HIROSHIMA MATH. J.
4 (1974), 425-434

Note on γ-Operations in KO-Theory

Teiichi KOBAYASHI

(Received January 17, 1974)

§1. Introduction

Let $p_i(\alpha)$ be the i-th (integral) Pontrjagin class of a real stable vector bundle α over a finite CW-complex X, and let γ^i be the Grothendieck γ-operation in KO-theory. Let k be a positive integer. Consider the two conditions: $p_k(\alpha)=0$ and $\gamma^{2k}(\alpha)=0$.

M. F. Atiyah has shown the following result in [3, §6] using the Chern character.

THEOREM 1.1. (M. F. Atiyah) *Suppose that $H^*(X; Z)$ is free. Then, for any real stable vector bundle α over X and for any positive integer k,*

$$\gamma^{2k}(\alpha) = 0 \Rightarrow p_k(\alpha) = 0 .$$

For integers $n>0$ and $q>1$, we denote by $L^n(q)(=S^{2n+1}/Z_q)$ the $(2n+1)$-dimensional standard lens space mod q and by $RP^n(=S^n/Z_2)$ the real projective n-space. The purpose of this note is to prove the following

THEOREM 1.2. (i) *Assume that q is an odd integer>1. Let α be any real stable vector bundle over $L^n(q)$ and k be any positive integer. Then*

$$\gamma^{2k}(\alpha) = 0 \Rightarrow p_k(\alpha) = 0 ,$$

while the converse does not hold in general.

(ii) *The same is true for RP^n.*

There are examples of vector bundles for which the equality $\gamma^{2k}(\alpha)=0$ does not imply the equality $p_k(\alpha)=0$. Let CP^n $(=S^{2n+1}/S^1)$ be the complex projective n-space, and $D(m, n)$ be the Dold manifold of dimension $m+2n$ obtained from $S^m\times CP^n$ by identifying (x, z) with $(-x, \bar{z})$, where $(x, z)\in S^m\times CP^n$.

THEOREM 1.3. *Assume that $n=2^r$ and $m=2^s$ $(r>s>1)$. Let $\tau_0=\tau-(m+2n)$ be the stable class of the tangent bundle τ of $D(m, n)$, and put $k=n/2+m/4$. Then $\gamma^{2i}(-\tau_0)=0$ for any $i\geqq k$, but $p_k(-\tau_0)\neq 0$.*

Let η be the canonical complex line bundle over $L^n(q)$. In §2, we calculate the Pontrjagin class of a real stable vector bundle $\alpha=r\sum_{i=1}^{q-1}a_i(\eta^i-1)$, where

a_i ($i=1, 2,\dots, q-1$) are integers and r denotes the real restriction. In § 3, following M. F. Atiyah [3], we recall the γ-operations in KO-theory and compute $\gamma_t(\alpha)$ for the stable class α. In § 4, we apply the results of § 2 and § 3 to the proof of Theorem 1.2. The proof is mainly based on the structure of $\widetilde{KO}(L^n(q))$ investigated by T. Kawaguchi and M. Sugawara [8], and that of $\widetilde{KO}(RP^n)$ investigated by J. F. Adams [1]. In § 5, we recall the cohomology structure of $D(m, n)$ according to A. Dold [4], M. Fujii [5] and J. J. Ucci [12]. We prove Theorem 1.3 in § 6 using the results in § 5 and the results on $\widetilde{KO}(D(m, n))$ (cf. M. Fujii and T. Yasui [6] and J. J. Ucci [12]). In the final section, § 7, we consider the problem of immersing $L^n(q)$ in CP^m.

The author wishes to express his sincere thanks to Professor M. Sugawara for valuable advice.

§ 2. Pontrjagin classes

According to N. Mahammed [11, Lemma 3.3], the following is known.

(2.1) *The K-ring $K(L^n(q))$ is a quotient ring*

$$Z[\eta]/<(\eta-1)^{n+1}, \eta^q-1>,$$

where $Z[\eta]$ is the polynomial ring generated by η and $<a, b>$ is its ideal generated by a and b.

