi_j\chi_i, \quad \chi_1^2 = \chi_2^2 = 1,$$

$$\chi_3 = \chi_1\chi_2, \quad \chi_1\chi_4 = \chi_4, \quad \chi_2\chi_4 = \chi_{2^{m-1}+2}$$

$$\chi_4^2 = \begin{cases} 1+\chi_1+\chi_2+\chi_3 & \text{for } m = 2, \\ 1+\chi_1+\chi_5 & \text{for } m \geqq 3, \end{cases}$$

$$\chi_{i+1} = \chi_4\chi_i - \chi_{i-1} \qquad \text{for } i \geqq 5,\ m \geqq 3.$$

LEMMA 3.4. (cf. [6, Prop. 1.5]) *By the monomorphism*

$$c\colon RO(H_m) \longrightarrow R(H_m),$$

$RO(H_m)$ may be considered as the subring of $R(H_m)$, generated by 1, χ_1, χ_2, χ_3, $2\chi_{2i+2}$ and χ_{2i+3} ($i \geqq 1$).

LEMMA 3.5. (cf. [6, Prop. 1.6]) *By the monomorphism*

$$c'\colon RSp(H_m) \longrightarrow R(H_m),$$

$RSp(H_m)$ may be considered as the free abelian subgroup of $R(H_m)$, generated by 2, $2\chi_1$, $2\chi_2$, $2\chi_3$, $2\chi_{2i+3}$ and χ_{2i+2} ($i \geqq 1$).

LEMMA 3.6. (cf. [4, Ch. 13, Th. 3.1])

$$R(S^3) = Z[\chi],$$

where χ is the c'-image $c'\chi$ of the identity symplectic representation χ: $S^3 = Sp(1)$.

LEMMA 3.7. *For the monomorphism c: $RO(S^3) \to R(S^3)$, we have*

$$2\chi^i, \quad \chi^{2i} \in \operatorname{Im} c, \quad \textit{for any } i \geqq 1.$$

PROOF. Since $\chi \in R(S^3)$ is self-conjugate, we have $2\chi^i = cr(\chi^i) \in \operatorname{Im} c$. By the commutative diagram

$$\begin{array}{ccc} RSp(S^3) \otimes_Z RSp(S^3) & \longrightarrow & RO(S^3) \\ {\scriptstyle c' \otimes c'} \downarrow & & \downarrow {\scriptstyle c} \\ R(S^3) \otimes_Z R(S^3) & \longrightarrow & R(S^3) \end{array}$$

of (3.2), we have $\chi^2 = c(\chi^2)$, where $\chi^2 \in RO(S^3)$ is the image of $\chi \otimes \chi \in RSp(S^3) \otimes_Z RSp(S^3)$. q.e.d.

It is clear that $\chi_4 \in R(H_m)$ is the c'-image of the symplectic representation of H_m given by the inclusion $H_m \subset S^3 = Sp(1)$, and we have

LEMMA 3.8. $$i(\chi) = \chi_4,$$

where i: $H_m \subset S^3$ is the inclusion.

For an n-dimensional representation ω of H_m, the n-plane bundle $\xi(\omega)$ is induced from the principal H_m-bundle ξ: $S^{4n+3} \to N^n(m)$ by the group homomorphism ω: $H_m \to GL(n, \boldsymbol{R})$, and we have a ring homomorphism

$$\xi \colon RO(H_m) \longrightarrow KO(N^n(m)). \tag{3.9}$$

LEMMA 3.10. *The elements α_0 and β_0 of (2.3) may be so taken*

$$\xi c^{-1}(\chi_1 - 1) = \alpha_0, \qquad \xi c^{-1}(\chi_2 - 1) = \beta_0.$$

Also, for the elements δ_0 and x_0 of (2.6), we have

$$\xi c^{-1}(2\chi_4 - 4) = \delta_0, \qquad \xi c^{-1}((\chi_4 - 2)^2) = x_0.$$

PROOF. The ring homomorphism ξ: $R(H_m) \to K(N^n(m))$ is defined in the same way as (3.9), and we have the commutative diagram

$$\begin{array}{ccc} RO(H_m) & \xrightarrow{c} & R(H_m) \\ \downarrow {\scriptstyle \xi} & & \downarrow {\scriptstyle \xi} \\ KO(N^n(m)) & \xrightarrow{c} & K(N^n(m)). \end{array}$$

