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A Characterization of Anosov Flows for Geodesic Flows

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Introduction

It is important and interesting to determine when the dynamical system is Anosov. J. N. Mather has already obtained a characterization of Anosov diffeomorphisms in [7], but the corresponding characterization of Anosov flows has not been established as yet.

In this paper, the author proposes a characterization of Anosov flows for a special flow, called a geodesic flow. For a geodesic flow, there is a canonical splitting $TSM = E \oplus \bar{S}$ (see Lemma 3) and when the geodesic flow is Anosov, one can obtain $E = E^s \oplus E^u$ (see Lemma 8), thus the question is whether we can obtain a splitting of E with some properties or not, so that one can obtain the result corresponding to [7].

In Section 1 the definition of geodesic flows is given and some lemmas are proved which provide useful information on the structure of geodesic flows. Extensive use is made of results on Riemannian geometry as developed by Anosov [2], Dombrowski [3] and Sasaki [8].

In Section 2 and 3 some lemmas are presented upon which the proof of the main theorem (Theorem 1) is based. All the tools used in these sections can be found in the theory of manifolds and in the theory of semigroups (see, e.g. [1], [4], [6]).

In Section 4 the main theorem is stated and proved.

§1. A treatment of geodesic flows

1.1 Riemannian connector K.

Let M be an n-dimensional complete connected Riemannian C^∞-manifold, TM_p be the tangent space at $p \in M$ and $TM = \bigcup_{p \in M} TM_p$ denote the tangent bundle on M with projection π. If $(x^1, .., x^n)$ is a local coordinate system at $p \in M$ and if we set $TM_p \ni v = \sum_{i=1}^n v^i \partial/\partial x^i$, then we can take $(x^1, .., x^n, v^1, ..., v^n)$ as a local coordinate system at $v \in TM_p$, so that we can induce a differentiable structure on TM.

In order to define a geodesic flow by some vector field on TM, we must consider the double tangent space $TTM \equiv T^2M$. Since the Riemannian manifold has the Riemannian connection ∇, we can interpret the double tangent space T^2M as the vector bundle over M as follows:

There is a mapping K: $T^2M \to TM$, called the Riemannian connector associated with the Riemannian connection ∇, such that the following diagram is commutative

$$\begin{array}{ccc} T^2M & \xrightarrow{K} & TM \\ \big\downarrow{\scriptstyle \pi_{TM}} & & \big\downarrow{\scriptstyle \pi_M} \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

and $\pi_{TM} \oplus \pi_* \oplus K$ maps T^2M to $TM \oplus TM \oplus TM$ isomorphically as vector bundles on M, where $\pi_M: TM \to M$ and $\pi_{TM}: T^2M \to TM$ are the projections for the tangent bundles on M and TM, respectively, and π_* is the differential of π_M. (For details, see [3].)

The following proposition due to P. Dombrowski [3] is important in our investigation.

Proposition 1. *For any triple X, Y, $Z \in TM_p$, there is a unique $A \in T^2M_Z$ such that $\pi_*A = X$ and $KA = Y$.*

Remark 1. By the local coordinate system, we can represent the mappings π_* and K as follows: If we set

$$TM_p \ni v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}, \quad T^2M_v \ni x = \sum_{i=1}^n \left(\xi^i \frac{\partial}{\partial x^i} + \xi^{n+i} \frac{\partial}{\partial v^i}\right),$$

then

$$\pi_* X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} \tag{1}$$

$$KX = \sum_{i=1}^n (\xi^{n+i} + \sum_{j,k=1}^n \Gamma^i_{jk} \xi^j v^k) \frac{\partial}{\partial x^i}, \tag{2}$$

where Γ^i_{jk} are Christoffel's symbols.

By using the mappings π_* and K, we can induce a Riemannian metric on TM, that is,

$$<X, Y>_{TM} = <\pi_*X, \pi_*Y>_M + <KX, KY>_M, \tag{3}$$

where X, $Y \in T^2M_v$ and $<\ ,\ >_M$ is the Riemannian metric on M. (We will not distinguish $<\ ,\ >_M$ and $<\ ,\ >_{TM}$, unless they are confused.)

1.2 Definition of geodesic flows.

Definition 1. *A geodesic flow* is the flow generated by the following vector field S:

(4) $$\begin{cases} \pi_* S_v = v\,, \\ K S_v = 0\,, \end{cases}$$

where $v \in TM$ and $S_v \in T^2 M_v$.

REMARK 2. (i) S is well defined because of Proposition 1 and S is a so-called geodesic spray (see [6]).

(ii) Let ω be the 2-form on TM defined as follows:

(5) $$\omega(X, Y) = <\pi_* X, KY> - <KX, \pi_* Y>,$$

where X, $Y \in \Gamma(T^2 M)$ (the set of vector fields on TM). Then (TM, ω) is a symplectic manifold and for every $Y \in \Gamma(T^2 M)$, S satisfies the equation

(6) $$\omega(S, Y) = \frac{1}{2} dH(Y)\,,$$

where $H(v) = <v, v>$, $v \in TM$. This shows that a geodesic flow is a global Hamiltonian flow. (For the definition of global Hamiltonian flows, see [1].)

Let us restrict S to the sphere bundle SM on M which consists of unit tangent vectors. We can do this because of the following lemma.

LEMMA 1. *The vector field X on TM is tangent to SM, if and only if*

(7) $$<v, KX_v> = 0.$$

PROOF. X is tangent to SM if and only if $X(H) = dH(X) = 0$, where H is the same as in Remark 2. By using (5) and (6) we can see that for all $v \in SM$,

$$\begin{aligned} X_v(H) = dH(X_v) = 2\omega(S_v, X_v) &= 2<\pi_* S_v, KX_v> - 2<KS_v, \pi_* X_v> \\ &= 2<v, KX_v>. \end{aligned}$$

This completes the proof.