Let $r: K(X)\to KO(X)$, $c: KO(X)\to K(X)$ and $t: K(X)\to K(X)$ denote the real restriction, the complexification and the conjugation, respectively. Then

(2.2) $$rc=2,\ cr=1+t \quad \text{(cf. [1, Lemma 3.9])}.$$

Let x be the first Chern class of η. Notice that $H^2(L^n(q); Z)=Z_q$ is generated by x and that $x^{n+1}=0$.

LEMMA 2.3. *Let d be any integer. The total Pontrjagin class $p=\sum_i p_i$ of the real 2-plane bundle $r\eta^d$ over $L^n(q)$ is given by $p(r\eta^d)=1+d^2x^2$.*

PROOF. Denote by $C=\sum_i c_i$ the total Chern class. Then $p_i(r\eta^d)=(-1)^i c_{2i}(cr\eta^d)=(-1)^i c_{2i}((1+t)\eta^d)=(-1)^i c_{2i}(\eta^d+\eta^{-d})$ by the definition and (2.2). But $C(\eta^d+\eta^{-d})=C(\eta^d)C(\eta^{-d})=(1+dx)(1-dx)=1-d^2x^2$, as desired. q.e.d.

PROPOSITION 2.4. *Suppose q is odd>1. The total Pontrjagin class of a real stable bundle $\alpha=r\sum_{i=0}^{q-1}a_i(\eta^i-1)$ ($a_i\in Z$) is given by $p(\alpha)=\sum_{l=0}^{[n/2]}A(l)x^{2l}$, where*

(2.5) $$A(l)=\sum_{j_1+\cdots+j_{q-1}=l}\prod_{i=1}^{q-1}\binom{a_i}{j_i}i^{2j_i}.$$

PROOF. Since q is odd, $H^*(L^n(q); Z)$ has no 2-torsion. Hence, by Lemma 2.3,

$$\begin{aligned} p(\alpha) &= \prod_{i=1}^{q-1} p(r\eta^i)^{a_i} = \prod_{i=1}^{q-1}(1+i^2x^2)^{a_i} \\ &= \prod_{i=1}^{q-1}\sum_{j_i=0}^{[n/2]}\binom{a_i}{j_i} i^{2j_i}x^{2j_i} \\ &= \sum_{l=0}^{[n/2]}\left\{\sum_{j_1+\cdots+j_{q-1}=l}\prod_{i=1}^{q-1}\binom{a_i}{j_i} i^{2j_i}\right\}x^{2l}. \end{aligned}$$

q.e.d.

§3. γ-operations

Following M. F. Atiyah [3, §2], we recall the γ-operations in KO-theory. Let λ^i: $KO(X)\to KO(X)$ be the exterior power operation and λ_t: $KO(X)\to A(X)$ be the homomorphism with $\lambda_t(\alpha)=\sum_{i=0}^{\infty}\lambda^i(\alpha)t^i$ for $\alpha\in KO(X)$, where $A(X)$ denotes the multiplicative group of formal power series in t with coefficients in $KO(X)$ and constant term 1. The homomorphism γ_t: $KO(X)\to A(X)$ is defined by $\gamma_t=\lambda_{t/1-t}$, and the operation γ^i: $KO(X)\to KO(X)$ is given by $\gamma_t(\alpha)=\sum_{i=0}^{\infty}\gamma^i(\alpha)t^i$.

The following is due to [7, Lemma (4.8)].

(3.1) *For the real* 2*-plane bundle* $r\eta^d$ *over* $L^n(q)$,

$$\gamma_t(r\eta^d-2) = 1+(r\eta^d-2)t-(r\eta^d-2)t^2.$$

Let Ψ_R^i: $KO(X)\to KO(X)$ (resp. Ψ_C^i: $K(X)\to K(X)$) denote the real (resp. complex) Adams operation.

