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Oscillatory and Asymptotic Behavior of Sublinear Retarded Differential Equations

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1. Introduction

The oscillatory behavior of differential equations with retarded arguments has drawn increasing attention in the last few years. There are two main directions in the investigation of the subject. The first direction is to generalize to retarded differential equations oscillation results already known for ordinary differential equations without delay. In the case of higher-order equations, this was done, for example, in [4–8, 13–17, 19, 21, 22]. The second direction is to establish results regarding the oscillation which is generated by the retarded arguments and which does not always occur for the corresponding ordinary differential equations. Efforts in this direction were undertaken in [3, 9–12, 18, 20, 23].

The purpose of this paper is to proceed in both directions to establish some oscillation and nonoscillation theorems for the retarded differential equations of the form

$$(*) \qquad x^{(n)}(t)+q(t)f(x(g_0(t)))\phi(x'(g_1(t)),\dots, x^{(n-1)}(g_{n-1}(t)))=0$$

an important particular case of which is the "sublinear" equation

$$x^{(n)}(t)+q(t)|x(g(t))|^{\alpha}\operatorname{sgn} x(g(t))=0, \qquad 0<\alpha<1.$$

It is tacitly assumed that under the initial condition

$$x^{(i)}(t)=\phi_i(t), \quad t\leqq t_0, \qquad i=0, 1,\dots, n-1,$$

equation $(*)$ has a solution which can be continued to $[t_0, \infty)$. We restrict attention to solutions $x(t)$ of $(*)$ which exist on some ray $[T_x, \infty)$ and are nontrivial for all large t. A solution $x(t)$ is called oscillatory if there is a sequence $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k=\infty$ and $x(t_k)=0$ for all k. Otherwise, a solution is called nonoscillatory.

In Section 2 we consider equation $(*)$ in which $q(t)$ is nonpositive and present results regarding oscillation and asymptotic behavior of its solutions. Our main concern is to extend some of the basic results of Kiguradze [2] for ordinary differential equations.

In Section 3 we study the effect of the delay on the oscillatory and asymptotic character of equation (*) in which $q(t)$ is nonnegative if n is odd and nonpositive if n is even. The results in this section are closely related to those recently obtained by Koplatadze [3] for second order equations and by Sficas and Staikos [20] for higher order equations.

In Section 4 we state a further oscillation theorem which can be obtained by combining results of Section 3 with those of Section 2 and of our previous paper [5].

2. Asymptotic Behavior and Oscillation

Let us consider the retarded differential equation

$$\text{(A)} \qquad x^{(n)}(t)-p(t)f(x(g_0(t)))\phi(x'(g_1(t)),\dots, x^{(n-1)}(g_{n-1}(t)))=0$$

where the following assumptions are assumed to hold:

(a) $p\in C[[0, \infty), R]$, $p(t)\geqq 0$, $p(t)\not\equiv 0$;

(b) $f\in C[R, R]$, $y_0 f(y_0)>0$ for $y_0\neq 0$, $f(y_0)$ is nondecreasing;

(c) $\phi\in C[R^{n-1}, R]$, $\phi(y_1,\dots, y_{n-1})>0$;

(d) $g_i\in C[[0, \infty), R]$, $g_i(t)\leqq t$, $\lim_{t\to\infty} g_i(t)=\infty$, $\quad i=0, 1,\dots, n-1$.

We shall need the following lemmas due to Kiguradze ([1], [2]).

LEMMA 1. *If $u(t)$ is a function such that it and all its derivatives up to order $(n-1)$ inclusive are absolutely continuous and of constant sign in the interval $[t_1, \infty)$, and $u(t)u^{(n)}(t)\leqq 0$, then there is an integer l, $0\leqq l\leqq n-1$, which is odd if n is even and even if n is odd, such that for $t\geqq t_1$*

$$u(t)u^{(i)}(t)\geqq 0, \qquad i=0, 1,\dots, l,$$

$$(-1)^{n+i-1}u(t)u^{(i)}(t)\geqq 0, \qquad i=l+1,\dots, n,$$

and if $l>0$,

$$\text{(1)} \qquad |u(t)|\geqq\frac{(t-t_1)^{n-1}}{(n-1)\cdots(n-l)}\,|u^{(n-1)}(2^{n-l-1}t)|\,.$$