By this lemma we can obtain the following corollary which will be needed later.

COROLEARY 1.

$$\Gamma(TSM) = \{X \in \Gamma(T^2 M);\ <v, KX_v> = 0, \quad \text{for all} \quad v \in SM\}\,,$$

where TSM is the tangent bundle of SM.

From now on, by a geodesic flow on M we mean the flow generated by the vector field S restricted to SM. Set $\phi_t = \exp tS$. Then ϕ_t is a one-parameter groups of diffeomorphisms on SM and the parameter t is equal to the arc-length, since $\|S\|=1$.

REMARK 3. Since a global Hamiltonian flow preserves the volume element $\Lambda^n\omega$, we can easily see that a geodesic flow preserves some measure by the theorem of Hamilton-Jacobi. (see [1], Theorem 16.27.)

1.3 Some lemmas for geodesic flows.

Let ϕ_t be a geodesic flow defined above. Then the differential $T\phi_t$ of ϕ_t is canonically defined. We will investigate $T\phi_t$ as follows.

Let (x^i, v^i) be a local coordinate system at $v\in TM_x$. Then from (1), (2) and (4), the geodesic spray S is locally represented as follows:

$$\sum_{i=1}^{n}\left(v^i\frac{\partial}{\partial x^i}-\sum_{j=1,k}^{n}\Gamma^i_{jk}v^iv^k\frac{\partial}{\partial v^i}\right). \tag{8}$$

Therefore the mapping ϕ_t: $SM\to SM$ is locally defined by the following differential equations

$$\begin{cases} \dfrac{dx^i}{dt}=v^i, & (9)\\[2ex] \dfrac{dv^i}{dt}=-\displaystyle\sum_{j,k=1}^{u}\Gamma^i_{jk}v^jv^k. & (10)\end{cases}$$

Let $(x^i, v^i, \xi^i, \xi^{n+i})$ be a local coordinate system at $\xi\in TSM_v$. Then the mapping $T\phi_t$: $TSM\to TSM$ is locally defined by the following equations

$$\begin{cases} \dfrac{d\xi^i}{dt}=\xi^{n+i}, & (11)\\[2ex] \dfrac{d\xi^{n+i}}{dt}=-\displaystyle\sum_{j,k,l=1}^{n}\frac{\partial\Gamma^i_{jk}}{\partial x^l}\xi^lv^jv^k-2\sum_{j,k=1}^{n}\Gamma^i_{jk}v^j\xi^{n+k}. & (12)\end{cases}$$

Now we can obtain the differential equations for $T\phi_t$ which are independent of the local coordinate system as follows:

LEMMA 2. *Let $\xi\in TSM_v$ be a tangent vector at $v\in SM$. Then $T\phi_t\xi$ satisfies the following equations*

$$\begin{cases} \dfrac{D}{dt}\pi_*T\phi_t\xi=KT\phi_t\xi, & (13)\\[2ex] \dfrac{D}{dt}KT\phi_t\xi+R(\phi_tv,\ \pi_*T\phi_t\xi)\phi_tv=0, & (14)\end{cases}$$

where $\dfrac{D}{dt}$ is the covariant derivative along the geodesic curve γ on M with the *initial conditions* $\gamma(0)=x$, $\dot{\gamma}(0)=v$ *and* $\dot{\gamma}(t)=\phi_t v$ *and* R *is the curvature tensor defined as: for* $X, Y, Z \in TM_x$

$$R(X, Y)Z=-\nabla_X \nabla_Y Z+\nabla_Y \nabla_X Z+\nabla_{[X,Y]} Z . \tag{15}$$

PROOF. Since the curve $t \to x(t)$ is equal to γ defined before because of (9) and (10), $\pi_* T\phi_t \xi$ and $KT\phi_t \xi$ are vector fields along γ. Thus the covariant derivatives of $\pi_* T\phi_t \xi$ and $KT\phi_t \xi$ along γ have a meaning. For brevity we represent $\pi\phi_t v$, $\phi_t v$ and $T\phi_t \xi$ as $(x^i(t)), (v^i(t))$ and $(\xi^i(t), \xi^{n+i}(t))$ respectively. Then from (1) and (2), $\pi_* T\phi_t \xi$ and $KT\phi_t \xi$ can be represented as $(\xi^i(t))$ and $(\xi^{n+i}(t)+\Gamma^i_{jk}(x(t)) v^j(t)\xi^k(t))$ by using Einstein's rule. Furthermore, so as to avoid complication we do not write t for $x^i(t)$, $v^i(t)$, $\xi^i(t)$ and $\xi^{n+i}(t)$. It follows from (9), (10), (11) and (12) that

$$\begin{aligned} \frac{D}{dt}\pi_*\xi &= \frac{d\xi^i}{dt}+\Gamma^i_{jk}\frac{dx^i}{dt}\xi^k \\ &= \xi^{n+i}+\Gamma^i_{jk}v^j\xi^k = K\xi . \end{aligned}$$

Further, by using classical relations (see [5]; note that our definition of R has the opposite sign),

$$\begin{aligned} \frac{D}{dt}K\xi &= \frac{d(\xi^{n+i}+\Gamma^i_{jk}v^j\xi^k)}{dt}+\Gamma^i_{jl}\frac{dx^j}{dt}(\xi^{n+l}+\Gamma^l_{km}v^k\xi^m) \\ &= -\frac{\partial \Gamma^i_{jk}}{\partial x^l}\xi^l v^j v^k - 2\Gamma^i_{jk}v^j\xi^{n+k}+\frac{\partial \Gamma^i_{jk}}{\partial x^l}v^l v^j \xi^k \\ &\quad -\Gamma^i_{jk}\Gamma^j_{lm}\xi^k v^l v^m+\Gamma^i_{jk}v^j\xi^{n+k} \\ &\quad +\Gamma^i_{jl}v^j\xi^{n+l}+\Gamma^i_{jl}\Gamma^l_{km}v^j v^k \xi^m \\ &= -\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l}-\frac{\partial \Gamma^i_{jl}}{\partial x^k}+\Gamma^i_{ml}\Gamma^m_{kj}-\Gamma^i_{mk}\Gamma^m_{jl}\right)v^j\xi^l v^k \\ &= -R^i_{jlk}v^j\xi^l v^k = -R(v, \pi_*\xi)v , \end{aligned}$$

where R^i_{jlk} is defined in [5] (Proposition 7.6). This completes the proof.

