HIROSHIMA MATH. J.
4 (1974), 441–457

Eigenfunctions of the Laplacian on a Hermitian Hyperbolic Space

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(Received January 17, 1974)

Let G be a connected real semisimple Lie group of real rank one with finite center, K a maximal compact subgroup, $G=KAN$ an Iwasawa decomposition and M the centralizer of A in K. We put $X=G/K$ and $B=K/M$. Let Δ denote the laplacian on X corresponding to the G-invariant riemannian metric on X induced by the Killing form of the Lie algebra of G. In [2, Chap. IV, Th. 1.8], S. Helgason proved that when $G=SU(1, 1)$, any eigenfunction of Δ can be given as the Poisson transform of a (Sato's) hyperfunction on B, and suggested the possibility of generalizing the theorem to the case of a (non-compact) symmetric space of rank one, which we shall call Helgason's conjecture.

The purpose of this paper is to prove that when X is a hermitian hyperbolic space $SU(n, 1)/S(U_n \times U_1)$, Helgason's conjecture is valid in a weak sense. That is, any eigenfunction of Δ with real eigenvalue $\mu \geqq -<\rho, \rho>$ can be given as the Poisson transform of a hyperfunction on B (Corollary 4.5). For a real hyperbolic space $SO_0(n, 1)/SO(n)$, the author proved in [7] that Helgason's conjecture is valid for any complex eigenvalue.

The construction of this paper is as follows. In § 1, we define the Poisson transform of a continuous function and state some results on this transform. In § 2, we review the structure of the Lie algebra $\mathfrak{su}(n, 1)$ and investigate the eigenvalues of some differential operators. In § 3, the Poisson transform of a K-finite function on B are determined explicitly. In the final section, by using the results in § 3 we prove that for $s \geqq 0$, Poisson transform $\mathscr{P}_s$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{H}_s(X)$ (Theorem 4.4), where $\mathscr{B}(B)$ is the space of hyperfunctions on B and $\mathscr{H}_s(X)$ is the space of eigenfunctions of Δ with eigenvalue $(s^2-1)<\rho, \rho>$. From this theorem Corollary 4.5 follows immediately.

We shall use the standard notation $\boldsymbol{N}$, $\boldsymbol{R}$, $\boldsymbol{C}$ for the set of natural numbers, the field of real numbers and the field of complex numbers respectively; $\boldsymbol{N}^0$ is the set of non-negative integers. If E is a differentiable manifold, $C(E)$ (resp. $C^\infty(E)$) denotes the space of all continuous (resp. smooth) functions on E.

§ 1. Poisson transform and its fundamental properties

In this section, we define the Posison transform and gather some results on this transform without proof. For details, see [7, § 1–§ 3].

Throughout this paper we assume that G is a connected real semisimple Lie group of real rank one with finite center. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ its complexification. Let K be a maximal compact subgroup of G, $\mathfrak{k}_0$ its Lie algebra and $\mathfrak{p}_0$ the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Killing form $< , >$ of $\mathfrak{g}_0$. Then $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$. Let θ denote the corresponding Cartan involution and $\mathfrak{a}_+$ be a maximal abelian subspace in $\mathfrak{p}_0$. Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{a}_+$ and put $\mathfrak{a}_-=\mathfrak{a}_0\cap\mathfrak{k}_0$. We denote the complexifications of $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_0$, $\mathfrak{a}_+$ and $\mathfrak{a}_-$ in $\mathfrak{g}$ by $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{k}$ respectively. Then Lie algebra $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. For $\lambda\in\mathfrak{a}^*$, let $\bar{\lambda}$ denote the restriction of λ to $\mathfrak{a}_\mathfrak{p}$ and let H_λ denote the element in $\mathfrak{a}$ determined by $<H_\lambda, H>=\lambda(H)$ for $H\in\mathfrak{a}$. For λ, $\mu\in\mathfrak{a}^*$, put $<\lambda, \mu>=<H_\lambda, H_\mu>$. We introduce and fix compatible orders in $(\mathfrak{a}_++\sqrt{-1}\,\mathfrak{a}_-)^*$ and $\mathfrak{a}_+^*$. Let P denote the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ under this ordering, P_+ the set of $\alpha\in P$ with $\bar{\alpha}\neq 0$ and Σ_+ the set of $\bar{\alpha}$ with $\alpha\in P_+$. Since $\dim\mathfrak{a}_+=1$, we can select $\mu_0\in\Sigma_+$ such that $2\mu_0$ is the only other possible element in Σ_+. Put P_{μ_0} (resp. $P_{2\mu_0}$) be the set of $\alpha\in P_+$ with $\bar{\alpha}=\mu_0$ (resp. $\bar{\alpha}=2\mu_0$) and p (resp. q) be the number of roots in P_{μ_0}(resp. $P_{2\mu_0}$). We put

$$\rho=\frac{1}{2}\sum_{\alpha\in P_+}\bar{\alpha},\quad \mathfrak{n}=\sum_{\alpha\in P_+}\mathfrak{g}^\alpha,\quad \mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0\,,$$

where $\mathfrak{g}^\alpha$ is the root subspace of α. Let K, A, N denote the analytic subgroups of G with Lie algebras $\mathfrak{k}_0$, $\mathfrak{a}_+$, $\mathfrak{n}_0$ respectively. Then $G=KAN$ is an Iwasawa decomposition. For $x\in G$, we define a unique element $H(x)\in\mathfrak{a}_+$ by $x\in K(\exp H(x))\,N$. Put $X=G/K$ and $B=K/M$, where M is the centralizer of A in K. Let db denote the normalized K-invariant measure on B. We introduce a parameter $s\in\boldsymbol{C}$ in $\mathfrak{a}_\mathfrak{p}^*$ by $\lambda=-\sqrt{-1}\,s\rho$. For $s\in\boldsymbol{C}$, we define a real-analytic function $P_s(z, b)$ on $X\times B$, called Poisson kernel, by

$$P_s(xK, kM)=\exp\{-(1+s)\rho(H(x^{-1}k))\}\,.$$

For $\phi\in C(B)$, the Poisson transform $\mathscr{P}_s(\phi)$ of ϕ is defined by

$$\mathscr{P}_s(\phi)(z)=\int_B P_s(z, b)\phi(b)db,\qquad z\in X\,.$$

Let R denote the set of equivalence classes of irreducible unitary representations of K and R^0 denote the subset of those which are of class one with respect to M. For each $\gamma\in R$, we take and fix a representative $(\tau^\gamma, W^\gamma)\in\gamma$ and choose an orthonormal base $\{w_1,\dots, w_{d(\gamma)}\}$ of W^γ with respect to the unitary inner product $(,)$ of W^γ so that w_1^γ is an M-fixed vector if $\gamma\in R^0$, where $d(\gamma)$ is the dimension of W^γ. Let π be the left regular representation of K on $C(K)$, $C^\infty(B)$ and $C^\infty(X)$, and put