LEMMA 2. *If $u(t)$ is a function such that it and all its derivatives up to order $(n-1)$ inclusive are absolutely continuous and of constant sign in the interval $[t_1, \infty)$ and $u(t)u^{(n)}(t)\geqq 0$, then either*

$$u(t)u^{(i)}(t)\geqq 0, \qquad i=0, 1,\dots, n,$$

or there is an integer l, $0\leqq l\leqq n-2$, which is even if n is even, and odd if n is odd, *such that for* $t \geqq t_1$

$$u(t)u^{(i)}(t) \geqq 0, \qquad i=0, 1, \ldots, l,$$

$$(-1)^{n+i}u(t)u^{(i)}(t) \geqq 0, \qquad i=l+1, \ldots, n,$$

and inequality (1) *holds.*

THEOREM 1. *Let assumptions* (a)–(d) *hold. A necessary and sufficient condition in order that*:

(i) *for n even, there exist a bounded nonoscillatory solution of* (A) *such that* $\lim_{t\to\infty} x(t) = a \neq 0$,

(ii) *for n odd, there exist a bounded nonoscillatory solution of* (A),

is that

$$\int^{\infty} t^{n-1} p(t) dt < \infty . \tag{2}$$

PROOF. (*Necessity*) Let $x(t)$ be a bounded nonoscillatory solution of (A) with the property as described in the theorem. We may assume without loss of generality that $x(t) > 0$ for $t \geqq t_0$. Since $\lim_{t\to\infty} g_0(t) = \infty$, there is $t_1 \geqq t_0$ such that $x(g_0(t)) > 0$ for $t \geqq t_1$. From (A) we have

$$x^{(n)}(t) = p(t) f(x(g_0(t))) \phi(x'(g_1(t)), \ldots, x^{(n-1)}(g_{n-1}(t))) \geqq 0 \tag{3}$$

for $t \geqq t_1$. By Lemma 2 and the boundedness of $x(t)$ it follows that

$$\lim_{t\to\infty} x(t) = c_0 > 0, \quad \lim_{t\to\infty} x^{(i)}(t) = 0, \qquad i=1, \ldots, n-1 .$$

Since f and ϕ are continuous, there is $t_2 \geqq t_1$ such that

$$f(x(g_0(t))) \phi(x'(g_1(t)), \ldots, x^{(n-1)}(g_{n-1}(t))) \geqq \frac{1}{2} f(c_0) \phi(0, \ldots, 0) \tag{4}$$

for $t \geqq t_2$. Integrating (3) n times from t to ∞, we obtain

$$x(t) = c_0 + \frac{(-1)^n}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} p(s) f(x(g_0(s))) \phi(x^{(i)}(g_i(s))) ds , \tag{5}$$

where we have put

$$\phi(x^{(i)}(g_i(t))) = \phi(x'(g_1(t)), \ldots, x^{(n-1)}(g_{n-1}(t))) .$$

Using (4) in (5), we obtain

$$(-1)^n [x(t) - c_0] \geqq \frac{f(c_0) \phi(0, \ldots, 0)}{2(n-1)!} \int_t^{\infty} (s-t)^{n-1} p(s) ds .$$

Since $x(t)$ is bounded, the above inequality yields

$$\int_t^\infty (s-t)^{n-1} p(s)ds < \infty$$

which implies (2).

(*Sufficiency*) It suffices to show that under condition (2) there exists a bounded continuous solution, defined for all sufficiently large t, of the integral equation

$$x(t) = c_0 + \frac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1} p(s) f(x(g_0(s)))\phi(x^{(i)}(g_i(s)))ds$$

where c_0 is an arbitrary but fixed nonzero constant. This can be done with the use of the fixed point technique as developed in [7]. We omit the details.

COROLLARY 1. *Let assumptions* (a)–(d) *hold. A necessary and sufficient condition in order that:*

(i) *for n even, every bounded solution of* (A) *either oscillate or tend monotonically to zero together with its first $n-1$ derivatives,*

(ii) *for n odd, every bounded solution of* (A) *oscillate,*

is that

$$\int^\infty t^{n-1} p(t)dt = \infty . \tag{6}$$

THEOREM 2. *In addition to* (a), (c), (d) *assume that*

(e) $f(y_0) = |y_0|^\alpha \operatorname{sgn} y_0, \qquad 0 < \alpha < 1;$

(f) *there exist positive constants γ, Γ such that*

$$\gamma \leqq \phi(y_1, \dots, y_{n-1}) \leqq \Gamma \qquad \text{for all} \quad y_1, \dots, y_{n-1} .$$