Since $c_1\xi(\chi_1)$ and $c_1\xi(\chi_2)$ generate $H^2(N^n(m); Z) = Z_2 \oplus Z_2$, (cf. [3, p. 259]), we can take a, $b \in H^1(N^n(m); Z_2)$ in (2.2) so that

$$\Delta a = c_1\xi(\chi_1), \qquad \Delta b = c_1\xi(\chi_2),$$

by Lemma 2.1. Then,

$$j_*\Delta a = j_*c_1\xi(\chi_1) = j_*c_1(c\xi c^{-1}(\chi_1)) = w_2(rc\xi c^{-1}(\chi_1))$$
$$= w_2(2\xi c^{-1}(\chi_1)) = w_1(\xi c^{-1}(\chi_1))^2 = j_*\Delta w_1(\xi c^{-1}(\chi_1)).$$

Therefore $w_1(\xi c^{-1}(\chi_1))=a$ by Lemma 2.1, and we have $\xi c^{-1}(\chi_1)=\alpha'_0$ by (2.2). In the same way as above, we have $\xi c^{-1}(\chi_2)=\beta'_0$.

Consider the commutative diagram

$$\begin{array}{ccccc} R(S^3) & \xrightarrow{i} & R(H_m) & \xrightarrow{r} & RO(H_m) \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ K(HP^n) & \xrightarrow{\pi!} & K(N^n(m)) & \xrightarrow{r} & KO(N^n(m)), \end{array}$$

where ξ' is the ring homomorphism defined in the same way as ξ of (3.9), using ξ': $S^{4n+3}\to HP^n$. Then,

$$\xi'(\chi) = \lambda, \qquad \xi'(\chi-2) = x$$

directly by definition. Therefore, by Lemma 3.8, (2.4) and (2.6), we have

$$\xi^{-1}(2\chi_4-4) = \xi r(\chi_4-2) = \xi ri(\chi-2) = r\pi^!\xi'(\chi-2) = r\pi^!z = \delta_0.$$

Finally, consider the commutatived iagram

$$\begin{array}{ccccccc} R(H_m) & \xleftarrow{c} & RO(H_m) & \xleftarrow{i} & RO(S^3) & \xrightarrow{c} & R(S^3) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi'} \\ K(N^n(m)) & \xleftarrow{c} & KO(N^n(m)) & \xleftarrow{\pi!} & KO(HP^n) & \xrightarrow{c} & K(HP^n). \end{array}$$

Then, by Lemma 3.8, (2.4) and (2.6), we have

$$\xi c^{-1}((\chi_4-2)^2) = \xi ic^{-1}((\chi-2)^2) = \pi^!c^{-1}\xi'((\chi-2)^2) = \pi^!c^{-1}(z^2) = x_0.$$

q.e.d.

Finally, we consider the representation ring of the cyclic group Z_4 of order 4. It is well known that

LEMMA 3.11. $$R(Z_4) = Z[\mu]/\langle\mu^4-1\rangle,$$

where μ is the unitary representation such that $\mu(g)=\exp(\pi i/2)$ for the generator g of Z_4.

Let $L^{2n+1}(4)=S^{4n+3}/Z_4$ be the standard lens space mod 4, and ζ: $S^{4n+3}\to L^{2n+1}(4)$ be the natural projection. Then, we have the commutative diagram

$$\begin{array}{ccc} R(Z_4) & \xrightarrow{\zeta} & K(L^{2n+1}(4)) \\ c\uparrow\downarrow r & & c\uparrow\downarrow r \\ RO(Z_4) & \xrightarrow{\zeta} & KO(L^{2n+1}(4)), \end{array}$$

where ζ's are the natural ring homomorphisms defined in the same way as ξ of (3.9).

LEMMA 3.12. *For the element* μ *of* Lemma 3.11,

$$\sigma+1=\zeta(\mu)\in K(L^{2n+1}(4))$$

is the complex line bundle whose first Chern class generates $H^2(L^{2n+1}(4); Z)=Z_4$. *Also* μ^2 *belongs to* $cRO(Z_4)$, *and*

$$\kappa+1=\zeta c^{-1}(\mu^2)\in KO(L^{2n+1}(4))$$

is the real line bundle whose first Stiefel-Whitney class generates $H^1(L^{2n+1}(4); Z_2)=Z_2$.

PROOF. The first half of the lemma is proved by Lemma 3.11 and [1, Appendix, (3)].