For every $v \in SM$, we define E_v as follows:

$$E_v=\{X \in T^2M_v;\ \langle \pi_* X, v\rangle = \langle KX, v\rangle = 0\} .$$

Then we obtain the following.

LEMMA 3. *The following statements are true.*

(i) $E=\bigcup_{v \in SM} E_v$ *is a subbundle of the tangent bundle* TSM *on* SM.

(ii) $TSM=E\oplus\bar{S}$ *is a continuous bundle splitting, where* $\oplus$ *means an* *orthogonal sum and* $\bar{S}$ *is the* 1*-dimensional vector bundle generated by the vector field* S.

(iii) E *is* $T\phi_t$*-invariant, namely* $T\phi_t E_v = E_{\phi_t v}$.

PROOF. The statement (i) is evident and (ii) follows directly from the following fact, together with Corollary 1: for every $X \in E_v$,

$$<X,\ S_v> = <\pi_* X,\ \pi_* S_v> + <KX,\ KS_v> = <\pi_* X,\ v> = 0.$$

To prove (iii) let us consider $X \in E_v$ and $T\phi_t X$. Then from Lemma 2, we have

$$\begin{aligned} \frac{d}{dt} <\pi_* T\phi_t X, \phi_t v> &= <\frac{D}{dt}\pi_* T\phi_t X,\ \phi_t v> + <\pi_* T\phi_t X,\ \frac{D}{dt}\phi_t v> \\ &= <K\pi_* T\phi_t X,\ \phi_t v> = 0\,, \\ \frac{d}{dt} <KT\phi_t X,\ \phi_t v> &= <\frac{D}{dt} KT\phi_t X,\ \phi_t v> + <KT\phi_t X,\ \frac{D}{dt}\phi_t v> \\ &= <-R(\phi_t v,\ \pi_* T\phi_t X)\phi_t v,\ \phi_t v> = 0\,. \end{aligned}$$

The first equation follows from the facts that the Riemannian connection is compatible with the Riemannian metric and $\phi_t v$ is the velocity vector of the arc-length parametrized geodesic curve, with $T\phi_t X \in TSM_{\phi_t v}$. The second equation follows from the skew symmetric property of the curvature tensor (i.e. $<R(X,\ Y)Z,\ W> + <R(X,\ Y)W,\ Z> = 0$).

This completes the proof.

Let us define the mapping $D\colon E_v \to E_v$ as follows: for $X \in E_v$, $v \in SM$,

$$\left\{ \begin{array}{l} \pi_* DX = KX, \\ KDX = -R(v,\ \pi_* X)v. \end{array} \right. \tag{16}$$

By Lemma 2 and Lemma 3, we can easily show that D is well defined and is a bundle mapping on E.

Now we get the following lemma.

LEMMA 4. *For every* $X \in E_v$ *and* $T\phi_t X$,

$$\frac{d}{dt}\|T\phi_t X\|^2 = 2<DT\phi_t X,\ T\phi_t X>, \tag{17}$$

where $\|X\|^2 = <X,\ X>$.

PROOF. Using Lemma 2, we obtain

$$\frac{d}{dt}<T\phi_t X,\ T\phi_t X> = \frac{d}{dt}<\pi_* T\phi_t X,\ \pi_* T\phi_t X> + \frac{d}{dt}<KT\phi_t K,\ KT\phi_t X>$$

$$=2<\frac{D}{dt}\pi_* T\phi_t X,\ \pi_* T\phi_t X> + 2<\frac{D}{dt}KT\phi_t X,\ KT\phi_t X>$$

$$=2<KT\phi_t X,\ \pi_* T\phi_t X> + 2<-R(\phi_t v,\ \pi_* T\phi_t X)\phi_t v,\ KT\phi_t X>$$

$$=2<\pi_* DT\phi_t X,\ \pi_* T\phi_t X> + 2<KDT\phi_t X,\ KT\phi_t X>$$

$$=2<DT\phi_t X,\ T\phi_t X>.$$

This completes the proof.

§2. Some lemmas from the theory of differentiable manifolds

Let M be a compact connected C^∞-manifold with some Riemannian metric $<\ ,\ >$ and ϕ_t be a one-parameter group of diffeomorphisms on M, so-called "flow", defined by some vector field X on M.

DEFINITION 2. A flow ϕ_t is called an *Anosov flow*, if there exists a continuous splitting of the tangent bundle $TM = E^s \oplus E^u \oplus \bar{X}$ such that:

(i) E^s and E^u are $T\phi_t$-invariant vector bundles of dimension larger than one;

(ii) there are positive constants c and ω such that for $Y \in E_x^s$, $\|T\phi_t Y\| \leqq ce^{-\omega t}\|Y\|$ in $t \geqq 0$, and for $Y \in E_x^u$, $\|T\phi_{-t} Y\| \leqq ce^{-\omega t}\|Y\|$ in $t \geqq 0$.

We will call E^s and E^u the stable and unstable bundles respectively.