A necessary and sufficient condition for (A) *to have an unbounded nonoscillatory solution such that $\lim_{t\to\infty} x^{(n-1)}(t) = b \neq 0$ is that*

$$\int^\infty [g_0(t)]^{\alpha(n-1)} p(t)dt < \infty . \tag{7}$$

PROOF. (*Necessity*) Let $x(t)$ be a nonoscillatory solution of (A) with the property $\lim_{t\to\infty} x^{(n-1)}(t) = b \neq 0$. We may suppose that $b > 0$. Since, by L'Hospital's rule, $\lim_{t\to\infty} x(t)/t^{n-1} = b/(n-1)!$, there are positive numbers h, k and t_0 such that

$$ht^{n-1} < x(t) < kt^{n-1} \qquad \text{for} \quad t \geqq t_0 . \tag{8}$$

Taking $t_1 \geqq t_0$ so large that $g_0(t) \geqq t_0$ for $t \geqq t_1$, integrating (A) from t_1 to t and using (8), we obtain

$$x^{(n-1)}(t)-x^{(n-1)}(t_1)=\int_{t_1}^{t} p(s)[x(g_0(s))]^{\alpha}\phi(x^{(i)}(g_i(s)))ds$$

$$\geqq \gamma h^{\alpha}\int_{t_1}^{t}[g_0(s)]^{\alpha(n-1)}p(s)ds,$$

which gives (7) in the limit as $t\to\infty$.

(*Sufficiency*) Assume that (7) holds. Let $x(t)$ be a solution of (A) satisfying the initial condition

(9) $$x(t)=x'(t)=\cdots=x^{(n-2)}(t)\equiv 0,\quad x^{(n-1)}(t)\equiv K>0,\quad t\leqq t_0.$$

By our hypothesis such a solution exists on $[t_0, \infty)$. It is easy to verify that $x^{(i)}(t)\geqq 0$, $i=0, 1,\dots, n-1$, and

(10) $$x(t)\leqq (t-t_0)^{n-1}x^{(n-1)}(t)\qquad \text{for}\quad t\geqq t_0.$$

From (10) and the increasing character of $x^{(n-1)}(t)$ we find

(11) $$x(g_0(t))\leqq [g_0(t)]^{n-1}x^{(n-1)}(t)\qquad \text{for}\quad t\geqq t_0.$$

Integrating (A) from t_0 to t and using (9), (11), we get

$$x^{(n-1)}(t)\leqq K+\int_{t_0}^{t} p(s)[x(g_0(s))]^{\alpha}\phi(x^{(i)}(g_i(s)))ds$$

$$\leqq K+\Gamma[x^{(n-1)}(t)]^{\alpha}\int_{t_0}^{t}[g_0(s)]^{\alpha(n-1)}p(s)ds.$$

Therefore, if K is sufficiently large, then we obtain

$$[x^{(n-1)}(t)]^{1-\alpha}\leqq K+\Gamma\int_{t_0}^{t}[g_0(s)]^{\alpha(n-1)}p(s)ds$$

which shows that $x^{(n-1)}(t)$ remains bounded as $t\to\infty$. Since $x^{(n-1)}(t)$ is nondecreasing, it follows that the nonzero limit $\lim_{t\to\infty} x^{(n-1)}(t)$ exists.

THEOREM 3. *Assume that the hypotheses of Theorem 2 are satisfied. A necessary and sufficient condition in order that:*

(i) *for n even, every solution of* (A) *either oscillate or else tend monotonically to zero or infinity as $t\to\infty$ together with its first $n-1$ derivatives,*

(ii) *for n odd, every solution of* (A) *either oscillate or else tend monotonically to infinity as $t\to\infty$ together with its first $n-1$ derivatives,*

is that

(12) $$\int^{\infty}[g_0(t)]^{\alpha(n-1)}p(t)dt=\infty.$$

PROOF. The necessity follows from Theorem 2. To prove the sufficiency, let $x(t)$ be a nonoscillatory solution of (A). We may suppose that $x(g_0(t))>0$ for $t \geqq t_1$. In view of (A), $x^{(n)}(t) \geqq 0$ for $t \geqq t_1$, so that from Lemma 2 it follows that either

$$x^{(i)}(t) \geqq 0 \quad \text{for} \quad t \geqq t_1, \quad i=0, 1,\dots, n-1, \tag{13}$$

or there is an integer l, $0 \leqq l \leqq n-2$, which is even if n is even and odd if n is odd, such that for $t \geqq t_1$

$$x^{(i)}(t) \geqq 0, \quad i=0, 1,\dots, l, \qquad (-1)^{n+i}x^{(i)}(t) \geqq 0, \quad i=l+1,\dots, n-1, \tag{14}$$

and if $l>0$

$$x(t) \geqq \frac{(t-t_1)^{n-1}}{(n-1)\cdots(n-l)}[-x^{(n-1)}(2^{n-l-1}t)]. \tag{15}$$