Since $\mu^2(g)=-1$ by Lemma 3.11, we have $\mu^2\in cRO(Z_4)$, and $\kappa+1$ is the real line bundle over $L^{2n+1}(4)$. Also, the first Chern class of $c(\kappa+1)=\zeta(\mu^2)=(\sigma+1)^2$ is equal to $2c_1(\sigma+1)$, which is not zero. Therefore, $\kappa+1$ is non-trivial.

q.e.d.

Let $i\colon Z_4\subset H_m$ and $i'\colon Z_4\subset H_m$ be the inclusions defined by $i(g)=x^{2^{m-2}}$ and $i'(g)=y$, and

(3.13) $\qquad \rho\colon L^{2n+1}(4)\longrightarrow N^n(m), \qquad \rho'\colon L^{2n+1}(4)\longrightarrow N^n(m)$

by the natural projections induced from i, i'.

LEMMA 3.14. *For the induced homomorphisms* $\rho^!$ *and* $\rho'^!$ *of* (3.13), *and the elements* α_0, β_0, δ_0, x_0 *of* (2.3) *and* (2.6), *we have*

$$\rho^!\alpha_0=0=\rho'^!\beta_0, \qquad \rho^!\beta_0=\kappa=\rho'^!\alpha_0,$$

$$\rho^!\delta_0=2r\sigma=\rho'^!\delta_0, \qquad \rho^!x_0=(r\sigma)^2=\rho'^!x_0.$$

PROOF. We prove the equalities for $\rho'^!$. Consider the commutative diagram

$$\begin{array}{ccccccc} R(H_m) & \xleftarrow{c} & RO(H_m) & \xrightarrow{\xi} & KO(N^n(m)) & \xleftarrow{r} & K(N^n(m)) \\ \downarrow i' & & \downarrow i' & & \downarrow \rho'^! & & \downarrow \rho'^! \\ R(Z_4) & \xleftarrow{c} & RO(Z_4) & \xrightarrow{\zeta} & KO(L^{2n+1}(4)) & \xleftarrow{r} & K(L^{2n+1}(4)) \end{array}$$

We notice that the following equalities hold by [3, Prop. 3.9, Lemma 4.8]:

$$(*)\qquad i'\chi_1=\mu^2,\quad i'\chi_2=1,\quad \rho'^!\delta=\sigma^2/(1+\sigma),\quad i'\chi_4=\mu+t\mu,$$

where t is the conjugation. Then, we have

$$\rho'^!\alpha_0=\rho'^!\xi c^{-1}(\chi_1-1)=\zeta c^{-1}i'(\chi_1-1)=\zeta c^{-1}(\mu^2-1)=\kappa,$$

$$\rho'^!\beta_0=\zeta c^{-1}i'(\chi_2-1)=0,$$

by Lemmas 3.10 and 3.12. Also,

$$\rho'^!\delta_0=\rho'^!r\delta=r\rho'^!\delta=r(\sigma^2/(1+\sigma))=r(\sigma+t\sigma)=rcr\sigma=2r\sigma,$$

by (2.6), the third equality of $(*)$ and the fact that $t\sigma=-\sigma/(1+\sigma)$. Finally, we have

$$\rho'^!x_0=\rho'^!\xi c^{-1}((\chi_4-2)^2)=\zeta c^{-1}i'((\chi_4-2)^2)=\zeta c^{-1}((\mu+t\mu-2)^2)$$
$$=\zeta c^{-1}((cr(\mu-1)^2)=\zeta((r(\mu-1))^2)=(r\zeta(\mu-1))^2=(r\sigma)^2,$$

by Lemmas 3.10, 3.12 and the last euqality of $(*)$.

We notice that the equalities

$$i\chi_1=1,\quad i\chi_2=\mu^2,\quad \rho^!\delta=\sigma^2/(1+\sigma),\quad i\chi_4=\mu+t\mu,$$

which are similar to $(*)$, can be proved in the same way as [3, Prop. 3.9, Lemma 4.7], using the inclusions

$$Z_4\subset H_2\subset H_m.$$

Therefore, the desired equalities for $\rho^!$ can be proved in the same way as above. q.e.d.

§4. Proof of Theorem 1.1

Let N^k be the k-skeleton of the CW-complex $N^n(m)$ in §2, and $i\colon N^k\to N^n(m)$ be the inclusion. For an element $a\in\widetilde{KO}(N^n(m))$, we denote its image $i^!a\in\widetilde{KO}(N^k)$ by the same letter a. Therefore, we have the elements

$$(4.1)\qquad \alpha_0,\ \beta_0,\ \delta_0,\ x_0\in\widetilde{KO}(N^k)\qquad\text{for any}\quad k\geqq 0,$$

from those of (2.3) and (2.6).