PROPOSITION 3.2. *Let q be an integer>1, and a_i $(i=1, 2,\dots, q-1)$ be integers. Denote by $\sigma=\eta-1$ the stable class of η. Then, for an element $\alpha=r\sum_{i=1}^{q-1}a_i(\eta^i-1)$, we obtain*

$$\begin{aligned} \gamma_t(\alpha) &= \sum_l\left\{\sum_{j_1+\cdots+j_{q-1}=l}\prod_{i=1}^{q-1}\binom{a_i}{j_i}(\Psi_R^i r\sigma)^{j_i}\right\}(t-t^2)^l \\ &= \sum_l\left\{\sum_{j_1+\cdots+j_{q-1}=l}\prod_{i=1}^{q-1}\binom{a_i}{j_i}\left(\sum_{s=1}^{i}\frac{i}{s}\binom{i+s-1}{2s-1}(r\sigma)^{s-1}\right)^{j_i}\right\} \\ &\quad (r\sigma)^l(t-t^2)^l. \end{aligned}$$

PROOF. Using (3.1), we have

$$\begin{aligned} \gamma_t(\alpha) &= \prod_{i=1}^{q-1}(\gamma_t(r\eta^i-2))^{a_i} = \prod_{i=1}^{q-1}(1+(r\eta^i-2)(t-t^2))^{a_i} \\ &= \prod_{i=1}^{q-1}\sum_{j_i}\binom{a_i}{j_i}(r\eta^i-2)^{j_i}(t-t^2)^{j_i} \\ &= \sum_l\left\{\sum_{j_1+\cdots+j_{q-1}=l}\prod_{i=1}^{q-1}\binom{a_i}{j_i}(r\eta^i-2)^{j_i}\right\}(t-t^2)^l. \end{aligned}$$

On the other hand,

$$\begin{aligned} r\eta^i - 2 &= r\Psi_C^i\eta - 2 && \text{by [1, Theorem 5.1, (iii)]} \\ &= \Psi_R^i r\eta - 2 = \Psi_R^i r\sigma && \text{by [2, Lemma A2]} \\ &= \textstyle\sum_{s=1}^i \frac{i}{s}\binom{i+s-1}{2s-1}(r\sigma)^s && \text{by [9, (4.2)]}. \end{aligned}$$

Thus we get the desired equalities. q.e.d.

§4. Proof of Theorem 1.2

For the proof of the first part of Theorem 1.2, we make use of the following results of T. Kawaguchi and M. Sugawara [8, Theorem 1.1, Propositions 2.6 and 2.11]. Let $L_0^n(q)$ be the $2n$-skeleton of $L^n(q)$.

THEOREM 4.1. (T. Kawaguchi and M. Sugawara) (i) *Let q be an odd integer >1. Then the ring $\widetilde{KO}(L_0^n(q))$ is generated by $r\sigma$, the element $(r\sigma)^{[n/2]}$ is of order q, and $(r\sigma)^{[n/2]+1}=0$.*

(ii) *Let p be an odd prime and $r\geqq 1$. Then the order of the element $(r\sigma)^i$ of $\widetilde{KO}(L^n(p^r))$ is equal to $p^{r+[(n-2i)/(p-1)]}$ for $1\leqq i\leqq[n/2]$.*

Also we need the results of J. F. Adams [1, Theorem 7.4]. Let ξ be the canonical line bundle over RP^n and let $\lambda=\xi-1$.

THEOREM 4.2. (J. F. Adams) *$\widetilde{KO}(RP^n)$ is a cyclic group of order $2^{\phi(n)}$ generated by λ, where $\phi(n)$ is defined as the number of integers s with $0<s\leqq n$ and $s\equiv 0, 1, 2$ or $4 \bmod 8$. The multiplicative structure is determined by $\lambda^2=-2\lambda$, $\lambda^{\phi(n)+1}=0$.*

PROOF OF THEOREM 1.2. (i) As is well-known [11],

$$\widetilde{KO}(L^n(q)) = \widetilde{KO}(L_0^n(q)) + \widetilde{KO}(S^{2n+1}),\ \widetilde{KO}(L_0^n(q)) = r\tilde{K}(L^n(q)),$$

$$\widetilde{KO}(S^{2n+1}) = Z_2 \quad \text{if} \quad n\equiv 0 \mod 4, \qquad =0 \quad \text{if} \quad n\not\equiv 0 \mod 4.$$