$$V^\gamma=\{\phi\in C^\infty(K)|\phi \text{ transforms according to } \gamma \text{ under } \pi\},$$

$$\tau^{\gamma}_{ij}(k)=(\tau^{\gamma}(k)w^{\gamma}_{j},\ w^{\gamma}_{i})\ ,$$

$$\phi^{\gamma}_{ij}=d(\gamma)^{1/2}\bar{\tau}^{\gamma}_{ij}\ ,$$

$$\phi^{\gamma}_{i}=\phi^{\gamma}_{i1}$$

for $\gamma\in R$, $1\leqq i\leqq d(\gamma)$. Then

$$\{\phi^{\gamma}_{i}|1\leqq i\leqq d(\gamma)\}$$

is an orthonormal base of $V^{\gamma}(\gamma\in R^{0})$ and

$$\{\phi^{\gamma}_{i}|\gamma\in R^{0},\ 1\leqq i\leqq d(\gamma)\}$$

is a complete orthonormal base of $L^{2}(B)$.

Let Δ be the laplacian corresponding to the G-invariant riemannian metric on X induced by the Killing form of $\mathfrak{g}_0$. Put

$$\mathscr{H}_s(X)=\{f\in C^{\infty}(X)|\Delta f=(s^2-1)<\rho,\ \rho>f\},$$

$$\mathscr{H}^{\gamma}_s(X)=\{f\in\mathscr{H}_s(X)|f \text{ transforms according to } \gamma \text{ under } \pi\}$$

and put

$$e(s)=\Gamma\left(\frac{1}{2}\left(\frac{p}{2}+1+\left(\frac{p}{2}+q\right)s\right)\right)^{-1}\Gamma\left(\frac{1}{2}\left(\frac{p}{2}+q+\left(\frac{p}{2}+q\right)s\right)\right)^{-1},$$

where Γ denotes the gamma function.

PROPOSITION 1.1 (Helgason).

(1) *$\mathscr{P}_s$ maps $C(B)$ into $\mathscr{H}_s(X)$ and V^{γ} into $\mathscr{H}^{\gamma}_s(X)$.*

(2) *$\mathscr{P}_s$ is injective on $C(B)$ if and only if $e(s)\neq 0$.*

(3) *If $\mathscr{H}^{\gamma}_s(X)\neq\{0\}$, then $\gamma\in R^{0}$.*

(4) *If $\mathscr{P}_s$ is injective on $C(B)$, $\mathscr{P}_s$ maps V^{γ} onto $\mathscr{H}^{\gamma}_s(X)$.*

We put $f^{\gamma}_{si}=\mathscr{P}_s(\phi^{\gamma}_{i})$.

PROPOSITION 1.2. *Suppose that $e(s)\neq 0$ and $f\in\mathscr{H}_s(X)$.*

(1) *There exist unique compelx numbers $a^{\gamma}_{i}(\gamma\in R^{0},\ 1\leqq i\leqq d(\gamma))$ such that*

$$f(z)=\sum_{\gamma\in R^0}\sum_{i=1}^{d(\gamma)}a^{\gamma}_{i}f^{\gamma}_{si}(z)\ ,$$

which is absolutely convergent for $z\in X$.

(2) *Put $\phi^{z}_{f}(k)=f(kz)$. Then*

$$\phi_f^z = \sum_{\gamma \in R^0} d(\gamma)^{-1/2} \sum_{i,j=1}^{d(\gamma)} a_i^\gamma f_{sj}^\gamma(z) \phi_{ij}^\gamma ,$$

which is absolutely and uniformly convergent on K.

(3) *Let* $\| \quad \|$ *denote the norm of* $L^2(B)$. *Then*

$$\|\phi_f^z\|^2 = \sum_{\gamma \in R^0} d(\gamma)^{-1} \Big(\sum_{i=1}^{d(\gamma)} |a_i^\gamma|^2 \Big) \Big(\sum_{j=1}^{d(\gamma)} |f_{sj}^\gamma(z)|^2 \Big).$$

Put $f_s = \mathscr{P}_s(1_B)$, where 1_B denotes the constant function identically equal to 1 on B. We remark that f_s coincides with Harish-Chandra's spherical function $\phi_\lambda (\lambda = -\sqrt{-1}\, s\rho)$.

THEOREM 1.3. *Assume that* $\mathrm{Re}(s) > 0$. *Then* $f_s(aK) (a \in A)$ *is not equal to zero when* $\rho(H(a))$ *is sufficiently large, and for* $\phi \in C(B)$

$$\lim_{\rho(H(a)) \to \infty} \frac{1}{f_s(aK)} \mathscr{P}_s(\phi)(kaK) = \phi(kM)$$

uniformly on B.

We denote by $\mathfrak{B}$ the univesal enveloping algebra of $\mathfrak{g}$, whose elements are regarded as the left G-invariant differential operators on G. For $\alpha \in P_+$, take and fix a root vector $X_\alpha \in \mathfrak{g}^\alpha$ such that $<X_\alpha, X_{-\alpha}> = 1$. Putting $Z_\alpha = 2^{-1}(X_\alpha + \theta X_\alpha)$, we define ω_{μ_0} and $\omega_{2\mu_0}$ in $\mathfrak{B}$ by

$$\omega_{\mu_0} = \sum_{\alpha \in P_{\mu_0}} (Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha),$$

$$\omega_{2\mu_0} = \sum_{\alpha \in P_{2\mu_0}} (Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha).$$

Let H_0 be the element of $\mathfrak{a}_+$ such that $\mu_0(H_0) = 1$. For $t \in \boldsymbol{R}$, we put $a_t = \exp tH_0$. Then t can be regarded as a coordinate function on the one-dimensional Lie group A. Let L be the differential of the left regular representation of G on $C^\infty(X)$.

