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A Note on Graded Gorenstein Modules

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Recently the following conjecture was proposed by M. Nagata in [6]. Let $A=\sum_{n\geqq 0} A_n$ be a commutative Noetherian graded ring. If $A_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$ with $\mathfrak{m}\supset \sum_{n\geqq 1} A_n$, then A is Cohen-Macaulay. This conjecture was solved affirmatively by J. Matijevic and P. Roberts in [5]. The aim of this paper is to prove the following theorem which generalizes the assertion in [5].

THEOREM. *Let* $A=\sum_{n\in Z} A_n$ *be a commutative Noetherian graded ring and* $M=\sum_{n\in Z} M_n$ *be a non-zero, finite graded A-module. If* $M_{\mathfrak{p}}$ *is a Gorenstein* $A_{\mathfrak{p}}$*-module (resp, a Cohen-Macaulay* $A_{\mathfrak{p}}$*-module) for every homogeneous prime ideal* $\mathfrak{p}\in \mathrm{Supp}(M)$, *then M is Gorenstein (resp. Cohen-Macaulay).*

1. We denote by $\mu^i(\mathfrak{p}, M)$ the dimension of the $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-vector space $\mathrm{Ext}^i_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}})$ (cf. [1]) and by $ht_M\mathfrak{p}$ the Krull dimension of the local ring $A_{\mathfrak{p}}/Ann(M)A_{\mathfrak{p}}$ (cf. [7]), where $Ann(M)$ is the annihilator of M and $\mathfrak{p}\in \mathrm{Supp}(M)$. The following lemma, due to Bass and Sharp, plays an important role in our discussion.

LEMMA 1 (Bass [1, (3.7)] and Sharp [7, (3.11)]). *Let M be a finite A-module.*

(i) *M is a Cohen-Macaulay module if and only if, for each* $\mathfrak{p}\in \mathrm{Supp}(M)$, $\mu^i(\mathfrak{p}, M)=0$ *whenever* $i<ht_M\mathfrak{p}$.

(ii) *The following conditions are equivalent.*

(1) *M is a Gorenstein module.*

(2) *For each* $\mathfrak{p}\in \mathrm{Supp}(M)$, $\mu^i(\mathfrak{p}, M)=0$ *if and only if* $i\neq ht_M\mathfrak{p}$.

For an ideal $\mathfrak{a}$ of the graded ring A we let $\mathfrak{a}^*$ denote the homogeneous ideal generated by homogeneous elements of $\mathfrak{a}$.

LEMMA 2. *Let M be a graded A-module and* $\mathfrak{p}$ *a prime ideal of A. Then* $\mathfrak{p}\in \mathrm{Supp}(M)$ *if and only if* $\mathfrak{p}^*\in \mathrm{Supp}(M)$.

PROOF. Suppose that $M_{\mathfrak{p}}=0$; then, for each homogeneous element m in M, there is a homogeneous component of s with $sm=0$, say s_u, which is not contained in $\mathfrak{p}$. Clearly $s_u m=0$ and this implies $M_{\mathfrak{p}^*}=0$. The converse is obvious. q.e.d.

2. Proof of the theorem. Let $\mathfrak{p}$ be a non-homogeneous prime ideal of Supp(M) and S be the multiplicative set of homogeneous elements in $A-\mathfrak{p}$. Then A_S becomes naturally a graded ring and M_S a non-zero, finite graded A_S-module (Bourbaki [2, Chap. 2, § 2, n°9]). Since, for $\mathfrak{q}\in\mathrm{Supp}(M)$ such that $\mathfrak{q}\cap S=\phi$, $ht_M\mathfrak{q}$ and $\mu^i(\mathfrak{q}, M)$ are invariant by localization, we may assume that $A=A_S$ and $M=M_S$. It follows from Lemme 4 of Bourbaki [3, Chap. 5, § 1, n°8] that $A/\mathfrak{p}^*=k[X, 1/X]$ where k is a field and $\deg X>0$. Therefore there exists an element x of $\mathfrak{p}$ such that $\mathfrak{p}=(\mathfrak{p}^*, x)$. The element x can be written uniquely as a sum of homogeneous constituents: $x=x_s+x_{s+1}+\cdots+x_t$, $\deg x_s<\deg x_{s+1}<\cdots<\deg x_t$. We may assume that the leading term x_s does not belong to $\mathfrak{p}^*$. However any homogeneous element of A which is not contained in $\mathfrak{p}^*$ is a unit. Therefore replacing x by x/x_s, if necessary, we may suppose that $x=1+x_1+\cdots+x_t$.

Now we consider the following exact sequence:

$$0\longrightarrow A/\mathfrak{p}^*\xrightarrow{\ x\ }A/\mathfrak{p}^*\longrightarrow A/\mathfrak{p}\longrightarrow 0,$$

where x means the multiplication by x. From this we can obtain the exact sequence

$$\longrightarrow \mathrm{Ext}_A^i(A/\mathfrak{p}, M)\longrightarrow \mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)\xrightarrow{\ x\ }\mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)\longrightarrow.$$

Since $A/\mathfrak{p}^*$ is a finite graded A-module, $\mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)$ is a graded A-module (see p. 14 of Eichler [4]). Since x is of the form $x=1+x_1+\cdots+x_t$, the sequence:

$$0\longrightarrow \mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)\xrightarrow{\ x\ }\mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)$$

is exact. Therefore we have the exact sequence

$$0\longrightarrow \mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)\xrightarrow{\ x\ }\mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)\longrightarrow \mathrm{Ext}_A^{i+1}(A/\mathfrak{p}, M)\longrightarrow 0.$$

for every $i\geqq 0$ and we have $\mathrm{Ext}_A^0(A/\mathfrak{p}, M)=0$. Since $A_\mathfrak{p}$ is flat over A,

$$0\longrightarrow \mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)_\mathfrak{p}\xrightarrow{\ x\ }\mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)_\mathfrak{p}\longrightarrow \mathrm{Ext}_A^{i+1}(A/\mathfrak{p}, M)_\mathfrak{p}\longrightarrow 0$$

is a exact sequence for every $i\geqq 0$ and $\mathrm{Ext}_A^0(A/\mathfrak{p}, M)_\mathfrak{p}=0$. Using Nakayama's lemma we can conclude that, for $i\geqq 0$, $\mu^{i+1}(\mathfrak{p}, M)=0$ if and only if $\mathrm{Ext}_A^i(A/\mathfrak{p}^*, M)_\mathfrak{p}=0$.
It follows from Lemma 2 that, for $i\geqq 0$, $\mu^{i+1}(\mathfrak{p}, M)=0$ if and only if $\mu^i(\mathfrak{p}^*, M)=0$. On the other hand we see that $ht_M\mathfrak{p}=ht_M\mathfrak{p}^*+1$ from Lemma 1 of Matijevic and Roberts [5]. Combining these facts with Lemma 1 we can complete the proof. q.e.d.

References

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