Thus we can write $\alpha=\alpha'+\beta$ where $\alpha'\in\widetilde{KO}(L_0^n(q))$ and $\beta\in\widetilde{KO}(S^{2n+1})$. It is easy to see that $\gamma^i(\beta)\in\widetilde{KO}(S^{2n+1})$ for $i>0$. Hence $\gamma^i(\alpha')\gamma^j(\beta)=0$ for $i>0$ and $j>0$, because $\gamma^i(\alpha')$ $(i>0)$ is zero or of odd order and $\gamma^i(\beta)$ $(i>0)$ is zero or of order 2. Consequently, we obtain

$$\gamma_t(\alpha) = \gamma_t(\alpha')\gamma_t(\beta) = 1+\textstyle\sum_{i>0}(\gamma^i(\alpha')+\gamma^i(\beta))t^i.$$

Thus $\gamma^i(\alpha)=0$ implies $\gamma^i(\alpha')=0$. Since $p(\alpha)=p(\alpha')p(\beta)=p(\alpha')$, we may assume that $\alpha=\alpha'\in\widetilde{KO}(L_0^n(q))=r\tilde{K}(L^n(q))$.

Let η be the canonical complex line bundle over $L^n(q)$. By (2.1) we can write $\alpha=r\sum_{i=1}^{q-1}a_i(\eta^i-1)$, $a_i\in Z$. Since $x^{n+1}=0$, we may assume that $n>1$ and that $k\leqq[n/2]$. By Proposition 3.2 and Theorem 4.1, (i) we have

$$\gamma^{2k}(\alpha)=(-1)^k A(k)(r\sigma)^k+\sum_{j=k+1}^{[n/2]}b_j(r\sigma)^j$$

for some coefficients b_j ($j=k+1, k+2,\dots, [n/2]$) (cf. (2.5)). Suppose that $\gamma^{2k}(\alpha)=0$. Multiplying $(r\sigma)^{[n/2]-k}$ on both sides of the equality, we obtain $A(k)$ $(r\sigma)^{[n/2]}=0$, and so $A(k)\equiv 0 \mod q$, by Theorem 4.1, (i). Therefore $p_k(\alpha)=0$ by Proposition 2.4.

In order to study the converse, assume that q is equal to the power $p^r(r>0)$ of an odd prime p (>1) and consider an element $\alpha=r(a(\eta-1))\in\widetilde{KO}(L^n(q))$, $a\in Z$. Then, by Proposition 3.2 and Theorem 4.1, (i),

$$\gamma^{2k}(\alpha)=\sum_{i=k}^{[n/2]}(-1)^i\binom{a}{i}\binom{i}{2i-2k}(r\sigma)^i\,.$$

Now, put $n=p^{r+1}-1$, $a=p^{r+1}$ and $k=(p^{r+1}-p)/2$. Then

$$\binom{a}{k}\not\equiv 0 \mod p^{r+1},\ \equiv 0 \mod p^r\,,$$

$$\binom{a}{i}\equiv 0 \mod p^{r+1} \qquad \text{for} \quad i=k+1,\ k+2,\dots,\ n/2\,.$$

Thus, by Theorem 4.1, (ii), we have $\gamma^{2k}(\alpha)\neq 0$. On the other hand, clearly, $p_i(\alpha)=0$ for any $i\geqq k$, by Proposition 2.4.

(ii) Let α be any real stable vector bundle over RP^n. According to Theorem 4.2, $\alpha=a\lambda$ for some $a\in Z$. Then

$$\gamma_t(\alpha)=(1+\lambda t)^a=\sum_{i=0}^{\infty}\binom{a}{i}\lambda^i t^i$$

by [3, §2]. Therefore, by Theorem 4.2, $\gamma^{2k}(\alpha)=-2^{2k-1}\binom{a}{2k}\lambda$, and hence the equality $\gamma^{2k}(\alpha)=0$ implies that

$$2^{2k-1}\binom{a}{2k}\equiv 0 \mod 2^{\phi(n)}\,.$$

If $4k<n+1$, then $2k-1<\phi(n)$, and so $\binom{a}{2k}\equiv 0 \mod 2$. Then $p_k(\alpha)=(-1)^k c_{2k}$ $(c\alpha)=\binom{a}{2k}x^{2k}=0$, where x is the generator of $H^2(RP^n;Z)=Z_2$. If $4k\geqq n+1$, it is obvious that $p_k(\alpha)=0$.