PROPOSITION 1.4. *Let* $f \in \mathscr{H}_s(X)$. *Then f satisfies*

$$\frac{d^2}{dt^2} f(a_tK) + (p \coth t + 2q \coth 2t) \frac{d}{dt} f(a_tK) - \frac{2p+8q}{(\sinh t)^2} \{L(\omega_{\mu_0}) f\}(a_tK)$$
$$- \frac{2p+8q}{(\sinh 2t)^2} \{L(\omega_{2\mu_0}) f\}(a_tK) + (1-s^2) \Big(\frac{p}{2} + q \Big)^2 f(a_tK) = 0 .$$

PROPOSITION 1.5. *Suppose that* $f_n (n \in \boldsymbol{N}^0)$ *are eigenfunctions of* Δ *with eigenvalue* $\mu \in \boldsymbol{C}$ *and that* $\sum_{n \in \boldsymbol{N}^0} f_n$ *is absolutely and uniformly convergent on every compact subset in X. Then* $\sum_{n \in \boldsymbol{N}^0} f_n$ *is also an eigenfunction of* Δ *with the same* *eigenvalue* μ.

§2. Hermitian hyperbolic spaces

From now on we deal with the Lie group $G=SU(n, 1)$ $(n\geqq 2)$. The associated symmetric space $X=G/K$ is called a hermitian hyperbolic space.

The Lie algebra $\mathfrak{g}_0=\mathfrak{su}(n, 1)$ is given by

$$\mathfrak{g}_0=\left\{\begin{pmatrix} Z & \eta \\ {}^t\bar{\eta} & z \end{pmatrix} \,\middle|\, \begin{array}{l} Z\in\mathfrak{u}(n),\ z\in\mathfrak{u}(1),\ \eta\in \boldsymbol{C}^n \\ \mathrm{Tr}(Z)+z=0 \end{array}\right\}.$$

Put

$$\mathfrak{k}_0=\left\{\begin{pmatrix} Z & 0 \\ 0 & z \end{pmatrix} \,\middle|\, \begin{array}{l} Z\in\mathfrak{u}(n),\ z\in\mathfrak{u}(1) \\ \mathrm{Tr}(Z)+z=0 \end{array}\right\},$$

$$\mathfrak{p}_0=\left\{\begin{pmatrix} 0 & \eta \\ {}^t\bar{\eta} & 0 \end{pmatrix} \,\middle|\, \eta\in \boldsymbol{C}^n\right\}.$$

Then $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ is a Cartan decomposition and negative conjugate transpose is the corresponding Cartan involution θ. The complexifications $\mathfrak{k}=\mathfrak{k}_0^{\boldsymbol{C}}$ and $\mathfrak{p}=\mathfrak{p}_0^{\boldsymbol{C}}$ in $\mathfrak{g}=\mathfrak{g}_0^{\boldsymbol{C}}=\mathfrak{sl}(n+1, \boldsymbol{C})$ are given by

$$\mathfrak{k}=\left\{\begin{pmatrix} Z & 0 \\ 0 & z \end{pmatrix} \,\middle|\, \begin{array}{l} Z\colon n\times n \text{ complex matrix},\ z\in\boldsymbol{C} \\ \mathrm{Tr}(Z)+z=0 \end{array}\right\},$$

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & \xi \\ {}^t\eta & 0 \end{pmatrix} \,\middle|\, \xi,\ \eta\in \boldsymbol{C}^n\right\}.$$

Let $\mathfrak{h}$ be the set of diagonal elements of $\mathfrak{k}$ and put $\mathfrak{h}_0=\mathfrak{k}_0\cap\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra both for $\mathfrak{k}$ and $\mathfrak{g}$. Let e_i $(1\leqq i\leqq n+1)$ be the linear form on $\mathfrak{h}$ whose value on a diagonal matrix is the i-th entry. Then roots of $(\mathfrak{g}, \mathfrak{h})$ are the differences e_i-e_j $(1\leqq i, j\leqq n+1)$. Choose an order in $(\sqrt{-1}\,\mathfrak{h}_0)^*$ so that the positive roots are e_i-e_j $(1\leqq i<j\leqq n+1)$. Let Q, Q_k and Q_n be the sets of positive, compact positive and non-compact positive roots respectively. Putting $\beta_{ij}=e_i-e_j$, we have

$$Q=\{\beta_{ij}|1\leqq i<j\leqq n+1\},$$

$$Q_k=\{\beta_{ij}|1\leqq i<j\leqq n\},$$

$$Q_n=\{\beta_{i,n+1}|1\leqq i\leqq n\}.$$

The root subspace $\mathfrak{g}^{\beta_{ij}}$ is equal to $\boldsymbol{C}E_{ij}$, where E_{ij} $(1 \leqq i,\ j \leqq n+1)$ is the matrix unit. We have

$$\mathfrak{g} = \mathfrak{h} + \sum_{\beta \in \pm Q} \mathfrak{g}^{\beta},$$

$$\mathfrak{k} = \mathfrak{h} + \sum_{\beta \in \pm Q_k} \mathfrak{g}^{\beta},$$

$$\mathfrak{p} = \sum_{\beta \in \pm Q_n} \mathfrak{g}^{\beta}.$$

The Killing form $<\ ,\ >$ of $\mathfrak{g}$ is given by

$$<X,\ Y> = 2(n+1)\mathrm{Tr}(XY)$$

for $X,\ Y \in \mathfrak{g}$. For $\beta \in \mathfrak{h}^*$, let H_β denote the element in $\mathfrak{h}$ determined by $<H_\beta, H> = \beta(H)$ for $H \in \mathfrak{h}$. For $\beta,\ \beta' \in \mathfrak{h}^*$, we put $<\beta, \beta'> = <H_\beta, H_{\beta'}>$.

For simplicity we write β_0 for $\beta_{1,n+1}$ and put $\mathfrak{h}_+ = \sqrt{-1}\,\boldsymbol{R}H_{\beta_0}$, $\mathfrak{h}_- = \{H \in \mathfrak{h}_0 | <H_{\beta_0}, H> = 0\}$. Then $\mathfrak{h}_0 = \mathfrak{h}_+ + \mathfrak{h}_-$ (direct sum). Put $E'_{\beta_0} = E_{1,n+1}$ and $E'_{-\beta_0} = E_{n+1,1}$. Then $<E'_{\beta_0}, E'_{-\beta_0}> = 2<\beta_0, \beta_0>^{-1}$, $E'_{\beta_0} - E'_{-\beta_0} \in \sqrt{-1}\,\mathfrak{p}_0$ and $E'_{\beta_0} + E'_{-\beta_0} \in \mathfrak{p}_0$. Put $\mathfrak{a}_+ = \boldsymbol{R}(E'_{\beta_0} + E'_{-\beta_0})$, $\mathfrak{a}_- = \mathfrak{h}_-$, $\mathfrak{a}_0 = \mathfrak{a}_+ + \mathfrak{a}_-$, $\mathfrak{a} = \mathfrak{a}_0^c$ and $u = \exp\left\{\frac{\pi}{4}\,\mathrm{ad}\,(E'_{\beta_0} - E'_{-\beta_0})\right\}$. Then $u \in \mathrm{Aut}\,(\mathfrak{g})$ is the identity on $\mathfrak{a}_-$, $u\mathfrak{a}_+ = \sqrt{-1}\mathfrak{h}_+$, $u\mathfrak{a} = \mathfrak{h}$ and $\mathfrak{a}_0$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ ([11]). Thus we can take these $\mathfrak{a}_+$, $\mathfrak{a}_-$ and $\mathfrak{a}_0$ as those defined in § 1. We introduce an order in $\mathfrak{a}_0^*$ from $(\sqrt{-1}\,\mathfrak{h}_0)^*$ by ${}^t u$, which is, as is easily seen, compatible. Put $\alpha_{ij} = {}^t u \beta_{ij}$ $(\alpha_0 = {}^t u \beta_0)$. Then $\mu_0 = \frac{1}{2}\bar{\alpha}_0$ ($\bar{\lambda}$ denotes the restriction of $\lambda \in \mathfrak{a}^*$ to $\mathfrak{a}_\mathfrak{p}$) and