Let F be a $T\phi_t$-invariant vector subbundle of TM and $\Gamma^0(F)$ be the set of continuous sections of F. For $\xi \in \Gamma^0(F)$, set $\|\xi\| = \sup_{x \in M}\|\xi_x\|$, where $\|Y\|^2 = <Y, Y>$ and set $\phi_t^\# \xi = T\phi_t \circ \xi \circ \phi_{-t}$. Then $\Gamma^0(F)$ is a normed space and $\phi_t^\#$ is a one-parameter group of linear operators on $\Gamma^0(F)$.

We can easily check the following properties of $\Gamma^0(F)$ and $\phi_t^\#$.

PROPOSITION 2. *The following statements are true.*

(i) *$\Gamma^0(F)$ is a Banach space over the reals.*

(ii) *$\phi_t^\#: \Gamma^0(F) \to \Gamma^0(F)$ is a toplinear automorphism on $\Gamma^0(F)$ for any $t \in R$.*

(iii) *ϕ_0 is the identity operator on $\Gamma^0(F)$.*

(iv) *For all $s, t \in R$, $\phi_t^\# \circ \phi_s^\# = \phi_{t+s}^\#$.*

(v) *$\phi_t^\#$ is strongly continuous in t, that is, $\lim_{t \to 0}\|\phi_t^\# \xi - \xi\| = 0$ for all $\xi \in \Gamma^0(F)$.*

PROOF. Since (i), (ii), (iii) and (iv) are obvious, we will only prove (v). Choose a finite cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of M by charts and choose $\gamma > 0$ so small that for $d(x, y) < \gamma$ there is a chart (α, U) of the cover with $x, y \in U$. Then we can consider that ϕ_t maps U into U for $|t| < \gamma$.

Take a chart (α, U), where U is an open subset of M and α is a C^∞-diffeomorphism mapping a neighborhood of $\bar{U}$ onto an open subset of $\mathscr{R}^n$, and $\xi \in \Gamma^0(F)$. Then by identifying ϕ_t with the induced mapping on the image of α, we have

$$\begin{aligned}
\|(D\phi_t)_{\phi_{-t}x}\xi_{\phi_{-t}x}-\xi_x\| &\leqq \|(D\phi_t)_{\phi_{-t}x}\xi_{\phi_{-t}x}-(D\phi_t)_{\phi_{-t}x}\xi_x\| \\
&\quad + \|(D\phi_t)_{\phi_{-t}x}\xi_x-(D\phi_t)_x\xi_x\| \\
&\quad + \|(D\phi_t)_x\xi_x-\xi_x\| \\
&\leqq \|(D\phi_t)_{\phi_{-t}x}\| \ \|\xi_{\phi_{-t}x}-\xi_x\| \\
&\quad + \|(D\phi_t)_{\phi_{-t}x}-(D\phi_t)_x\| \ \|\xi\| + \|(D\phi_t)_x-I\| \ \|\xi\| \\
&\leqq C_1\|\xi_{\phi_{-t}x}-\xi_x\| + C_2\|(D\phi_t)_{\phi_{-t}x}-(D\phi_t)_x\| \\
&\quad + C_3\|(D\phi_t)_x-I\|,
\end{aligned}$$

where D is the derivation and C_1, C_2 and C_3 are constants independent of t.

For arbitrary $\varepsilon>0$, when $|t|$ is sufficiently small, the first term is smaller than ε, because ξ_x is equicontinuous in $x\in M$. For the second and third terms, by taking the Taylor expansion of first order with respect to t, we have

$$(D\phi_t)_x = I+(DX)_{x_1}t,$$

$$(D\phi_t)_{\phi_{-t}x} = I-(DX)_{x_2}\cdot X_{x_2}t,$$

for some x_1 and x_2, where X is the vector field which defines ϕ_t. Therefore

$$\|(D\phi_t)_{\phi_{-t}x}-(D\phi_t)_x\| \leqq C_4|t|,$$

$$\|(D\phi_t)_x-I\| \leqq C_5|t|.$$

This implies that $\|\phi_t^\#\xi-\xi\|$ converges to zero as t tends to zero, because $\|\phi_t^\#\xi-\xi\|$ is equivalent to

$$\sup_{\lambda\in\Lambda,\, x\in\bar{U}_\lambda} \|(D\phi_t)_{\phi_{-t}x}\xi_{\phi_{-t}x}-\xi_x\|.$$

This completes the proof.
Now let us obtain the relation between $T\phi_t$ and $\phi_t^\#$.

LEMMA 5. *The following statements are equivalent.*

(i) *There are positive constants c and ω such that for every $x\in M$ and every $X\in F_x$,*

$$\|T\phi_t X\| \leqq ce^{-\omega t}\|X\| \quad in \quad t\geqq 0.$$

(ii) *There are positive constants c and ω such that for every $\xi \in \Gamma^0(F)$,*

$$\|\phi_t^{\#}\xi\| \leqq ce^{-\omega t}\|\xi\| \quad in \quad t \geqq 0$$

(namely $\|\phi_t^{\#}\| \leqq ce^{-\omega t}$).

PROOF. For $x \in M$ and $\xi \in \Gamma^0(F)$

$$\|T\phi_t \xi_{\phi_{-t}x}\| \leqq ce^{-\omega t}\|\xi_{\phi_{-t}x}\| \leqq ce^{-\omega t}\|\xi\|,$$

and therefore,

$$\|\phi_t^{\#}\xi\| = \sup_{x \in M}\|T\phi_t \xi_{\phi_{-t}x}\| \leqq ce^{-\omega t}\|\xi\|.$$

Thus (i) implies (ii).

Conversely, for $X \in F_x$, we can take $\xi \in \Gamma^0(F)$ such that $\xi_x = X$ and $\|\xi\| = \|X\|$. Then we have

$$\|T\phi_t X\| = \|T\phi_t \xi_x\| = \|(\phi_t^{\#}\xi)_{\phi_t x}\| \leqq \|\phi_t^{\#}\xi\|$$
$$\leqq ce^{-\omega t}\|\xi\| = ce^{-\omega}{}_t\|X\|.$$

Thus (ii) implies (i), which completes the proof.