We obtain an example, for which the converse does not hold, by setting $n=2^r-1$, $a=2^r$ and $k=2^{r-3}$ $(r>3)$. q.e.d.

§5. Dold manifold $D(m, n)$

We recall the cohomology of the Dold manifold $D(m, n)$ according to A. Dold [4, Satz 1] and M. Fujii [5, Proposition (1.6)].

Let (c^i, d^j) be the $(i+2j)$-dimensional cohomology class of $D(m, n)$ which is dual to the homology class determined by the $(i+2j)$-cell (C^i, D^j) (cf. [4] or [5]). For the simplicity, we use the same notation for the integral class and its mod 2 reduction.

THEOREM 5.1. (M. Fujii) *$H^*(D(m, n); Z)$ is a direct sum of a free abelian group generated by elements (c^0, d^{2j}) and $(c^m, d^{2j+\varepsilon})$ ($\varepsilon=0$ for odd m, $\varepsilon=1$ for even m), and a torsion group generated by elements (c^{2i}, d^{2j}) and (c^{2i-1}, d^{2j+1}) of order 2, where $i=1, 2,\dots, [m/2]$, and $j=0, 1,\dots, [n/2]$.*

Let (c'^i, d'^j) be the corresponding cohomology class for $D(m', n')$ where $m'\leqq m$ and $n'\leqq n$. If h: $D(m', n')\rightarrow D(m, n)$ is the standard inclusion, then it holds that $h^*(c^i, d^j)=(c'^i, d'^j)$.

THEOREM 5.2. (A. Dold) *$H^*(D(m, n); Z_2)=Z_2[\boldsymbol{c}, \boldsymbol{d}]/(\boldsymbol{c}^{m+1}, \boldsymbol{d}^{n+1})$, where* $\boldsymbol{c}=(c^1, d^0)\in H^1(D(m, n); Z_2)$ and $\boldsymbol{d}=(c^0, d^1)\in H^2(D(m, n); Z_2)$.

Let $\boldsymbol{c}'$ and $\boldsymbol{d}'$ be the corresponding cohomology classes for $D(m', n')$ where $m'\leqq m$ and $n'\leqq n$. If h: $D(m', n')\rightarrow D(m, n)$ is the standard inclusion, then it holds that $h^*(\boldsymbol{c}^i\boldsymbol{d}^j)=\boldsymbol{c}'^i\boldsymbol{d}'^j$.

Let π: $D(m, n)\rightarrow RP^m$ be the natural projection. Then π is the projection of the fibre bundle with fibre CP^n. Let i: $CP^n\rightarrow D(m, n)$ be the inclusion of the fibre in the total spec. The following results are due to [12, Proposition (1.4)].

THEOREM 5.3. (J. J. Ucci) (i) *Let $\xi_1=\pi^!\xi$ be the real line bundle over $D(m, n)$ induced by π from the canonical line bundle ξ over RP^m. Then the total Stiefel-Whitney class $w=\sum_i w_i$ is given by $w(\xi_1)=1+\boldsymbol{c}$.*

(ii) *There exists a real 2-plane bundle μ_1 over $D(m, n)$ such that $i^!\mu_1=r\mu$ and $w(\mu_1)=1+\boldsymbol{c}+\boldsymbol{d}$, where $r\mu$ is the real restriction of the complex line bundle over CP^n.*

Let c denote the complexification and $C=\sum_i c_i$ denote the total Chern class.

LEMMA 5.4. (i) $C(c\xi_1) = 1+\boldsymbol{c}^2$ $(m\geqq 2)$,

(ii) $C(c\mu_1) = 1+\boldsymbol{c}^2-\boldsymbol{d}^2$ $(m\geqq 2,\ n\geqq 2)$,

where $\boldsymbol{c}^2=(c^2, d^0)\in H^2(D(m, n); Z)$ and $\boldsymbol{d}^2=(c^0, d^2)\in H^4(D(m, n); Z)$.