$$P_+ \ = \{\alpha_0 = \alpha_{1,n+1},\ \alpha_{1i},\ \alpha_{j,n+1} (1 < i,\ j < n+1)\},$$

$$P_{\mu_0} = \{\alpha_{1i},\ \alpha_{j,n+1} (1 < i,\ j < n+1)\},$$

$$P_{2\mu_0} = \{\alpha_0\}.$$

Put $E_{\beta_{ij}} = (2n+2)^{-1/2} E_{ij}$ and $X_{\alpha_{ij}} = u^{-1} E_{\beta_{ij}}$ $(1 \leqq i,\ j \leqq n+1)$. Since $E_{\beta_{ij}} \in \mathfrak{g}^{\beta_{ij}}$ and $<E_{\beta_{ij}}, E_{-\beta_{ij}}> = 1$, we obtain that $X_{\alpha_{ij}} \in \mathfrak{g}^{\alpha_{ij}}$ and $<X_{\alpha_{ij}}, X_{-\alpha_{ij}}> = 1$. By a direct calculation on the $\mathfrak{k}$-component $Z_\alpha = \frac{1}{2}(X_\alpha + \theta X_\alpha)$ of X_α, we have

LEMMA 2.1.

$$Z_{\alpha_0} = Z_{-\alpha_0} = -\{(n+1)/2\}^{1/2} H_{\beta_0},$$

$$Z_{\alpha_{1i}} = \frac{\sqrt{2}}{2} E_{\beta_{1i}}\ (1 < i < n+1),$$

$$Z_{-\alpha_{1i}}=\frac{\sqrt{2}}{2}E_{-\beta_{1i}}\ (1<i<n+1)\,,$$

$$Z_{\alpha_{j,n+1}}=-\frac{\sqrt{2}}{2}E_{-\beta_{1j}}\ (1<j<n+1)\,,$$

$$Z_{-\alpha_{j,n+1}}=-\frac{\sqrt{2}}{2}E_{\beta_{1j}}\ (1<j<n+1)\,.$$

Let $\mathfrak{m}$ be the Lie algebra of $M=Z_K(A)$, where K(resp. A) denotes the analytic subgroup in G with Lie algebra $\mathfrak{k}_0$ (resp. $\mathfrak{a}_+$). Then, putting $P_-=P-P_+$, we have

$$\mathfrak{m}=\sum_{\alpha\in\pm P_-}\mathfrak{g}^{\alpha}=\sum_{2\leq i,j\leq n}\mathfrak{g}^{\beta_{ij}}\,,$$

since u is the identity on $\mathfrak{a}_-$. Put

$$Z_c=(n+1)^{-1}\Big(\sum_{i=1}^{n}E_{ii}-nE_{n+1,n+1}\Big)\,,$$

$$Z_m=(n+1)^{-1}\{(n-1)E_{11}+(n-1)E_{n+1,n+1}-2\sum_{i=2}^{n}E_{ii}\}\,.$$

Then Z_c lies in the center of $\mathfrak{k}$, Z_m lies in $\mathfrak{m}$ and

$$H_{\beta_0}=(2n+2)^{-1}(2Z_c+Z_m)\,,$$

as $H_{\beta_0}=(2n+2)^{-1}(E_{11}-E_{n+1,n+1})$. Hence we have

$$H_{\beta_0}\equiv(n+1)^{-1}Z_c \mod \mathfrak{m}\mathfrak{B}\,. \tag{2.1}$$

By Lemma 2.1 and (2.1) we have

$$\begin{aligned}\omega_{2\mu_0}&=\sum_{\alpha\in P_{2\mu_0}}(Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha)\\&=2\Big(\frac{n+1}{2}\Big)H_{\beta_0}^2\equiv(n+1)^{-1}Z_c^2 \mod \mathfrak{m}\mathfrak{B}\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\omega_{\mu_0}&=\sum_{\alpha\in P_{\mu_0}}(Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha)\\&=2\sum_{1<i<n+1}\Big(\frac{1}{2}E_{\beta_{1i}}E_{-\beta_{1i}}+\frac{1}{2}E_{-\beta_{1i}}E_{\beta_{1i}}\Big)\\&\equiv\sum_{\beta\in Q_k}(E_\beta E_{-\beta}+E_{-\beta}E_\beta) \mod \mathfrak{m}\mathfrak{B}\end{aligned} \tag{2.3}$$

Let ω_K denote the Casimir operator on K corresponding to the restriction of the Killing form of $\mathfrak{g}$ on $\mathfrak{k}$. Since $<E_\beta, E_{-\beta}>=1$ for $\beta\in Q_k$, $<\mathfrak{h}_+, \mathfrak{h}_->=0$, $\mathfrak{h}_-\subset\mathfrak{m}$ and $<\sqrt{n+1}\,H_{\beta_0}, \sqrt{n+1}\,H_{\beta_0}>=1$, we have

$$\omega_K\equiv\sum_{\beta\in Q_k}(E_\beta E_{-\beta}+E_{-\beta}E_\beta))+(n+1)H_{\beta_0}^2 \qquad \mathrm{mod}\ \mathfrak{m}\mathfrak{B}. \tag{2.4}$$

From (2.2), (2.3) and (2.4), it follows that

$$\omega_K\equiv\omega_{\mu_0}+\omega_{2\mu_0} \qquad \mathrm{mod}\ \mathfrak{m}\mathfrak{B}.$$

LEMMA 2.2. *For* $\mathfrak{g}=\mathfrak{su}(n, 1)$, *we have*

$$\omega_{2\mu_0}\equiv(n+1)^{-1}Z_c^2 \qquad \mathrm{mod}\ \mathfrak{m}\mathfrak{B},$$

$$\omega_{\mu_0}+\omega_{2\mu_0}\equiv\omega_K \qquad \mathrm{mod}\ \mathfrak{m}\mathfrak{B}.$$

Let L^0 be the set of highest weights of $\gamma\in R^0$. Then by the theory of Kostant-Rallis ([4]), L^0 is given as the set

$$\{\Lambda=\Lambda_{l,m}=(l-m)\Lambda_1+m\Lambda_{n-1}+(l-3m)\Lambda_n | l,\ m\in\boldsymbol{N}^0,\quad l\geqq m\}$$

for $G=SU(n, 1)$ $(n\geqq 2)$, where $\Lambda_i=e_1+\cdots+e_i$.