Suppose that (13) holds. Then, for some $c>0$ and $t_2 \geqq t_1$, we have

$$x(t) \geqq ct^{n-1} \quad \text{for} \quad t \geqq t_2. \tag{16}$$

We choose $t_3 \geqq t_2$ so that $g_0(t) \geqq t_2$ for $t \geqq t_3$, integrate (A) from t_3 to t and use (16) to obtain

$$x^{(n-1)}(t) \geqq x^{(n-1)}(t_3) + \gamma c^\alpha \int_{t_3}^t [g_0(s)]^{\alpha(n-1)} p(s)ds$$

which, by (12), implies that $\lim_{t\to\infty} x^{(n-1)}(t)=\infty$, and consequently,

$$\lim_{t\to\infty} x^{(i)}(t)=\infty, \qquad i=0, 1,\dots, n-1.$$

Suppose now that both (14) and (15) hold. Then, from (15) we have

$$x(t) \geqq At^{n-1}[-x^{(n-1)}(t)] \quad \text{for} \quad t \geqq t_4=2t_1,$$

where $A=2^{(l-n+1)}/(n-1)\dots(n-l)$, and we can choose $t_5 \geqq t_4$ such that

$$x(g_0(t)) \geqq A[g_0(t)]^{n-1}[-x^{(n-1)}(t)] \quad \text{for} \quad t \geqq t_5. \tag{17}$$

Combining (A) with (17), we have

$$x^{(n)}(t) \geqq \gamma A^\alpha p(t)[g_0(t)]^{\alpha(n-1)}[-x^{(n-1)}(t)]^\alpha. \tag{18}$$

Dividing both sides of (18) by $[-x^{(n-1)}(t)]^\alpha$ and integrating from t_5 to t we obtain

$$\frac{[-x^{(n-1)}(t_5)]^{1-\alpha}}{1-\alpha} - \frac{[-x^{(n-1)}(t)]^{1-\alpha}}{1-\alpha} \geqq \gamma A^\alpha \int_{t_5}^t [g_0(s)]^{\alpha(n-1)} p(s)ds,$$

which implies $\int_{t_5}^\infty [g_0(t)]^{\alpha(n-1)}p(t)dt<\infty$, a contradiction. Thus, we must have

$l=0$. (We observe that this can happen only when n is even.) In this case $x(t)$ decreases to a finite limit $c_0 \geqq 0$ as t grows to infinity. We claim that $c_0=0$. In fact, if $c_0>0$, then from (A) we find

$$x^{(n)}(t) \geqq \gamma c_0^{\alpha} p(t) \qquad \text{for} \quad t \geqq t_5 . \tag{19}$$

Multiplying both sides of (19) by t^{n-1} and integrating from t_5 to t we obtain

$$P(t)-P(t_5)+(-1)^{n-1}(n-1)![x(t)-x(t_5)] \geqq \gamma c_0^{\alpha} \int_{t_5}^{t} s^{n-1} p(s) ds , \tag{20}$$

where

$$P(t)=\sum_{i=1}^{n-1} (-1)^{i-1}(n-1)(n-2)\cdots(n-i+1) t^{n-i} x^{(n-i)}(t)$$

which is nonpositive on account of (14). Since $x(t)$ is bounded, from (20) we conclude that $\int_{t_5}^{\infty} t^{n-1} p(t) dt < \infty$, which clearly contradicts (12).

REMARK 1. When $g_i(t) \equiv t$, $i=0, 1,\dots, n-1$, Theorems 2 and 3 reduce to the analogues of Lemma 5 and Theorem 2 of Kiguradze [2], respectively.

3. Effect of the Delay

In this section, motivated by a recent paper by Koplatadze [3], we investigate the effect of the delay on the oscillatory and asymptotic behavior of the retarded differential equation

$$x^{(n)}(t)+(-1)^{n+1} p(t) f(x(g_0(t))) \phi(x'(g_1(t)),\dots, x^{(n-1)}(g_{n-1}(t)))=0 . \tag{B}$$

THEOREM 4. *In addition to* (a)–(d) *assume that*:

(g) $|f(yz)| \geqq |f(y)f(z)|$ *for all* y, z;

(h) $\int_{+0}^{+a} \frac{dy}{f(y)} < \infty, \qquad \int_{-0}^{-a} \frac{dy}{f(y)} < \infty$ *for some* $a>0$.