Lemma 4.2. $\alpha_0^i\beta_0^j\delta_0^k x_0^l=0$ *in* $\widetilde{KO}(N^{i+j+4k+8l-1})$.

Proof. α_0 and β_0 are zero in $\widetilde{KO}(N^0)=0$, and δ_0 and x_0 are zero in $\widetilde{KO}(N^3)$

$=\widetilde{KO}(N^0(m))$ and $\widetilde{KO}(N^7)=\widetilde{KO}(N^1(m))$ respectively, by (2.4). Therefore, the desired results follow from the obvious fact that ab is zero in $\widetilde{KO}(N^{p+q-1})$ if a is zero in $\widetilde{KO}(N^{p-1})$ and b is zero in $\widetilde{KO}(N^{q-1})$. q.e.d.

LEMMA 4.3. *If the ring $\widetilde{KO}(N^{4n+2})$ is generated by α_0, β_0, δ_0 and x_0, then $i^!: \widetilde{KO}(N^{4n+3})\to\widetilde{KO}(N^{4n+2})$ is an isomorphism.*

PROOF. Consider the Puppe sequence

$$0 \longrightarrow \widetilde{KO}(N^{4n+3}) \xrightarrow{i^!} \widetilde{KO}(N^{4n+2}).$$

Since the elements α_0, β_0, δ_0 and x_0 in $\widetilde{KO}(N^{4n+2})$ are the $i^!$-images of those in $\widetilde{KO}(N^{4n+3})$, we have the lemma. q.e.d.

LEMMA 4.4. *$i^!: \widetilde{KO}(N^{8n+6})\to\widetilde{KO}(N^{8n+5})$ is an isomorphism.*

PROOF. By the Puppe seuqence, the lemma follows immediately. q.e.d.

LEMMA 4.5. *If the ring $\widetilde{KO}(N^{8n+1})$ is generated by α_0, β_0, δ_0 and x_0, then the ring $\widetilde{KO}(N^{8n+2})$ is so.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc}
\widetilde{KO}(S^{8n+2}\vee S^{8n+2}) & \xrightarrow{p^!} & \widetilde{KO}(N^{8n+2}) & \xrightarrow{i^!} & \widetilde{KO}(N^{8n+1}) \\
\uparrow r & & \uparrow r & & \uparrow r \\
\widetilde{K}(S^{8n+2}\vee S^{8n+2}) & \xrightarrow{p^!} & \widetilde{K}(N^{8n+2}) & \xrightarrow{i^!} & \widetilde{K}(N^{8n+1}),
\end{array}$$

In the lower sequence, $\ker i^!=\operatorname{Im} p^!=Z_2\oplus Z_2$ is generated by $\alpha\delta^{2n}$ and $\beta\delta^{2n}$ (cf. [3, p 263]). Since r in the left is an epimorphism, $\operatorname{Ker} i^!=\operatorname{Im} p^!$ is generated by $r(\alpha\delta^{2n})$ and $r(\beta\delta^{2n})$ in the upper exact sequence. Since $c(\alpha_0x_0^n)=\alpha\delta^{2n}$ by Lemma 2.7, we have $r(\alpha\delta^{2n})=rc(\alpha_0x_0^n)=2\alpha_0x_0^n$, and also $r(\beta\delta^{2n})=2\beta_0x_0^n$. These imply the desired result. q.e.d.

LEMMA 4.6. *If the ring $\widetilde{KO}(N^{8n+4})$ is generated by α_0, β_0, δ_0 and x_0, then $i^!: \widetilde{KO}(N^{8n+5})\to\widetilde{KO}(N^{8n+4})$ is an isomorphism.*

PROOF. We have the desired result in the same way as Lemma 4.3. q.e.d.