§3. K-finite eigenfunctions on a hermitian hyperbolic space

In this section we determine the Poisson transform of a K-finite function on B for a hermitian hyperbolic space $X=SU(n, 1)/S(U_n\times U_1)$.

From now on, for $\gamma\in R^0$ with the highest weight $\Lambda_{l,m}$, we write $\tau^{l,m}$, $V^{l,m}$, $d(l, m)$, $\mathscr{H}_s^{l,m}$, $\phi_i^{l,m}$ and $f_{si}^{l,m}$ instead of τ^γ. V^γ, $d(\gamma)$, $\mathscr{H}_s^\gamma$, ϕ_i^γ and f_{si}^γ respectively. We identify L^0 with the set of the pairs (l, m) such that l, $m\in\boldsymbol{N}^0$ and $l\geqq m$. Put

$$\rho_K=2^{-1}\sum_{\beta\in Q_k}\beta.$$

LEMMA 3.1. *Let* $f\in\mathscr{H}_s^{l,m}$. *Then, for* $a\in A$,

$$\{L(\omega_{\mu_0})f\}\ (aK)=\left\{\frac{1}{n+1}(lm-m^2)+\frac{n-1}{2(n+1)}l\right\}f(aK),$$

$$\{L(\omega_{2\mu_0})f\}(aK)=\frac{1}{n+1}(l-2m)^2f(aK).$$

PROOF. Since f transforms according to $\tau^{l,m}$ under π,

$$L(\omega_K)f=<\Lambda_{l,m}+2\rho_K, \Lambda_{l,m}>f$$

$$L(Z_c^2)=\Lambda_{l,m}(Z_c)^2f.$$

On the other hand, since M normalizes A,

$$f((\exp tY)aK)=f(a(\exp tY)K)=f(aK)$$

for $f\in C^{\infty}(X)$, $a\in A$, $Y\in\mathfrak{m}$ and $t\in\boldsymbol{R}$. Therefore we have

$$(L(u)f)(aK)=0$$

for $u\in\mathfrak{m}\mathfrak{B}$ and $f\in C^{\infty}(X)$. Hence from Lemma 2.2 it follows that

$$(3.1)\qquad (L(\omega_{\mu_0})f)(aK)=\left\{L\left(\omega_K-\frac{1}{n+1}Z_c^2\right)f\right\}(aK),$$

$$(L(\omega_{2\mu_0})f)(aK)=\frac{1}{n+1}\{L(Z_c^2)\}(aK).$$

By a simple computation, we have

$$(3.2)\qquad <\Lambda_{l,m}+2\rho_K, \Lambda_{l,m}>=\frac{1}{n+1}(l^2-3lm+3m^2)+\frac{n-1}{2(n+1)}l,$$

$$\Lambda_{l,m}(Z_c)=l-2m.$$

From (3.1) and (3.2) we obtain this Lemma.

LEMMA 3.2. *Let $s\in\boldsymbol{C}$ and put $f_s^{l,m}=f_{s1}^{l,m}$. Then*

(3.3)

$$f_s^{l,m}(a_tK)=d(l,m)^{1/2}\frac{\Gamma(n)}{\Gamma(l+n)}\cdot\frac{\Gamma\left(l-m+\frac{n}{2}(1+s)\right)}{\Gamma\left(\frac{n}{2}(1+s)\right)}\cdot\frac{\Gamma\left(m+\frac{n}{2}(1+s)\right)}{\Gamma\left(\frac{n}{2}(1+s)\right)}$$

$$\times(\tanh t)^l(\cosh t)^{n(s-1)}F\left(l-m+\frac{n}{2}(1-s),\ m+\frac{n}{2}(1-s),\ l+n;\ (\tanh t)^2\right),$$

$$f_{si}^{l,m}(a_tK)=0\quad(2\leqq i\leqq d(l,m)).$$

PROOF. From Proposition 1.4 and Lemma 3.1, $f=f_{si}^{l,m}$ satisfies the differential equation

$$\frac{d^2f}{dt^2}+2\{(n-1)\coth t+\coth 2t\}\frac{df}{dt}-\frac{4}{(\sinh t)^2}\left(lm-m^2+\frac{n-1}{2}l\right)f$$
$$-\frac{4(l-2m)^2}{(\sinh 2t)^2}f+(1-s^2)n^2f=0.$$

By a new parameter $z=(\tanh t)^2$, the above differential equation turns into

$$\frac{d^2f}{dz}+\frac{n-z}{z(1-z)}\frac{df}{dz}-\frac{1}{z^2(1-z)}\left(lm-m^2+\frac{n-1}{2}l\right)f$$
$$-\frac{(l-2m)^2}{4z^2}f+\frac{(1-s^2)n^2}{4z(1-z)^2}f=0.$$

By a routine argument (cf. [7]), f can be written as

$$(3.4)\qquad f_{si}^{l,m}(a_tK)=c_i^{l,m}(\tanh t)^l(\cosh t)^{n(s-1)}$$
$$\times F\left(l-m+\frac{n}{2}(1-s),\ m+\frac{n}{2}(1-s),\ l+n;\ (\tanh t)^2\right)$$

with a constant $c_i^{l,m}$. We notice that $f_s^{0,0}$ is equal to $f_s=\mathscr{P}_s(1_B)$ defined in §1. Since $f_s\in\mathscr{H}_s^{0,0}$ and $f_s(eK)=1$,

$$(3.5)\qquad f_s(a_tK)=(\cosh t)^{n(s-1)}F\left(\frac{n}{2}(1-s),\ \frac{n}{2}(1-s),\ n;\ (\tanh t)^2\right).$$

Now we assume that $\mathrm{Re}(s)>0$. Then from Theorem 1.3,

$$\lim_{t\to\infty}\frac{f_{si}^{l,m}(a_tK)}{f_s(a_tK)}=\lim_{t\to\infty}\frac{\mathscr{P}_s(\phi_i^{l,m})(a_tK)}{f_s(a_tK)}=\phi_i^{l,m}(eM)=d(l,m)^{1/2}\delta_{i1}.$$