If

$$\int^{\infty} p(t) f([t-g_0(t)]^{n-1}) dt = \infty , \tag{21}$$

then, every solution of (B) *is either oscillatory or tending monotonically to infinity as* $t \to \infty$.

PROOF. Let $x(t)$ be a nonoscillatory solution of (B) such that $x(g_0(t))>0$ for $t \geqq t_1$. From (A) we see that $x^{(n)}(t) \geqq 0$ if n is even and $x^{(n)}(t) \leqq 0$ if n is odd.

If n is even, then from Lemma 2 it follows that either

$$x^{(i)}(t) \geqq 0 \quad \text{for} \quad t \geqq t_1, \qquad i=0, 1, \dots, n-1, \tag{22}$$

or there exists an even integer l, $0 \leqq l \leqq n-2$, such that for $t \geqq t_1$

$$x^{(i)}(t) \geqq 0, \quad i=0, 1, \dots, l, \qquad (-1)^i x^{(i)}(t) \geqq 0, \quad i=l+1, \dots, n-1. \tag{23}$$

If n is odd, then, by Lemma 1, there exists an even integer m, $0 \leqq m \leqq n-1$, such that for $t \geqq t_1$

$$x^{(i)}(t) \geqq 0, \quad i=0, 1, \dots, m, \qquad (-1)^i x^{(i)}(t) \geqq 0, \quad i=m+1, \dots, n-1. \tag{24}$$

It is easy to see that if (22) holds or $l<0$ [or $m>0$], then $x(t)$ tends to infinity as $t \to \infty$. Therefore, it remains to examine the case where $l=0$ [or $m=0$]. In this case, by (23) [or (24)], we have

$$(-1)^i x^{(i)}(t) \geqq 0 \quad \text{for} \quad t \geqq t_1, \qquad i=0, 1, \dots, n-1, \tag{25}$$

from which we conclude that $x(t)$ is decreasing as $t \to \infty$ and

$$\lim_{t \to \infty} x^{(i)}(t) = 0, \qquad i=1, \dots, n-1. \tag{26}$$

Applying Taylor's theorem to the function $x(s)$ about the point t we obtain

$$x(s) = \sum_{i=0}^{n-1} \frac{x^{(i)}(t)}{i!}(s-t)^i + \frac{x^{(n)}(\tau)}{n!}(s-t)^n, \; s, t \geqq t_1, \tag{27}$$

where τ is a point between s and t. In view of (25) we get from (27)

$$x(s) \geqq (-1)^{n-1} \frac{x^{(n-1)}(t)}{(n-1)!}(t-s)^{n-1} \text{ for } t \geqq s \geqq t_1. \tag{28}$$

Since $g_0(t) \leqq t$ and $\lim_{t \to \infty} g_0(t) = \infty$, it follows from (28) that there exists $t_2 \geqq t_1$ such that

$$x(g_0(t)) \geqq (-1)^{n+1} \frac{x^{(n-1)}(t)}{(n-1)!}[t - g_0(t)]^{n-1}, \; t \geqq t_2. \tag{29}$$

Choosing t_2 sufficiently large, if necessary, and using (26) and (d), we find

$$\phi(x'(g_1(t)), \dots, x^{(n-1)}(g_{n-1}(t))) \geqq \frac{1}{2} \phi(0, \dots, 0) > 0, \qquad t \geqq t_2. \tag{30}$$

Substituting (29), (30) in (B) and using (g), we obtain

$$(-1)^n x^{(n)}(t) \geqq kp(t) f((-1)^{n-1} x^{(n-1)}(t)) f([t - g_0(t)]^{n-1}) \tag{31}$$

for $t \geqq t_2$, where $k = \phi(0, \dots, 0) f(1/(n-1)!)/2$. Dividing (31) by $f((-1)^{n-1} x^{(n-1)}(t))$

and integrating from t_2 to t, we obtain

$$\int_{(-1)^{n-1}x^{(n-1)}(t)}^{(-1)^{n-1}x^{(n-1)}(t_2)} \frac{dy}{f(y)} \geqq k\int_{t_2}^{t} p(s)f([s-g_0(s)]^{n-1})ds$$

which in view of (26), (h) produces a contradiction to (21). Thus, under the assumptions of the theorem, a nonoscillatory solution of (B) must tend to infinity as $t\to\infty$. This completes the proof.