LEMMA 4.7. *If the ring $\widetilde{KO}(N^{8n})$ is generated by α_0, β_0, δ_0 and x_0, then the ring $\widetilde{KO}(N^{8n+1})$ is also so. In particular, $\widetilde{KO}(N^1)=\widetilde{KO}(S^1\vee S^1)=Z_2\oplus Z_2$ is generated by α_0 and β_0.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc}
\widetilde{KO}(S^{8n+1}\vee S^{8n+1}) & \xrightarrow{p^!} & \widetilde{KO}(N^{8n+1}) & \xrightarrow{i^!} & \widetilde{KO}(N^{8n}) \\
\downarrow{\scriptstyle\rho^!} & & \downarrow{\scriptstyle\rho^!} & & \downarrow{\scriptstyle\rho^!} \\
\widetilde{KO}(S^{8n+1}) & \xrightarrow{p^!} & \widetilde{KO}(L^{4n}(4)) & \xrightarrow{i^!} & \widetilde{KO}(L_0^{4n}(4)).
\end{array}$$

Since $i^!(\alpha_0 x_0^n)=i^!(\beta_0 x_0^n)=0$ by Lemma 4.2, we have $\alpha_0 x_0^n$, $\beta_0 x_0^n \in \operatorname{Ker} i^! = \operatorname{Im} p^!$. On the other hand,

$$\rho^!(\alpha_0 x_0^n)=0, \quad \rho^!(\beta_0 x_0^n)=\rho'^!(\alpha_0 x_0^n)=2^{2n}\kappa$$

by Lemma 3.14. Also, $2^{2n}\kappa$ is not zero in $\widetilde{KO}(L^{4n}(4))$ by [5, Th. B]. Therefore, we have $\alpha_0 x_0^n \neq 0$, $\beta_0 x_0^n \neq 0$ and $\alpha_0 x_0^n \neq \beta_0 x_0^n$. Since $\widetilde{KO}(S^{8n+1}\vee S^{8n+1})= Z_2 \oplus Z_2$, these imply the desired result. q.e.d.

LEMMA 4.8. *If the ring $\widetilde{KO}(N^{4n-1})$ is generated by $\alpha_0, \beta_0, \delta_0$ and x_0, then the ring $\widetilde{KO}(N^{4n})$ is so.*

PROOF. We consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & KO(S^{4n}) & \xrightarrow{p^!} & KO(N^{4n}) & \xrightarrow{i^!} & KO(N^{4n-1}) & & \\
 & & \| & & \uparrow{\scriptstyle\pi^!} & & \uparrow{\scriptstyle\pi^!} & & \\
0 & \longrightarrow & KO(S^{4n}) & \xrightarrow{p^!} & KO(HP^n) & \xrightarrow{i^!} & KO(HP^{n-1}) & \longrightarrow & 0,
\end{array}$$

induced by $\pi=\pi|N^{4n}: (N^{4n}, N^{4n-1})\to(HP^n, HP^{n-1})$, which is a relative homeomorphism. In the lower sequence, $\operatorname{Ker} i^! = \operatorname{Im} p^! = Z$ is generated by

$$x^k \quad (\text{if } n=2k), \qquad z_0 x^k \quad (\text{if } n=2k+1)$$

by [7, p. 145]. Therefore, $\operatorname{Ker} i^! = \operatorname{Im} p^!$ in the upper sequence is generated by

$$\pi^!(x^k)=x_0^k \quad (\text{if } n=2k), \qquad \pi^!(z_0 x^k)=\delta_0 x_0^k \quad (\text{if } n=2k+1).$$

These complete the proof. q.e.d.

PROOF OF THEOREM 1.1. Starting from the latter half of Lemma 4.7, we have Theorem 1.1 for $\widetilde{KO}(N^k)$ by the induction on k, using Lemmas 4.3–4.8.

q.e.d.

By Theorem 1.1 and Lemma 3.10, we see that the ring homomorphism

$$\xi: RO(H_m) \longrightarrow KO(N^n(m))$$

of (3.9) is an epimorphism.

On the other hand the following theorem is proved by D. Pitt :

THEOREM 4.9. [6, Th. 2.5]

$$\operatorname{Im}\xi \cong \begin{cases} RO(H_m)/c^{-1}((\chi_4-2)^{n+1})RO(H_m) & \textit{if n is odd,} \\ RO(H_m)/c^{-1}((\chi_4-2)^{n+1}c'RSp(H_m)) & \textit{if n is even,} \end{cases}$$

where $(\chi_4-2)^{n+1} \in cRO(H_m)$ *if n is odd, by Lemma* 2.4.

Therefore, we have Corollary 1.2 in §1.

§5. Proof of Theorem 1.3

In this section, we deal with the special case

$$N^n(2) = S^{4n+3}/H_2,$$

where $H_2=\{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group.