On the other hand, from (3.4) and (3.5) it follows that

$$\lim_{t\to\infty}\frac{f_{si}^{l,m}(a_tK)}{f_s(a_tK)}$$
$$=c_i^{l,m}\frac{\Gamma(l+n)}{\Gamma\left(m+\frac{n}{2}(1+s)\right)}\cdot\frac{\Gamma(ns)}{\Gamma\left(l-m+\frac{n}{2}(1+s)\right)}\Big/\frac{\Gamma(n)\Gamma(ns)}{\Gamma\left(\frac{n}{2}(1+s)\right)^2},$$

since

$$F\left(l-m+\frac{n}{2}(1-s),\ m+\frac{n}{2}(1-s),\ l+n;\ 1\right)$$
$$=\frac{\Gamma(l+n)\Gamma(ns)}{\Gamma\left(m+\frac{n}{2}(1+s)\right)\Gamma\left(l-m+\frac{n}{2}(1+s)\right)}$$

for $\mathrm{Re}(s)>0$. Therefore we obtain (3.3) for $\mathrm{Re}(s)>0$. But both sides of (3.3) are entire functions in s for any fixed t. Hence (3.3) is valid for any $s\in\boldsymbol{C}$ from the uniqueness of analytic continuation, which finishes the proof.

§4. Poisson transform of a hyperfunction

In this section we define the Poisson transform $\mathscr{P}_s(T)$,which is a function on a hermitian hyperbolic space $X=SU(n, 1)/K$, of a hyperfunction T on $B=K/M$, and prove that for $s\geqq 0$, $\mathscr{P}_s$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{H}_s(X)$, where $\mathscr{B}(B)$ denotes the space of Sato's hyperfunctions on B and $\mathscr{H}_s(X)$ is the space of eigenfunctions of Δ on X with eigenvalue $(s^2-1)<\rho, \rho>$.

Let $\mathscr{A}(B)$ denote the space of real-analytic functions on B with the natural topology ([6]) and $\mathscr{A}'(B)$ the space of continuous linear functions of $\mathscr{A}(B)$ into $\boldsymbol{C}$. Since $B=K/M$ is real-analytically isomorphic to the $(2n-1)$-dimensional sphere S^{2n-1}, $\mathscr{A}'(B)$ is canonically isomorphic to $\mathscr{B}(B)$, the space of Sato's hyperfunctions on B([10]). Henceforth we write $\mathscr{B}(B)$ for $\mathscr{A}'(B)$ and call the elements of $\mathscr{A}'(B)$ hyperfunctions on B. We denote the value of $T\in\mathscr{B}(B)$ at $\phi\in\mathscr{A}(B)$ by

$$\int_B \phi(b)dT(b)\,.$$

We define a subspace $\mathscr{F}_b(B)$ in $\boldsymbol{C}^N=\prod_{(l,m)\in L^0}\boldsymbol{C}^{d(l,m)}$ by

$$\mathscr{F}_b(B)=\{(a_i^{l,m})\in\boldsymbol{C}^N\,|\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|\exp(-\eta\lambda_{l,m}^{1/2})<\infty \text{ for any } \eta>0\}\,,$$

where $\lambda_{l,m}=(n+1)^{-1}(l^2-3lm+3m^2)+(2n+2)^{-1}(n-1)l$ (the eigenvalue of ω_K on $V^{l,m}$) and define a mapping Ψ of $\mathscr{B}(B)$ into $\boldsymbol{C}^N$ by

$$\Psi(T)=(a_i^{l,m}),\ a_i^{l,m}=\int_B\bar{\phi}_i^{l,m}(b)dT(b)\,,$$

for $T\in\mathscr{B}(B)$. Then by Theorem 1.8 and the remark in [1, § 1], Ψ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{F}_b(B)$, and $\mathscr{F}_b(B)$ is also given by

$$\mathscr{F}_b(B)=\{(a_i^{l,m})\in\boldsymbol{C}^N\,|\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|^2\exp(-\eta\lambda_{l,m}^{1/2})<\infty \text{ for any } \eta>0\}\,.$$

On the other hand, it is easy to see that

$$\frac{l}{\sqrt{2}}\geqq\lambda_{l,m}^{1/2}\geqq\frac{1}{\sqrt{n+1}}\frac{l}{2}$$

for all $(l, m)\in L^0$. Therefore $\mathscr{F}_b(B)$ can be characterized as

$$(4.1)\quad \mathscr{F}_b(B)=\{(a_i^{l,m})\,|\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|\exp(-\eta l)<\infty \text{ for any } \eta>0\}$$

$$=\{(a_i^{l,m})\,|\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|^2\exp(-\eta l)<\infty \text{ for any } \eta>0\}\,.$$

LEMMA 4.1. *Let* $(a_i^{l,m})=\Psi(T)$ $(T\in\mathscr{B}(B))$. *Then*

$$\mathscr{P}_s(T)(z)=\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}a_i^{l,m}f_{si}^{l,m}(z),\qquad z\in X\,.$$

PROOF. For any fixed $z\in X$, $P_s(z, b)$ can be expanded in an absolutely and uniformly convergent series

$$P_s(z, b)=\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}\bar{\phi}_i^{l,m}(b)\int_B P_s(z, b)\phi_i^{l,m}(b)db\,,$$

which converges also in $\mathscr{A}(B)$ ([1, Corollary 1 to Proposition 1.7]). From the continuity of T on $\mathscr{A}(B)$, we have

$$\mathscr{P}_s(T)(z)=\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}\int_B\bar{\phi}_i^{l,m}(b)dT(b)\int_B P_s(z, b)\phi_i^{l,m}(b)db\,.$$

Since

$$a_i^{l,m}=\int_B\bar{\phi}_i^{l,m}(b)dT(b)\,,$$

$$f_{si}^{l,m}(z)=\int_B P_s(z, b)\phi_i^{l,m}(b)db\,,$$

we obtain this lemma.