COROLLARY 2. *Consider the equation*

$$\text{(C)} \qquad x^{(n)}(t)+(-1)^{n+1}p(t)|x(g_0(t))|^{\alpha}\operatorname{sgn} x(g_0(t))=0\,,$$

where $0<\alpha<1$, *and* $p(t)$ *and* $g_0(t)$ *satisfy* (a) *and* (d), *respectively.*
If

$$\text{(32)} \qquad \int^{\infty} p(t)[t-g_0(t)]^{\alpha(n-1)}dt=\infty\,,$$

then every solution of (C) *is either oscillatory or tending monotonically to infinity as* $t\to\infty$.

REMARK 2. Theorem 4 is closely related to a recent result of Sficas and Staikos [20] for the sublinear delay equation (B) with $\phi\equiv 1$ but with less restrictive assumption on f.

We shall give some generalizations of Theorem 4. Namely, let us consider the retarded differential equation

$$\text{(B}'\text{)} \qquad x^{(n)}(t)+(-1)^{n+1}p(t)f(\tilde{x}(\tilde{g}_0(t)))\phi(\tilde{x}'(\tilde{g}_1(t)),\dots,\tilde{x}^{(n-1)}(\tilde{g}_{n-1}(t)))=0$$

where

$$\tilde{x}^{(i)}(\tilde{g}_i(t))\equiv(x^{(i)}(g_{i1}(t)),\dots,x^{(i)}(g_{im_i}(t))),\qquad i=0,1,\dots,n-1\,.$$

The following vector notation will be used. R^d denotes the real d dimensional space of vectors $\tilde{\xi}=(\xi_1,\dots,\xi_d)$. The zero vector in R^d is denoted by $\tilde{O}$. Inequality between vectors $\tilde{\xi}$ and $\tilde{\eta}=(\eta_1,\dots,\eta_d)$ is defined as

$$\tilde{\xi}\geqq\tilde{\eta}\ [\tilde{\xi}>\tilde{\eta}]\ \text{ equivalent to }\ \xi_j\geqq\eta_j\ [\xi_j>\eta_j]\ \text{ for } j=1,\dots,d\,.$$

With regard to (B′) we make the following assumptions:

(a′) $p\in C[[0,\infty),R]$, $p(t)\geqq 0$, $p(t)\not\equiv 0$;

(*b*′) $f\in C[R^{m_0},R]$, $f(\tilde{y}_0)>0$ for $\tilde{y}_0>\tilde{O}$, $f(\tilde{y}_0)<0$ for $\tilde{y}_0<\tilde{O}$,

$$f(\tilde{y}_0)\geqq f(\tilde{z}_0) \text{ for } \tilde{y}_0\geqq\tilde{z}_0\geqq\tilde{O},\ f(\tilde{y}_0)\geqq f(\tilde{z}_0) \text{ for } \tilde{y}_0\leqq\tilde{z}_0\leqq\tilde{O};$$

(c') $\phi\in C[R^{m_1+\cdots+m_{n-1}}, R]$, $\phi(\tilde{y}_1,\dots,\tilde{y}_{n-1})>0$,

(d') $g_{ij}\in C[[0,\infty), R]$, $g_{ij}(t)\leqq t$, $\lim_{t\to\infty} g_{ij}(t)=\infty$,

$$j=1,\dots, m_i,\quad i=0,1,\dots,n-1.$$

A generalization of Theorem 4 is given in the following

THEOREM 5. *In addition to* (a')–(d') *assume that*:

(g') $|f(y_1z_1,\dots,y_{m_0}z_{m_0})|\geqq|f(y_1,\dots,y_{m_0})f(z_1,\dots,z_{m_0})|$;

(h') $\int_{+0}^{+a}\frac{dy}{f(y,\dots,y)}<\infty$, $\int_{-0}^{-a}\frac{dy}{f(y,\dots,y)}<\infty$ *for some* $a>0$;

If

$$\int^{\infty}p(t)f([t-g_{01}(t)]^{n-1},\dots,[t-g_{0m_0}(t)]^{n-1})dt=\infty,$$

then, every solution of (B') *either oscillates or tends monotonically to infinity as* $t\to\infty$.

This theorem can be proved by an argument similar to that in the proof of Theorem 4. So, we omit the details.