Here we will state a result of R. G. Swan [9] which will be needed later.

PROPOSITION 3. *If M is compact and E, E_1 and E_2 are vector bundles on M, then the correspondence which sends a vector bundle splitting $E = E_1 \oplus E_2$ to the corresponding splitting $\Gamma^0(E) = \Gamma^0(E_1) \oplus \Gamma^0(E_2)$ of $C^0(M)$-module is bijective, where $C^0(M)$ is the ring of all continuous functions on M.*

§3. Some lemmas from the general spectral theory and the theory of semi-groups

Let $\mathfrak{X}$ be a Banach space over the complex numbers $\mathscr{C}$ and A be a bounded linear operator on $\mathfrak{X}$.

DEFINITION 3. The *spectrum* of the bounded linear operator A is the set of complex numbers λ such that $A - \lambda I$ is not a toplinear automorphism on $\mathfrak{X}$. We denote this set by $\sigma(A)$. The set $\rho(A) = \mathscr{C} - \sigma(A)$ is called the resolvent set of A and $R(\lambda, A) = (A - \lambda I)^{-1}$ is called the resolvent of A for $\lambda \in \rho(A)$.

$\sigma(A)$, $\rho(A)$ and $R(\lambda, A)$ have the following properties (see [5], VII 3.4).

PROPOSITION 4. *If $\mathfrak{X} \neq \{0\}$, then the following statements are true.*

(i) *$\sigma(A)$ is a closed and bounded non-void subset of $\mathscr{C}$.*

(ii) $\sup|\sigma(A)| = \lim_{n\to\infty} n\sqrt{\|A\|^n} \leqq \|A\|$.

(iii) *$R(\lambda, A)$ is an operator valued analytic function on $\rho(A)$.*

Let T_t be a strongly continuous semi-group of bounded linear operators on $\mathfrak{X}$. Then we have the following lemma.

LEMMA 6. *If $\|T_n\| \leqq c_1 e^{-n\omega}$ for all positive integers n, where c_1 and ω are some positive constants, then there is a positive constant c such that $\|T_t\| \leqq ce^{-\omega t}$ for all $t \geqq 0$.*

PROOF. Since T_t is strongly continuous, by Lemma VIII 1.3 in [4], we have

$$\sup_{t\in[0,1]} \|T_t\| \leqq c_2,$$

where c_2 is some constant.

Set $c = c_1 c_2 e^{\omega}$. Then $\|T_t\| \leqq ce^{-\omega t}$ for all $t \geqq 0$, which completes the proof.

When $\rho(T_1) \supset \{\lambda \in \mathscr{C}: |\lambda| = 1\}$, if we set

$$P = -\frac{1}{2\pi i}\int_{|\lambda|=1} R(\lambda, T_1)d\lambda,$$

then P is a bounded projection operator on $\mathfrak{X}$ because $\{\lambda \in \sigma(T_1); |\lambda| < 1\}$ is a spectral set (see [4], p. 573), hence $P\mathfrak{X}$ is a closed subspace of $\mathfrak{X}$.

Now we obtain an important lemma.

LEMMA 7. *$P\mathfrak{X}$ is T_t-invariant for all $t \geqq 0$.*

PROOF. If $\rho(T_1) \supset \{\lambda \in \mathscr{C}; |\lambda| = 1\}$, then $\rho(T_{1/n}) \supset \{\lambda \in \mathscr{C}; |\lambda| = 1\}$ for every positive integer n, because it follows that $\sigma(T_1) = \sigma(T_{1/n})^n$ from the spectral mapping theorem (see [4], p. 569).

Since $T_{1/n}$ is a bounded operator, we can choose three constants $0 < r_1 < 1 < r_2 < r_3$ such that $\rho(T_{1/n}) \supset \{\lambda \in \mathscr{C}; r_1 \leqq |\lambda| \leqq r_2\}$ and $\sigma(T_{1/n}) \subset \{\lambda \in \mathscr{C}; |\lambda| < r_3\}$. By Dunford's integral (see [4], p. 568), we have

$$R(\lambda, T_1) = -\frac{1}{2\pi i}\int_{|\alpha|=r_1} \frac{R(\alpha, T_{1/n})}{\alpha^n - \lambda}d\alpha + \frac{1}{2\pi i}\int_{|\alpha|=r_2} \frac{R(\alpha, T_{1/n})}{\alpha^n - \lambda}d\alpha$$
$$-\frac{1}{2\pi i}\int_{|\alpha|=r_3} \frac{R(\alpha, T_{1/n})}{\alpha^n - \lambda}d\alpha,$$

for $|\lambda| = 1$. Therefore, by Proposition 4 (iii), we have

$$P = -\frac{1}{2\pi i}\int_{|\lambda|=1} R(\lambda, T_1)d\lambda$$
$$= -\left(\frac{1}{2\pi i}\right)^2 \int_{|\lambda|=1}\left\{-\int_{|\alpha|=r_1} + \int_{|\alpha|=r_2} - \int_{|\alpha|=r_3}\right\}\frac{R(\alpha, T_{1/n})}{\alpha^n - \lambda}d\lambda$$

$$= -\frac{1}{2\pi i}\int_{|\alpha|=r_1}\left\{\frac{1}{2\pi i}\int_{|\lambda|=1}\frac{d\lambda}{\lambda-\alpha^n}\right\}R(\alpha,\ T_{1/n})\,da$$

$$= -\frac{1}{2\pi i}\int_{|\alpha|=r_1}R(\alpha,\ T_{1/n})\,d\alpha = -\frac{1}{2\pi i}\int_{|\alpha|=1}R(\alpha,\ T_{1/n})\,d\alpha\,.$$

This shows that $T_{1/n}P = PT_{1/n}$ for every positive integer n, because $T_{1/n}R(\alpha, T_{1/n}) = R(\alpha,\ T_{1/n})T_{1/n}$.