PROPOSITION 4.2. (1) *For any $s\in\boldsymbol{C}$ and any $(a_i^{l,m})\in\mathscr{F}_b(B)$, the series*

$$\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}a_i^{l,m}f_{si}^{l,m}$$

is absolutely and uniformly convergent on every compact subset of X.
(2) *Suppose that $s\geqq 0$ and expand $f\in\mathscr{H}_s(X)$ as*

$$f=\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}a_i^{l,m}f_{si}^{l,m}$$

by Proposition 1.2, *which is possible as $e(s)\neq 0$ for $s\geqq 0$. Then $(a_i^{l,m})\in\mathscr{F}_b(B)$.*
For the proof, we need the following

LEMMA 4.3. *For $(l, m)\in L^0$ and $u\in\boldsymbol{C}$, put*

$$G_u^{l,m}(r)=r^l\frac{\Gamma(n)}{\Gamma(l+n)}\cdot\frac{\Gamma(l-m+u)}{\Gamma(u)}\cdot\frac{\Gamma(m+u)}{\Gamma(u)}$$
$$\times F(l-m+u,\, m+u,\, l+n;\, r^2)\quad(|r|<1)\,.$$

(1) *For any fixed h with $0<h<1$, there exists an l_0 such that for any $(l, m)\in L^0$ with $l\geqq l_0$,*

$$|G_u^{l,m}(r)|\leqq|r|^l\,\frac{\Gamma(n)}{\Gamma(l+n)}\cdot\frac{\Gamma(l+|u|)}{\Gamma(|u|)}\cdot(1-h)^{-|u|}\quad(|r|\leqq h)\,.$$

(2) *Assume that $u\geqq\frac{n}{2}$ and $t>0$. Then for any $(l, m)\in L^0$,*

$$G_u^{l,m}(\tanh t) \geqq \left(\tanh \frac{t}{2}\right)^l \frac{2^l \Gamma(l/2+u)^2}{\Gamma(l+2u+1/2)} \cdot \frac{\Gamma(n)}{\Gamma(u)^2} .$$

PROOF. First we notice that

$$\Gamma(l+v)\Gamma(v) \geqq \Gamma(l-m+v)\Gamma(m+v) \geqq \Gamma\left(\frac{l}{2}+v\right)^2 \tag{4.2}$$

for $l \geqq m \geqq 0$ and $v>0$. From the definition of the hypergeometric function, it follows that

(4.3)

$$G_u^{l,m}(r) = r^l \Gamma(n) \sum_{k=0}^{\infty} \frac{\Gamma(l-m+u+k)}{\Gamma(u)} \cdot \frac{\Gamma(m+u+k)}{\Gamma(u)} \cdot \frac{1}{\Gamma(l+n+k)} \cdot \frac{r^{2k}}{k!} .$$

Therefore using (4.2) we have

$$\begin{aligned} |G_u^{l,m}(r)| &\leqq |r|^l \Gamma(n) \sum_{k=0}^{\infty} \frac{\Gamma(l-m+|u|+k)}{\Gamma(|u|)} \cdot \frac{\Gamma(m+|u|+k)}{\Gamma(|u|)} \\ &\qquad \cdot \frac{1}{\Gamma(l+n+k)} \cdot \frac{r^{2k}}{k!} \\ &\leqq |r|^l \frac{\Gamma(n)}{\Gamma(l+n)} \cdot \frac{\Gamma(l+|u|)}{\Gamma(|u|)} \cdot F(l+|u|, |u|, l+n; r^2) . \end{aligned}$$

On the other hand if we put $l_0 = (h|u|-n)/(1-h)$, it can be shown ([7, Lemma 5.3]) that for any $l \geqq l_0$,

$$F(l+|u|, |u|, l+n; r^2) \leqq (1-h)^{-|u|} \qquad (|r| \leqq h) ,$$

which proves the first assertion of the lemma.

Next, putting $r = \tanh t$, we have from (4.2) and (4.3) that

$$\begin{aligned} G_u^{l,m}(r) &\geqq r^l \Gamma(n) \sum_{k=0}^{\infty} \frac{\Gamma(l/2+u+k)^2}{\Gamma(u)^2} \cdot \frac{1}{\Gamma(l+n+k)} \cdot \frac{r^{2k}}{k!} \\ &\geqq r^l \Gamma(n) \sum_{k=0}^{\infty} \frac{\Gamma(l/2+u+k)^2}{\Gamma(u)^2} \cdot \frac{1}{\Gamma(l+2u+1/2+k)} \cdot \frac{r^{2k}}{k!} , \end{aligned}$$

since $\Gamma(l+2u+1/2+k) \geqq \Gamma(l+n+k)$. Therefore we obtain

$$\begin{aligned} G_u^{l,m}(r) \geqq r^l \frac{\Gamma(u)}{\Gamma(l+2u+1/2)} \cdot \frac{\Gamma(l/2+u)^2}{\Gamma(u)^2} \\ \times F(l/2+u, l/2+u, l+2u+1/2; r^2) . \end{aligned}$$

By using the equality

$$F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; z\right)$$

$$=\left(\frac{1+\sqrt{1-z}}{2}\right)^{1/2-\alpha-\beta} F\left(\alpha-\beta+\frac{1}{2}, \beta-\alpha+\frac{1}{2}, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-z}}{2}\right)$$

in [5, p. 251] and considering that

$$F\left(\frac{1}{2}, \frac{1}{2}, l+2u+\frac{1}{2}; \frac{1-\sqrt{1-r^2}}{2}\right) \geqq 1,$$

$$\frac{r}{1+\sqrt{1-r^2}} = \tanh \frac{t}{2},$$

$$\left(\frac{2}{1+\sqrt{1-r^2}}\right)^{2u-1/2} \geqq 1,$$

we get

$$G_u^{l,m}(\tanh t) \geqq \left(\tanh \frac{t}{2}\right)^l \cdot \frac{2^l \Gamma(l/2+u)^2}{\Gamma(l+2u+1/2)} \cdot \frac{\Gamma(n)}{\Gamma(u)^2},$$

which completes the proof.

Proof of Proposition 4.2. We put $u=\frac{n}{2}(1+s)$. First we recall (Lemma 3.2) that

$$f_s^{l,m}(a_t K) = d(l, m)^{1/2} \frac{\Gamma(u)}{\Gamma(l+m)} \cdot \frac{\Gamma(l-m+u)}{\Gamma(u)} \cdot \frac{\Gamma(m+u)}{\Gamma(u)}$$

$$\times (\tanh t)^l (\cosh t)^{n(s-1)} F(l-m+n-u, m+n-u, l+n; (\tanh t)^2),$$

$$f_{si}^{l,m}(a_t K) = 0 \qquad (2 \leqq i \leqq d(l, m)).$$

Noticing (cf. [5, p. 248]) that

$$F(\alpha, \beta, \gamma; z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma: z)$$

and using the function $G_u^{l,m}$ defined in Lemma 4.3, we have

$$(4.4) \qquad f_s^{l,m}(a_t K) = d(l, m)^{1/2} \frac{\Gamma(n)}{\Gamma(l+n)} \cdot \frac{\Gamma(l-m+u)}{\Gamma(u)} \cdot \frac{\Gamma(m+u)}{\Gamma(u)}$$