COROLLARY 3. *Consider the equation*

$$x^{(n)}(t)+(-1)^{n+1}p(t)\prod_{j=1}^{m_0}|x(g_{0j}(t))|^{\alpha_j}\operatorname{sgn} x(g_{01}(t))=0, \tag{C'}$$

where $\alpha_j>0$, $\alpha_1+\cdots+\alpha_{m_0}<1$ *and* $p(t)$ *and* $g_{0j}(t)$ *satisfy* (a') *and* (d'), *respectively. If*

$$\int^{\infty}p(t)\prod_{j=1}^{m_0}[t-g_{0j}(t)]^{\alpha_j(n-1)}dt=\infty, \tag{33}$$

then every solution of (C') *either oscillates or tends monotonically to infinity as* $t\to\infty$.

PROOF. We need only to observe that the function

$$f(y_1,\dots,y_{m_0})\equiv\prod_{j=1}^{m_0}|y_j|^{\alpha_j}\operatorname{sgn} y_1$$

satisfies conditions (b'), (g') and (h') of Theorem 5 provided $\alpha_j>0$, $\alpha_1+\cdots+\alpha_{m_0}<1$.

The following corollary can easily be proved with the aid of the result of Corollary 3.

COROLLARY 4. *Consider the equation*

(D) $$x^{(n)}(t)+(-1)^{n+1}F(t, \tilde{x}(\tilde{g}_0(t)), \tilde{x}'(\tilde{g}_1(t)),\dots, \tilde{x}^{(n-1)}(\tilde{g}_{n-1}(t)))=0$$

where $F\in C[[0, \infty)\times R^N, R]$, $N=m_0+m_1+\cdots+m_{n-1}$, *and condition* (d') *is satisfied. Assume that there exist functions* $p(t)$ *and* $\phi(\tilde{y}_1,\dots, \tilde{y}_{n-1})$ *satisfying* (a') *and* (c'), *respectively, and positive constants* α_j *with* $\alpha_1+\cdots+\alpha_{m_0}<1$, *such that*

$$F(t, \tilde{y}_0, \tilde{y}_1,\dots, \tilde{y}_{n-1})\operatorname{sgn} y_{01}\geqq p(t)\prod_{j=1}^{m_0}|y_{0_j}|^{\alpha_j}\phi(\tilde{y}_1,\dots, \tilde{y}_{n-1})$$

for all $(t, \tilde{y}_0, \tilde{y}_1,\dots, \tilde{y}_{n-1})\in[0, \infty)\times R^N$, *where* $\tilde{y}_0=(y_{01},\dots, y_{0m_0})$. *Then,* (33) *is a sufficient condition that every solution of* (D) *be either oscillatory or tending monotonically to infinity as* $t\to\infty$.

REMARK 3. Corollary 4 is an extension of a recent result of Koplatadze [3, Corollary to Theorem 1] for the second order equation

$$x''(t)+F(t, x(g_1(t)), \dots, x(g_m(t)), x'(g_1(t)),\dots, x'(g_m(t)))=0.$$

4. Concluding Remarks

Let us further consider equation (B) for which assumptions (a), (c)–(f) are satisfied. Suppose that n is even. Combining Theorem 3 with Theorem 4, we see that if both (12) and (32) hold, then every solution of (B) either oscillates or tends monotonically to infinity as $t\to\infty$ together with its first $n-1$ derivatives. Suppose now that n is odd. We have recently shown in [5] that (12) is a necessary and sufficient condition for every solution of (B) to oscillate or tend monotonically to zero as $t\to\infty$ together with its first $n-1$ derivatives. From this and Theorem 4 it follows that if both (12) and (32) hold, then all solutions of (B) are oscillatory. We summarize these facts in the following theorem.

THEOREM 6. *Consider equation* (B) *for which* (a), (c)–(f) *are satisfied. Assume that both* (12) *and* (32) *hold.*

(i) *If n is even, then every solution of* (B) *either oscillates or tends monotonically to infinity as $t\to\infty$ together with its first $n-1$ derivatives.*

(ii) *If n is odd, then every solution of* (B) *is oscillatory.*

We observe that if $g(t)$ is such that

$$0<\liminf_{t\to\infty}\frac{g(t)}{t}\leqq\limsup_{t\to\infty}\frac{g(t)}{t}<1$$

and if

(34) $$\int^{\infty}t^{\alpha(n-1)}p(t)dt=\infty,$$

then both (12) and (32) are satisfied.