Thus by using the fact that T_t is strongly continuous, we have shown that $T_tP = PT_t$ for all $t \geqq 0$. This means that $P\mathfrak{X}$ is T_t-invariant for all $t \geqq 0$.

Next we will consider the case where $\mathfrak{X}$ is a Banach space over the reals $\mathscr{R}$ in order to apply the general spectral theory to $\Gamma^0(F)$.

Let A be a bounded linear operator on $\mathfrak{X}$ and set

$$\tilde{\mathfrak{X}} = \{\xi_1 + i\xi_2;\ \xi_1,\ \xi_2 \in \mathfrak{X}\},$$

$$\tilde{A}\xi = A\xi_1 + iA\xi_2 \quad \text{for} \quad \xi = \xi_1 + i\xi_2 \in \tilde{\mathfrak{X}}.$$

Then $\tilde{\mathfrak{X}}$ is a Banach space over $\mathscr{C}$ with norm $\|\xi\|^2 = \|\xi_1\|^2 + \|\xi_2\|^2$ and $\tilde{A}$ is a $\mathscr{C}$-linear bounded operator on $\tilde{\mathfrak{X}}$. ($\tilde{\mathfrak{X}}$ and $\tilde{A}$ are the so-called complexifications of $\mathfrak{X}$ and A, respectively.)

§4. A characterization of Anosov flows for geodesic flows

Let M be an n-dimensional compact connected Riemannian C^∞-manifold and let ϕ_t be the geodesic flow on M. Recall that ϕ_t is a one-parameter diffeomorphisms on SM. Let E be the subbundle of the tangent bundle TSM defined in §1 (1.3), and consider the Banach space $\Gamma^0(E)$ defined in §2. From now on, we will write $\Gamma^0(E)$ as $\mathfrak{X}$ for brevity.

Since E is $T\phi_t$-invariant, $\phi_t^\#$ operates on $\mathfrak{X}$ and $\phi_t^\#$ is a strongly continuous semi-group of bounded operators in both sides because of Proposition 2.

The following lemma is brief, but fundamental in our consideration.

LEMMA 8. *If the geodesic flow ϕ_t is Anosov and if E^s and E_u are the stable and unstable bundles, respectively, then $E = E^s \oplus E^u$.*

PROOF. It follows from Lemma 3 and Definition 2 that:

(a) $TSM = E^s \oplus E^u \oplus \bar{S} = E \oplus \bar{S}$ and especially $E \oplus \bar{S}$ is an orthogonal sum.

(b) E, $E^s \oplus E^u$ and $\bar{S}$ are $T\phi_t$-invariant.

(c) $\|T\phi_tX\| = \|X\|$ for $X \in \bar{S}$.

From (a), it is sufficient to show that $E^s \subset E$ and $E^u \subset E$. If $E^s \not\subset E$, then there is a non-zero vector $X \in E^s - E$. We can take the decomposition $X = X_1 + X_2$ where $X_1 \in E$, $X_2 \in \bar{S}$ and $X_2 \neq 0$. By (a) and (b), we have

$$\|T\phi_t X\|^2=\|T\phi_t X_1\|^2+\|T\phi_t X_2\|^2.$$

However, from the definition of E^s, the left hand side converges to zero as t tends to $+\infty$ while, in view of (c), the right hand side does not converge to zero as t tends to $+\infty$. This contradiction shows that $E^s\subset E$. In the same way, only replacing t by $-t$, we can show that $E^u\subset E$, which completes the proof.

Now let us consider the complexifications $\tilde{\mathfrak{X}}$, $\widetilde{\phi_t^\#}$ of $\mathfrak{X}$, $\phi_t^\#$ respectively as defined in § 3. Then we can state the main theorem in this paper.

THEOREM 1. *For the geodesic flow ϕ_t, the following statements are equivalent.*

(i) *$\phi_1^\#-I$ is a toplinear automorphism on $\mathfrak{X}$.*

(ii) *$1\notin\sigma(\widetilde{\phi_1^\#})$.*

(iii) *$\lambda\notin\sigma(\widetilde{\phi_1^\#})$ if $|\lambda|=1$.*

(iv) *ϕ_t is an Anosov flow.*

(v) *There exist a direct sum splitting $TSM=E^s\oplus E^u\oplus\bar{S}$ such that E^s and E^u are preserved by $T\phi_t$, a Riemannian metric $\|\ \|$ on SM, and a positive constant ω such that $\|T\phi_t X\|\leqq e^{-\omega t}\|X\|$ for all $X\in E^s$, $t\geqq 0$ and $\|T\phi_{-t}X\|\leqq e^{-\omega t}\|X\|$ for all $X\in E^u$, $t\geqq 0$.*

PROOF. The proof goes (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i).

1. (i)$\Leftrightarrow$(ii) is evident from the definition of the complexification.

2. (ii)$\Rightarrow$(iii) is a direct consequence of the following proposition which is an adaptation of a result of J. Mather [7].

PROPOSITION. *If $1\notin\sigma(\widetilde{\phi_1^\#})$, then $\mu\in\sigma(\widetilde{\phi_1^\#})$ and $|\lambda|=1$ imply that $\lambda\mu\in\sigma(\widetilde{\phi_1^\#})$.*

3. We will show that (iii)$\Rightarrow$(iv). In this step we denote $\widetilde{\phi_1^\#}$ by $\tilde{A}$ for brevity.

If $\rho(\tilde{A})\supset\{\lambda\in\mathscr{C};\ |\lambda|=1\}$, then because of Lemma 7 there is a projection operator P such that $P\tilde{\mathfrak{X}}$ is $\widetilde{\phi_t^\#}$-invariant for all t. Hence $Q=I-P$ is also a projection operator and $Q\tilde{\mathfrak{X}}$ is $\widetilde{\phi_t^\#}$-invariant for all t as well.