Consider the ring homomorphism

$$\xi\colon RO(H_2) \longrightarrow KO(N^n(2))$$

of (3.9), and set also

(5.1)
$$\alpha_0 = c^{-1}(\chi_1-1), \qquad \beta_0 = c^{-1}(\chi_2-1),$$
$$\delta_0 = c^{-1}(2\chi_4-4), \qquad x_0 = c^{-1}((\chi_4-2)^2)$$

in $RO(H_2)$. Then

$$\xi\alpha_0 = \alpha_0, \quad \xi\beta_0 = \beta_0, \quad \xi\delta_0 = \delta_0, \quad \xi x_0 = x_0,$$

by Lemma 3.10. Furthermore, by Lemmas 3.3 and 3.4, we see easily that

(5.2) $\widetilde{RO}(H_2)$ is the free Z-module with bases

$$1, \quad \alpha_0, \quad \beta_0, \quad \delta_0, \quad x_0,$$

and the multiplicative structure is given by

(5.3)
$$\alpha_0^2 = -2\alpha_0, \quad \beta_0^2 = -2\beta_0, \quad \delta_0^2 = 4x_0,$$
$$\alpha_0\beta_0 = -2\alpha_0-2\beta_0+x_0+2\delta_0, \quad \alpha_0\delta_0 = -4\alpha_0,$$
$$\beta_0\delta_0 = -4\beta_0, \quad \alpha_0 x_0 = 4\alpha_0, \quad \beta_0 x_0 = 4\beta_0.$$

By these relations, we have easily

(5.4)
$$\delta_0 x_0 + 12x_0 + 8\delta_0 = 0,$$

(5.5)
$$x_0^2 + 3\delta_0 x_0 + 8x_0 = 0.$$

LEMMA 5.6. $\alpha_0\delta_0^i x_0^j = (-1)^i 2^{2(i+j)}\alpha_0, \quad \beta_0\delta_0^i x_0^j = (-1)^i 2^{2(i+j)}\beta_0.$

PROOF. These equalities follow from the last four equalities of (5.3).

LEMMA 5.7. $\delta_0(1)\delta_0^i=(-1)^i\delta_0(1)x_0^i=(-1)^i2^{2i}\delta_0(1)$, *where* $\delta_0(1)=x_0+2\delta_0$.

PROOF. We see $\delta_0(1)\delta_0=-\delta_0(1)x_0=-2^2\delta_0(1)$ by (5.3), (5.4) and (5.5). These imply the desired results by the induction on i. q.e.d.

(I) The case $n=2m+1$

By Corollary 1.2 and (5.1), we have

$$\widetilde{KO}(N^n(2))\cong\widetilde{RO}(H_2)/x_0^{m+1}RO(H_2)\,.$$

By (5.2), $\widetilde{RO}(H_2)$ is the free Z-module with bases

$$\alpha_0,\quad \beta_0,\quad \delta_0,\quad \delta_0(1)+2^n\delta_0=x_0+(2+2^n)\delta_0,$$

and the ideal $x_0^{m+1}RO(H_2)$ is generated by

$$x_0^{m+1},\quad \alpha_0x_0^{m+1},\quad \beta_0x_0^{m+1},\quad \delta_0x_0^{m+1},\quad x_0^{m+2}. \tag{5.8}$$

Therefore, Theorem 1.3 for $n=2m+1$ follows immediately from

LEMMA 5.9. *The elements of* (5.8) *are linear combinations of*

$$2^{n+1}\alpha_0,\quad 2^{n+1}\beta_0,\quad 2^{2n+1}\delta_0,\quad 2^{n-1}(\delta_0(1)+2^n\delta_0)\,, \tag{5.10}$$

and the elements of (5.10) *are also so of* (5.8).

We prove this lemma by the following routine calculations.

LEMMA 5.11. (i) $2^{4i+3}\delta_0x_0^{m-i}\equiv 0$ $\quad(0\leqq i\leqq m)$,

(ii) $2^{4i+6}x_0^{m-i}\equiv 0$ $\quad(0\leqq i\leqq m-1)$,

(iii) $2\delta_0x_0^m\equiv 2^4x_0^m$,

(iv) $2^{4i+4}x_0^{m-i}+2^{4i+5}\delta_0x_0^{m-i-1}\equiv 0$ $\quad(0\leqq i\leqq m-1)$,

(v) $2^{4i+5}\delta_0x_0^{m-i-1}+2^{4i+8}x_0^{m-i-1}\equiv 0$ $\quad(0\leqq i\leqq m-2)$,

(vi) $2^{n-1}(\delta_0(1)+2^n\delta_0)\equiv 0$,

where $\equiv$ *means modulo the ideal generated by* $\{x_0^{m+1},\ \delta_0x_0^{m-1},\ x_0^{m+2}\}$.