PROOF. As (i) is obtained immediately from Theorem 5.3, (i), we only give a proof of (ii). Notice that

$$w(rc\mu_1) = w(2\mu_1) = w(\mu_1)^2 = (1+\boldsymbol{c}+\boldsymbol{d})^2 = 1+\boldsymbol{c}^2+\boldsymbol{d}^2,$$

by (2.2) and Theorem 5.3, (ii). Hence the mod 2 reduction of $c_1(c\mu)$ is $\boldsymbol{c}^2$ and that of $c_2(c\mu_1)$ is $\boldsymbol{d}^2$. Since the mod 2 reduction $H^2(D(m, n); Z) \to H^2(D(m, n); Z_2)$ is isomorphic, we have $C(c\mu_1) = 1 + \boldsymbol{c}^2 + l\boldsymbol{d}^2$, where l is some odd integer. On the other hand, by (2.2) and Theorem 5.3, (ii), $i^*C(c\mu_1) = C(i^! c\mu_1) = C(ci^! \mu_1) = C(cr\mu) = C(\mu)C(\bar{\mu}) = (1+z)(1-z) = 1 - z^2$, where z is the generator of $H^2(CP^n; Z)$. Since $i^*\boldsymbol{d}^2 = z^2$, we have $l = -1$, as desired. q.e.d.

§6. Proof of Theorem 1.3

LEMMA 6.1. *Let m and n be positive integers such that*

$$[m/2] \equiv \binom{m}{i} \equiv 0 \mod 2 \quad \textit{for any } i \textit{ with } 0 < i \leqq [m/2], \textit{ and}$$

$$\binom{n + [n/2] + [m/2]}{n} \binom{[n/2] + [m/2]}{[m/2]} \not\equiv 0 \mod 2.$$

Put $k = [n/2] + [m/4]$. Then $p_k(-\tau_0) \neq 0$, where $\tau_0 = \tau - (m + 2n)$ is the stable class of the tangent bundle τ of $D(m, n)$.

PROOF. According to [5, Theorem (2.8)] or [12, Theorem (1.5)]

$$-\tau_0 = -m(\xi_1 - 1) - (n+1)(\mu_1 - 2).$$

Therefore, by Lemma 5.4,

$$C(-c\tau_0) = C(-mc\xi_1)C(-(n+1)c\mu_1) = (1 + \boldsymbol{c}^2)^{-m}(1 + \boldsymbol{c}^2 - \boldsymbol{d}^2)^{-n-1}.$$

Now, $(1 + \boldsymbol{c}^2)^m = 1$, by the assumption, since $\boldsymbol{c}^2$ is of order 2 and $(\boldsymbol{c}^2)^{[m/2]+1} = 0$. While,

$$(1 + \boldsymbol{c}^2 - \boldsymbol{d}^2)^{-n-1} = \sum_i (-1)^i \binom{n+i}{i} \sum_{j=0}^{i} \binom{i}{j} (\boldsymbol{c}^2)^j (-\boldsymbol{d}^2)^{i-j}.$$

The coefficient of the monomial $(\boldsymbol{c}^2)^{[m/2]}(\boldsymbol{d}^2)^{[n/2]}$ in this expansion is

$$(-1)^{[m/2]} \binom{n + [n/2] + [m/2]}{[n/2] + [m/2]} \binom{[n/2] + [m/2]}{[m/2]}$$

and this is odd by the assumption. Thus $p_k(-\tau_0) = (-1)^k c_{2k}(-c\tau_0) \neq 0$, as desired. q.e.d.

We need the following results on the structure of $\widetilde{KO}(D(m, n))$ (cf. [12, Theorem (2.8)] and [6, Theorems 5 and 6]).