Set $\sigma_+=\{\lambda\in\sigma(\tilde{A});\ |\lambda|>1\}$ and $\sigma_-=\{\lambda\in\sigma(\tilde{A});\ |\lambda|<1\}$. Then $\sigma_\pm$ have the following properties:

(a) $\sigma_\pm$ are spectral sets, namely open and closed sets in $\sigma(\tilde{A})$;

(b) $\sigma_+\cup\sigma_-=\sigma(\tilde{A})$ (disjoint union);

(c) $\sigma_\pm$ are non-void.

(a) and (b) are evident from the fact that $\sigma(\tilde{A})$ is a closed bounded set and $\sigma(\tilde{A})\cap\{\lambda\in\mathscr{C};\ |\lambda|=1\}=\phi$. (c) will be proved later by Lemma 9.

Applying the general spectral theory (see [4], VII 3.20, VII 3.21), we get the following statements:

(d) If we denote the restricted poerators of $\tilde{A}$ on $P\tilde{\mathfrak{X}}$ and $Q\tilde{\mathfrak{X}}$ by $\tilde{A}_-$ and $\tilde{A}_+$ respectively, then $\sigma(\tilde{A}_\pm)=\sigma_\pm$;

(e) $\tilde{\mathfrak{X}}=P\tilde{\mathfrak{X}}\oplus Q\tilde{\mathfrak{X}}$.

Set $r_-=\sup\{|\lambda|;\ \sigma(\tilde{A})\ni\lambda,\ |\lambda|<1\}$ and $r_+=\sup\{|\lambda^{-1}|;\ \sigma(\tilde{A})\ni\lambda,\ |\lambda|>1\}$. Then it follows from (d) and Proposition 4 (ii) that

$$\text{(f)}\quad \begin{cases} \lim_{n\to\infty} \sqrt[n]{\|\tilde{A}_-^n\|}=r_-\,, \\ \lim_{n\to\infty} \sqrt[n]{\|\tilde{A}_+^{-n}\|}=r_+\,; \end{cases}$$

$$\text{(g)}\quad \begin{cases} P\tilde{\mathfrak{X}}=\{\xi\in\tilde{\mathfrak{X}};\ \limsup_{n\to+\infty} \sqrt[n]{\|\tilde{A}^n\xi\|}\leqq r_-\}\,, \\ Q\tilde{\mathfrak{X}}=\{\xi\in\tilde{\mathfrak{X}};\ \limsup_{n\to+\infty} \sqrt[n]{\|\tilde{A}^{-n}\xi\|}\leqq r_+\}\,. \end{cases}$$

(f) implies that there are positive constants c_1 and ω such that $\|\tilde{A}_-^n\|\leqq c_1e^{-n\omega}$ and $\|\tilde{A}_+^{-n}\|\leqq c_1e^{-n\omega}$ for all n. Hence by Lemma 6, there are positive constants c and ω such that

$$\text{(h)}\quad \begin{cases} \|\widetilde{\phi_t^{\#}}\,|\,P\tilde{\mathfrak{X}}\|\leqq \mathrm{c}e^{-\omega t} & \text{for}\quad t\geqq 0\,, \\ \|\widetilde{\phi_{-t}^{\#}}\,|\,Q\tilde{\mathfrak{X}}\|\leqq \mathrm{c}e^{-\omega t} & \text{for}\quad t\geqq 0\,. \end{cases}$$

Since $\phi_t^{\#}$ commutes with conjugation, there are closed subspaces $\mathfrak{X}_\pm$ of $\mathfrak{X}$ such that the complexifications of $\mathfrak{X}_-$ and $\mathfrak{X}_+$ are $P\tilde{\mathfrak{X}}$ and $Q\tilde{\mathfrak{X}}$ respectively, $\mathfrak{X}_\pm$ are $\phi_t^{\#}$-invariant and $\mathfrak{X}=\mathfrak{X}_-\oplus\mathfrak{X}_+$.

By (g) we can easily check that

$$\text{(i)}\quad \begin{cases} \mathfrak{X}_-=\{\xi\in\mathfrak{X};\ \limsup_{t\to+\infty} \sqrt[t]{\|\phi_t^{\#}\xi\|}\leqq r_-\}\,, \\ \mathfrak{X}_+=\{\xi\in\mathfrak{X};\ \limsup_{t\to+\infty} \sqrt[t]{\|\phi_{-t}^{\#}\xi\|}\leqq r_+\}\,. \end{cases}$$

It follows from (i) that $\mathfrak{X}_\pm$ are $C^0(SM)$-modules and moreover $\mathfrak{X}=\mathfrak{X}_-\oplus\mathfrak{X}_+$ is a splitting as $C^0(SM)$-module. Therefore according to Proposition 3, we see that there are two subbundles of E, denoted by E^s and E^u, such that

$$\text{(j)}\quad \begin{cases} \mathfrak{X}_-=\Gamma^0(E^s),\ \mathfrak{X}_+=\Gamma^0(E^u)\,, \\ E=E^s\oplus E^u\,. \end{cases}$$

Applying Lemma 5 in this case and using (h) we see that: for every $v\in SM$,

$$\text{(k)}\quad \begin{cases} \|T\phi_tX\|\leqq ce^{-\omega t}\|X\| & \text{for}\quad X\in E_v^s\quad\text{and}\quad t\geqq 0\,, \\ \|T\phi_{-t}X\|\leqq ce^{-\omega t}\|X\| & \text{for}\quad X\in E_v^u\quad\text{and}\quad t\geqq 0\,. \end{cases}$$

Thus (j) and (k) show that (iii) implies (iv).