PROOF. (i), (ii) We have the desired equalities by the induction on i, using the equalities $(5.4)\times 2^{4i}x_0^{m-i}$ and $(5.4)\times 2^{4i+1}\delta_0x_0^{m-i-1}$.

(iii) The equality follows from $(5.5)\times x_0^{m-1}$ and (i).

(iv) By (5.4) and (5.5), we have easily

$$x_0^2 = 28x_0 + 24\delta_0 = 2^4x_0 + 3\cdot 2^2\delta_0(1)\,, \tag{5.12}$$

and (iv) is obtained from (5.12) $\times 2^{4i+2}x_0^{m-i-1}$, using (i) and (ii).

(v) The equality follows from (5.12) $\times 2^{4i+3}\delta_0 x_0^{m-i-2}$, using (i) and (ii).

(vi) By Lemma 5.7 and (iii)—(v), we have

$$2^{n-1}\delta_0(1) = \delta_0(1)x_0^m \equiv 2\delta_0 x_0^m \equiv 2^4x_0^m \equiv -2^5\delta_0x_0^{m-1} \equiv 2^8x_0^{m-1} \equiv \cdots \equiv -2^{4m+1}\delta_0.$$

q.e.d.

LEMMA 5.13. (i) $x_0^{m+1} = 2^{n-1}(2^n-1)(\delta_0(1)+2^n\delta_0) - 2^{3n-1}\delta_0$,

(ii) $\delta_0x_0^{m+1} = 2^{2n+1}(2^{n+1}+1)\delta_0 - 2^{n+1}(2^{n+1}-1)(\delta_0(1)+2^n\delta_0)$,

(iii) $x_0^{m+2} = 2^{n+1}(2^{n+2}-1)(\delta_0(1)+2^n\delta_0) - 2^{2n+1}(2^{n+2}+3)\delta_0$.

PROOF. (i) From (5.12) $\times 2^{4i}x_0^{m-i-1}$, we have easily

$$2^{4i}x_0^{m+i-1} = 2^{4(i+1)}x_0^{m-i} + 3\cdot 2^{n-1+2i}\delta_0(1)\,,$$

using Lemma 5.7. Therefore, we have

$$\begin{aligned} x_0^{m+1} &= 2^{4m}x_0 + 3\cdot 2^{n-1}(1+2^2+2^4+\cdots+2^{2(m-1)})\delta_0(1)\,, \\ &= 2^{2(n-1)}x_0 + 2^{n-1}(2^{n-1}-1)\delta_0(1) \\ &= 2^{n-1}(2^n-1)(\delta_0(1)+2^n\delta_0) - 2^{3n-1}\delta_0. \end{aligned}$$

(ii), (iii) These are obtained easily from (i) $\times\,\delta_0$ and (i) $\times\, x_0$, using Lemma 5.7, (5.12) and (5.4). q.e.d.

PROOF OF LEMMA 5.9. By Lemma 5.6,

$$\alpha_0x_0^{m+1} = 2^{n+1}\alpha_0, \qquad \beta_0x_0^{m+1} = 2^{n+1}\beta_0.$$

The other elements of (5.8) are linear combinations of those of (5.10) by Lemma 5.13. Conversely, by Lemma 5.11 (i) and (vi), we have

$$2^{2n+1}\delta_0 \equiv 0, \qquad 2^{n-1}(\delta_0(1)+2^n\delta_0) \equiv 0,$$

modulo the ideal $x_0^{m+1}RO(H_2)$, as desired. q.e.d.