$$\times (\tanh t)^l (\cosh t)^{-2u} F(l-m+u, m+u, l+n; (\tanh t)^2)$$

$$= d(l, m)^{1/2} (\cosh t)^{-2u} G_u^{l,m}(\tanh t).$$

For h with $0<h<1$, we define a compact set U_h of X by

$$U_h = \{z = k a_t K \,|\, |\tanh t| \leqq h\}.$$

Let l_0 be as in Lemma 4.3 and consider the series

$$S(z)=\sum_{(l,m)\in L^0, l\geqq l_0}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|\;|f_{si}^{l,m}(z)|$$

in U_h for $(a_i^{l,m})\in\mathscr{F}_b(B)$. From Lemma 4.3, (4.4) and $|\tau^{l,m}(k)|\leqq 1$, we have

$$\begin{aligned}S(ka_tK)&\leqq\sum_{(l,m)\in L^0, l\geqq l_0}\sum_{i,j=1}^{d(l,m)}|a_i^{l,m}|\;|f_{sj}^{l,m}(a_tK)|\;|\tau_{ij}^{l,m}(k)|\\&\leqq\sum_{(l,m)\in L^0, l\geqq l_0}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|\;|f_s^{l,m}(a_tK)|\\&\leqq c\sum_{(l,m)\in L^0, l\geqq l_0}d(l,m)^{1/2}\sum_{i=1}^{d(l,m)}|a_i^{l,m}|\,|\tanh t|^l\\&\quad\times\frac{\Gamma(n)}{\Gamma(l+n)}\frac{\Gamma(l+|u|)}{\Gamma(|u|)}\cdot(1-h)^{-|u|},\end{aligned}$$

where we put

$$c=\sup_{a_tK\in U_h}(\cosh t)^{-2\mathrm{Re}(u)}.$$

Since

$$\lim_{l\to\infty}\left(\frac{\Gamma(n)}{\Gamma(l+n)}\cdot\frac{\Gamma(l+|u|)}{\Gamma(|u|)}\right)^{1/l}=\lim_{l\to\infty}d(l,m)^{1/l}=1,$$

it follows from (4.1) that $S(z)$ converges uniformly in U_h.

(2) Let $\eta>0$ and choose a $t>0$ such that $\tanh(t/2)=\exp(-\eta/2)$. From Proposition 1.2, we have

$$\|\phi_f^z\|^2=\sum_{l(,m)\in L^0}d(l,m)^{-1}\left(\sum_{i=1}^{d(l,m)}|a_i^{l,m}|^2\right)\left(\sum_{j=1}^{d(l,m)}|f_{sj}^{l,m}(z)|^2\right)$$

for $z\in X$. Putting $z=a_tK$, by Lemma 4.3 and (4.4) we obtain

$$\begin{aligned}\|\phi_f^z\|^2\geqq\sum_{(l,m)\in L^0}&\left(\sum_{i=1}^{d(l,m)}|a_i^{l,m}|^2\right)\left(\tanh\frac{t}{2}\right)^{2l}(\cosh t)^{-4u}\\&\times\left\{\frac{2^l\Gamma(l/2+u)^2}{\Gamma(l+2u+1/2)}\cdot\frac{\Gamma(n)}{\Gamma(u)^2}\right\}^2.\end{aligned}$$

Since

$$\lim_{l\to\infty}\left\{\frac{2^l\Gamma(l/2+u)^2}{\Gamma(l+2u+1/2)}\cdot\frac{\Gamma(n)}{\Gamma(u)^2}\right\}^{1/l}=1,$$

it follows that

$$\sum_{(l,m)\in L^0}\left(\sum_{i=1}^{d(l,m)}|a_i^{l,m}|^2\right)\exp(-\eta l)<\infty\ ,$$

which implies by (4.1) that $(a_i^{l,m})\in\mathscr{F}_b(B)$. This completes the proof.

THEOREM 4.4. *Let X be a hermitian hyperbolic space.*

(1) *The Poisson transform $\mathscr{P}_s$ maps $\mathscr{B}(B)$ into $\mathscr{H}_s(X)$.*

(2) *For $s\geqq 0$, $\mathscr{P}_s$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{H}_s(X)$.*

COROLLARY 4.5. *For a hermitian hyperbolic space, any eigenfunction f of Δ with eigenvalue $\mu\geqq -<\rho, \rho>$ can be represented as*

$$f(z)=\int_B P_s(z, b)dT(b)$$

with some $s\geqq 0$ and some $T\in\mathscr{B}(B)$.

PROOF. Assume that $\Delta f=\mu f$. We can select an $s\geqq 0$ such that $\mu=(s^2-1)<\rho, \rho>$. Then we have only to apply Theorem 4.4 to f.

PROOF OF THEOREM 4.4. (1) Let $T\in\mathscr{B}(B)$ and put $\Psi(T)=(a_i^{l,m})$. By Lemma 4.1 and Proposition 4.2,

$$\mathscr{P}_s(T)(z)=\sum_{(l,m)\in L^0}\sum_{i=1}^{d(l,m)}a_i^{l,m}f_{si}^{l,m}(z)$$

is absolutely and uniformly convergent in every compact subset of X. Then by Proposition 1.5, $\mathscr{P}_s(T)$ belongs to $\mathscr{H}_s(X)$.
(2) The surjectivity of $\mathscr{P}_s$ $(s\geqq 0)$ is clear from Lemma 4.1 and Proposition 4.2, (2). Assume that $\mathscr{P}_s(T)=0$. Then putting $\Psi(T)=(a_i^{l,m})$, we have

$$\sum_{(l,m)\in L^0}\sum_{i,j=1}^{d(l,m)}a_i^{l,m}f_{sj}^{l,m}(z)\phi_{ij}^{l,m}(k)=0\ .$$

Since $\phi_{ij}^{l,m}$ are linearly independent and $f_{sj}^{l,m}$ are not identically equal to zero on X, we get $a_i^{l,m}=0$, which finishes the proof of the theorem.

REMARK. The set L^0 defined in [1, § 3] should be replaced by the L^0 defined in § 3 in this paper. But Theorem 4.5 in [1] is valid and is a special case of $s=1$ in Theorem 4.4 of this paper.

Added in proof.

Recently S. Helgasan has proved that the same result as in Corollary 4.5 in this paper holds also for the quaternion hyperbolic spaces and the exceptional symmetric space of type FII in the preprint "Eigenspaces of the Laplacian; integral representations and irreducibility".

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