4. That (iv)$\Rightarrow$(v) is easily shown as follows. Take any μ satisfying $0<\mu<\omega$ and choose τ so that $ce^{(\mu-\omega)\tau}<1$, where c and ω are constants defined in Definition 2.

We define a new Riemannian metric $\| \ \|_1$, by

$$\|X\|_1^2=\int_0^\tau \|e^{\mu t}T\phi_t X\|^2 dt \qquad \text{for} \quad X\in E^s,$$

$$\|X\|_1^2=\int_0^\tau \|e^{\mu t}T\phi_{-t} X\|^2 dt \qquad \text{for} \quad X\in E^u,$$

and generally for $X=X_-+X_++X_0$, $X_-\in E^s$, $X_+\in E^u$, $X_0\in\bar{S}$,

$$\|X\|_1^2=\|X_-\|_1^2+\|X_+\|_1^2+\|X_0\|_1^2.$$

Let $X\in E^s$. Then

$$\begin{aligned}\|T\phi_t X\|_1^2 &=\int_0^\tau \|e^{\mu s}T\phi_{t+s}X\|^2 ds\\ &=e^{-2\mu t}\Big(\|X\|_1^2-\int_0^t \|e^{\mu s}T\phi_s X\|^2 ds\\ &\qquad +\int_0^t \|e^{\mu(s+\tau)}T\phi_{x+\tau}X\|^2 ds\Big)\\ &\leqq e^{-2\mu t}\|X\|_1^2,\end{aligned}$$

because

$$\begin{aligned}\int_0^t \|e^{\mu(s+\tau)}T\phi_{s+\tau}X\|^2 ds &=(ce^{(\mu-\omega)\tau})^2\int_0^t \|e^{\mu s}T\phi_s X\|^2 ds\\ &\leqq \int_0^t \|e^{\mu s}T\phi_s X\|^2 ds.\end{aligned}$$

Hence $\|T\phi_t X\|_1\leqq e^{-\mu t}\|X\|_1$ for $t\geqq 0$.

The same argument applied to ϕ_{-t} shows that $\|T\phi_{-t}X\|_1\leqq e^{-\mu t}\|X\|_1$ for $t\geqq 0$ and $X\in E^u$.

5. We will finally show that (v)$\Rightarrow$(i). Set $A=\phi_1^\#$, $A_-=A|\Gamma^0(E^s)$ and $A_+=A|\Gamma^0(E^u)$. Then $\|A_-\|<1$, and hence $-\sum_{k=0}^\infty A_-^k$ converges and equals $(A_--I)^{-1}$. Also $\|A_+^{-1}\|<1$, so $(A_1^{-1}-I)^{-1}$ exists. Therefore $(A_+-I)^{-1}=-A_+^{-1}(A_+^{-1}-I)^{-1}$ exists. Since $(A_--I)^{-1}$ and $(A_+-I)^{-1}$ exist, $(A-I)^{-1}$ exists.

This completes the proof of Theorem 1.

Now it only remains to prove Lemma 9.

LEMMA 9. *$\sigma_\pm$ are non-void.*

PROOF. We will prove this lemma by contradiction. If we assume that $\sigma_+ = \phi$, then by (h) in the proof of Theorem 1, we have $\|\phi_t^\#\| \leqq ce^{-\omega t}$ for $t \geqq 0$. Therefore, by Lemma 5, we see that for every $v \in SM$ and every $X \in E_v$, $\|T\phi_t X\| \leqq ce^{-\omega t}\|X\|$.

On the other hand, if we set $\pi_* X' = \pi_* X$, $KX' = -KX$, then $X' \in E_v$ and applying Lemma 4, we have

$$\begin{aligned}
\frac{d}{dt}\|T\phi_t X'\|^2 &= 2 < DT\phi_t X',\ T\phi_t X' > \\
&= 2\{ < \pi_* DT\phi_t X',\ \pi_* T\phi X' > + < KDT\phi_t X',\ KT\phi_t X' > \} \\
&= 2\{ < KT\phi_t X',\ \pi_* T\phi_t X' > + < -R(\phi_t v,\ \pi_* T\phi_t X')\phi_t v,\ KT\phi_t X' > \} \\
&= -2\{ < KT\phi_t X,\ \pi_* T\phi_t X > + < -R(\phi_t v,\ \pi_* T\phi_t X)\phi_t v,\ KT\phi_t X > \} \\
&= -\frac{d}{dt}\|T\phi_t X\|^2 .
\end{aligned}$$

This is contradictory. To see this fact, it is sufficient to observe the following fact.

If $r(t)$ is a C^1-function on $[0, \infty)$ and if $0 \leqq r(t) \leqq ce^{-\omega t}$ for some positive constants c and ω, then we have

$$r(t) = r(0) + \int_0^t \dot{r}(t)dt \leqq ce^{-\omega t} r(0) ,$$

which implies

$$\int_0^t \dot{r}(t)dt \leqq (ce^{-\omega t} - 1)r(0) .$$

This shows that $\int_0^t \dot{r}(t)dt < 0$ for sufficiently large $t \geqq 0$. Thus the proof of Lemma 9 is complete.

This proof of Lemma 9 gives us the following important corollary.

COROLLARY 2. *If the geodesic flow ϕ_t is Anosov, and if E^s and E^u are the stable and unstable bundles, respectively, then* $\dim E^s = \dim E^u = n-1$, *where* $\dim M = n$.

PROOF. If $X \in E^s$, then we have $\|T\phi_t X\| \leqq ce^{-\omega t}\|X\|$ for $t \geqq 0$. If we set $\pi_* X' = \pi_* X$, $KX' = -KX$, then, as already shown, we conclude that

$$\|T\phi_{-t} X'\| \leqq ce^{-\omega t}\|X'\| \qquad \text{in} \quad t \geqq 0.$$

This shows that $X' \in E^u$, which completes the proof.

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