(II) The case $n=2m$

By Corollary 1.2, we have

$$\widetilde{KO}(N^n(2)) \cong \widetilde{RO}(H_2)/c^{-1}((\chi_4-2)^{2m+1}c'RSp(H_2)).$$

By Lemma 3.5, the ideal $(\chi_4-2)^{2m+1}c'RSp(H_2)$ of $R(H_2)$ is generated by

$$2(\chi_4-2)^{2m+1}, \quad 2(\chi_i-1)(\chi_4-2)^{2m+1} \ (i=1, 2, 3), \quad (\chi_4-2)^{2m+2}.$$

On the other hand, by Lemma 3.3, we have

$$2(\chi_3-1)(\chi_4-2)^{2m+1} = 2((\chi_4-2)^2+4(\chi_4-2)-(\chi_1-1)-(\chi_2-1))(\chi_4-2)^{2m+1},$$

whose c^{-1}-image is equal to

$$\delta_0 x_0^{m+1} + 8x_0^{m+1} - \alpha_0\delta_0 x_0^m - \beta_0\delta_0 x_0^m = -4x_0^{m+1} - 8\delta_0 x_0^m - \alpha_0\delta_0 x_0^m - \beta_0\delta_0 x_0^m,$$

by (5.1) and (5.4). Therefore, we see that the ideal $c^{-1}((\chi_4-2)^{2m+1}c'RSp(H_2))$ of $RO(H_2)$ is generated by

(5.14) $$\delta_0 x_0^m, \quad \alpha_0\delta_0 x_0^m, \quad \beta_0\delta_0 x_0^m, \quad x_0^{m+1},$$

by the above facts and (5.1).

Also, $\widetilde{RO}(H_2)$ is the free Z-module with bases

$$\alpha_0, \quad \beta_0, \quad \delta_0, \quad \delta_0(1) = x_0 + 2\delta_0,$$

by (5.2). Therefore, Theorem 1.3 for $n=2m$ follows immediately from

LEMMA 5.15. *The elements of* (5.14) *are linear combinations of*

(5.16) $$2^{n+2}\alpha_0, \quad 2^{n+2}\beta_0, \quad 2^{2n}\delta_0, \quad 2^n\delta_0(1),$$

and the elements of (5.16) *are also so of* (5.14).

By Lemma 5.6, we have

$$\alpha_0\delta_0 x_0^m = -2^{n+2}\alpha_0, \qquad \beta_0\delta_0 x_0^m = -2^{n+2}\beta_0.$$

Therefore, Lemma 5.15 follows immediately from the following

LEMMA 5.17. (i) $\delta_0 x_0^m = 2^{2n}\delta_0 - 2^n(2^n-1)\delta_0(1)$,

(ii) $x_0^{m+1} = 2^n(2^{n+1}-1)\delta_0(1) - 2^{2n+1}\delta_0$,

(iii) $2^n\delta_0(1) = x_0^{m+1} + 2\delta_0 x_0^m$,

(iv) $2^{2n}\delta_0 = (2^n-1)x_0^{m+1} + (2^{n+1}-1)\delta_0 x_0^m$.

PROOF. (i) By $(5.4) \times x_0^{m-1}$, we have

$$-\delta_0 x_0^m = 12x_0^m + 8\delta_0 x_0^{m-1} = 8x_0^m + 4x_0^{m-1}\delta_0(1) = 8x_0^m + 2^n\delta_0(1)\,,$$

using Lemma 5.7. While, by $(5.12)\times 2^{4i+3}x_0^{m-i-2}$, we have

$$2^{4i+3}x_0^{m-i} = 2^{4(i+1)+3}x_0^{m-i-1} + 3\cdot 2^{n+1+2i}\delta_0(1)\,.$$

Therefore, we have (i), since

$$\begin{aligned} 8x_0^m &= 2^{4m-1}x_0 + 3\cdot 2^{n+1}(1+2^2+2^4+\cdots+2^{2(m-2)})\delta_0(1) \\ &= 2^{2n-1}x_0 + (2^{2n-1}-2^{n+1})\delta_0(1) \\ &= (2^{2n}-2^{n+1})\delta_0(1) - 2^{2n}\delta_0. \end{aligned}$$

(ii) By $(5.12)\times 2^{4i}x_0^{m-i-1}$, we have

$$2^{4i}x_0^{m+1-i} = 2^{4(i+1)}x_0^{m-i} + 3\cdot 2^{n+2i}\delta_0(1)\,.$$

Therefore, we have (ii), since

$$\begin{aligned} x_0^{m+1} &= 2^{2n}x_0 + 3\cdot 2^n(1+2^2+2^4+\cdots+2^{2(m-1)})\delta_0(1) \\ &= 2^n(2^{n+1}-1)\delta_0(1) - 2^{2n+1}\delta_0. \end{aligned}$$

(iii) follows immediately by Lemma 5.7, and (iv) follows from (i) and (ii). q.e.d.

These complete the proof of Theorem 1.3.

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