THEOREM 6.2. (J. J. Ucci, M. Fujii and T. Yasui) *Set $\xi_1 - 1 = v$ and $\mu_1 - \xi_1 - 1 = y$. Then $\widetilde{KO}(D(m, n))$ contains a summand isomorphic to*

$$Z_{2^{\phi(m)}} + Z + \cdots + Z \quad ([n/2]\textit{-copies})$$

generated by v, y, $y^2,\dots, y^{[n/2]}$ *with the relations*: $v^2=-2v$, $v^{\phi(m)+1}=0$, $vy=0$, $y^{[n/2]+1+\varepsilon}=0$, *where* $\varepsilon=0$ *if* $n\not\equiv 1 \bmod 4$, $\varepsilon=1$ *if* $n\equiv 1 \bmod 4$.

LEMMA 6.3. *Let* m *and* n *be positive integers with* $2[n/2]+1\geqq\phi(m)$. *Then* $\gamma^i(-\tau_0)=0$ *for any* $i\geqq 2[n/2]+2+2\varepsilon$, *where* ε *is as in Theorem* 6.2.

PROOF. According to J. J. Ucci [12, p. 289]

$$\gamma^i(-\tau_0)=\pm 2^{i-1}\binom{m+n+i}{i}v+\sum_{j=[(i+1)/2]}^{i}\alpha_{ij}y^j,$$

where α_{ij} are non-zero integers. If $i\geqq 2[n/2]+2+2\varepsilon$, we see, by the assumption and Theorem 6.2, $2^{i-1}v=0$ and $y^j=0$ for any $j\geqq[(i+1)/2]$. Thus $\gamma^i(-\tau_0)=0$.

q. e. d.

PROOF OF THEOREM 1.3. Let $n=2^r$ and $m=2^s$ $(r>s>1)$, and put $k=n/2+m/4$. Then $p_k(-\tau_0)\neq 0$ by Lemma 6.1, and $\gamma^{2i}(-\tau_0)=0$ for any $i\geqq k$ by Lemma 6.3.

q. e. d.

§7. Immersions of lens spaces in complex projective spaces

The results of §2 and §3 can be used to study the problem of finding a condition that a map of $L^n(q)$ in some manifold is homotopic to a differentiable immersion. In [10], we have been concerned with immersions and embeddings of $L^n(q)$ in $L^m(q)$.

In this section we consider the immersions of $L^n(q)$ in CP^m. For a given integer d, a continuous map $f\colon L^n(q)\to CP^m$ is said to have degree d(written $\deg(f)$), if $f^*z=dx$ for the distinguished generators $z\in H^2(CP^m;Z)$ and $x\in H^2(L^n(q);Z)$. If $n<m$, the homotopy classes of maps of $L^n(q)$ in CP^m are in one-to-one correspondence with $H^2(L^n(q);Z)=Z_q$. Thus the homotopy class of a map $f\colon L^n(q)\to CP^m$, $n<m$, is determined by $\deg(f)\in Z_q$.

In a way similar to [10, (2.3)], we have

PROPOSITION 7.1. *Suppose* q *is odd*>1. *If* $m\geqq n+[n/2]+1$, *any map of* $L^n(q)$ *in* CP^m *is homotopic to an immersion.*

Let μ and η be the canonical complex line bundles over CP^m and $L^n(q)$ respectively. The following is evident.

(7.2) *Let* $f\colon L^n(q)\to CP^m$ *be a map with degree* d. *Then* $f^!\mu=\eta^d$.

THEOREM 7.3. *Suppose* q *is odd*>1. *Let* n *and* m *be integers such that* $m\leqq n+[n/2]$. *If a map* $f\colon L^n(q)\to CP^m$ *with degree* d *is homotopic to an immersion, then*

$$\sum_{i+j=l}(-1)^i\binom{n+i}{i}\binom{m+1}{j}(\Psi_R^d r\sigma)^j(r\sigma)^i = 0$$

for any $l \geqq m-n$.