It would be of interest to compare this theorem with Theorem 3 of Kiguradze [2] to the effect that the ordinary differential equation (B) with $g_i(t)\equiv t$, $i=0, 1,\dots, n-1$, has nontrivial solutions which tend monotonically to zero as $t\to\infty$ provided $n\geqq 3$ and (12) [i.e. (34)] holds. We then conclude that the absence of such solutions in the statement of Theorem 6 is caused by (32), that is, the effect of the delay $t-g_0(t)$.

References

[1] I. T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m}+a(t)\,|u|^n \operatorname{sign} u=0$, Mat. Sb. **65** (1964), 172–187. (Russian)

[2] I. T. Kiguradze, The problem of oscillation of solutions of nonlinear differential equations, Differencial'nye Uravnenija **1** (1965), 995–1006. (Russian) = Differential Equations **1** (1965), 773–782.

[3] R. G. Koplatadze, On the existence of oscillatory solutions of second order nonlinear differential equations with retarded argument, Dokl. Akad. Nauk SSSR **210** (1973), 260–262. (Russian)

[4] T. Kusano and H. Onose, Oscillation of solutions of nonlinear differential delay equations of arbitrary order, Hiroshima Math. J. **2** (1972), 1–13.

[5] T. Kusano and H. Onose, Nonlinear oscillation of a sublinear delay equation of arbitrary order, Proc. Amer. Math. Soc. **40** (1973), 219–224.

[6] T. Kusano and H. Onose, Oscillations of functional differential equations with retarded argument, J. Differential Equations **15** (1974), 269–277.

[7] T. Kusano and H. Onose, Nonoscillatory solutions of differential equations with retarded arguments, Submitted for publication.

[8] G. Ladas, Oscillation and asymptotic behavior of solutions of differential equations with retarded argument, J. Differential Equations **10** (1971), 281–290.

[9] G. Ladas and V. Lakshmikantham, Oscillations caused by retarded actions, Applicable Anal. (to appear).

[10] G. Ladas, G. Ladde and J. S. Papadakis, Oscillations of functional-differential equations generated by delays, J. Differential Equations **12** (1972), 385–395.

[11] G. Ladas, V. Lakshmikantham and J. S. Papadakis, Oscillations of higher-order retarded differential equations generated by the retarded argument, "Delay and Functional Differential Equations and Their Applications" (K. Schmitt, Ed.), pp. 219–231, Academic Press, New York, 1972.

[12] G. S. Ladde, Oscillations of nonlinear functional differential equations generated by retarded actions, "Delay and Functional Differential Equations and Their Applications" (K. Schmitt, Ed.), pp. 355–365, Academic Press, New York, 1972.

[13] H. Onose, Some oscillation criteria for n-th order nonlinear delay-differential equa tions, Hiroshima Math. J. **1** (1971), 171–176.

[14] H. Onose, Oscillation and asymptotic behavior of solutions of retarded differential equations of arbitrary order, Hiroshima Math. J. **3** (1973), 333–360.

[15] V. N. Ševelo and N. V. Vareh, On the oscillation of solutions of higher order linear differential equations with retarded argument, Ukrain. Mat. Ž. **24** (1972), 513–520. (Russian)

[16] V. N. Ševelo and N. V. Vareh, On some properties of solutions of differential equations with delay, Ukrain. Mat. Ž. **24** (1972), 807–813. (Russian)
[17] Y. G. Sficas, On oscillation and asymptotic behavior of a certain class of differential equations with retarded argument, Utilitas Mathematica **3** (1973), 239–249.
[18] Y. G. Sficas, The effect of the delay on the oscillatory and asymptotic behavior of n-th order retarded differential equations, University of Ioannina, Technical Report No. 10, October, 1973.
[19] Y. G. Sficas and V. A. Staikos, Oscillation of retarded differential equations, Proc. Cambridge Philos. Soc. **75** (1974), 95–101.
[20] Y. G. Sficas and V. A. Staikos, The effect of retarded actions on nonlinear oscillations, Proc. Amer. Math. Soc. (to appear)
[21] V. A. Staikos and Y. G. Sficas, Oscillatory and asymptotic behavior of functional differential equations, J. Differential Equations **12** (1972), 426–437.
[22] V. A. Staikos and Y. G. Sficas, Oscillatory and asymptotic properties of differential equations with retarded arguments, Applicable Anal. (to appear).
[23] J. S. W. Wong, Second order oscillation with retarded arguments, "Ordinary Differential Equations" (L. Weiss, Ed.), pp. 581–596, Academic Press, New York, 1972.

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