PROOF. Let g be an immersion which is homotopic to f. Then g is of degree d. As g has the maximal rank $2n+1$, we must have $m-n>0$. Let ν be the normal bundle of g. Then $\nu+\tau(L^n(q))=g^!\tau(CP^m)$, where $\tau(M)$ denotes the tangent bundle of M. Since $\tau(L^n(q))+1=(n+1)r\eta$ by [7, (4.6)] and $g^!(\tau(CP^m)+2)=g^!((m+1)r\mu)=(m+1)rg^!\mu=(m+1)r\eta^d$ by (7.2), we obtain $\nu+1+(n+1)r\eta=(m+1)r\eta^d$. Let $\alpha=\nu-(2m-2n-1)$ be the stable class of ν. Then

$$\alpha = -(n+1)(r\eta-2)+(m+1)(r\eta^d-2)$$

and $g\text{-}\dim\alpha \leqq 2m-2n-1$. Taking account of the fact that $\gamma^i(\alpha)=0$ for $i>g\text{-}\dim\alpha$ [3, Proposition (2.3)], we find the result from Proposition 3.2. q.e.d.

COROLLARY 7.4. *Let p be an odd prime* >1. Set

$$m = n+\max\left\{l \leqq [n/2] \,\middle|\, \binom{n+l}{l} \not\equiv 0 \bmod p^{r+[(n-2l)/(p-1)]}\right\}.$$

If a map f: $L^n(p^r)\to CP^m$ *has degree* 0, *then* f *is not homotopic to an immersion.*

PROOF. This follows from Theorems 4.1 and 7.3. q.e.d.

Since the existence of an immersion $L^n(q)\to CP^m$ with degree 0 is equivalent to the existence of an immersion of $L^n(q)$ in Euclidean $2m$-space, Corollary 7.4 has already been obtained by T. Kawaguchi and M. Sugawara [8, Corollary 3.6].

COROLLARY 7.5. *Suppose q is odd* >1. *Let n and m be integers with* $m \leqq n+[n/2]$. *Then a map* f: $L^n(q) \to CP^m$ *with degree* ± 1 *is not homotopic to an immersion.*

PROOF. If f is homotopic to an immersion, we have $\binom{m-n}{l}(r\sigma)^l=0$ for any $l \geqq m-n$, since $\Psi_R^{\pm 1}$ is the identity (cf. [1, Theorem 5.1, (vii)]). Thus $(r\sigma)^{m-n}=0$, which contradicts to the fact that $(r\sigma)^i \neq 0$ for $0<i\leqq[n/2]$ (cf. Theorem 4.1, (i)). q.e.d.

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.

[2] J. F. Adams and G. Walker, *On complex Stiefel manifolds*, Proc. Camb. Phil. Soc. **61** (1965), 81–103.

[3] M. F. Atiyah, *Immersions and embeddings of manifolds*, Topology **1** (1962), 125–132.

[4] A. Dold, *Erzeugende der Thomschen Algebra* $\mathfrak{N}$, Math. Zeit. **65** (1956), 25–35.

[5] M. Fujii, K_U*-groups of Dold manifolds*, Osaka J. Math. **3** (1966), 49–64.

[6] M. Fujii and T. Yasui, K_O*-cohomologies of the Dold manifolds*, Math. J. Okayama Univ. **16** (1973), 55–84.

[7] T. Kambe, *The structure of* K_A*-rings of the lens space and their applications*, J. Math. Soc. Japan **18** (1966), 135–146.

[8] T. Kawaguchi and M. Sugawara, *K- and KO-rings of the lens space* $L^n(p^2)$ *for odd prime* p, Hiroshima Math. J. **1** (1971), 273–286.

[9] T. Kobayashi, *Non-immersion theorems for lens spaces*. II, J. Sci. Hiroshima Univ. Ser. A-I **32** (1968), 285–292.

[10] T. Kobayashi, *Immersions and embeddings of lens spaces*, Hiroshima Math. J. **2** (1972), 345–352.

[11] N. Mahammed, *A propos de la K-théorie des espaces lenticulaires*, C. R. Acad. Sci. Paris **271** (1970), 639–642.

[12] J. J. Ucci, *Immersions and embeddings of Dold manifolds*, Topology **4** (1965), 283–293.

Department of Mathematics,
Faculty of Science,
Hiroshima University