HIROSHIMA MATH. J.
4 (1974), 679–684

On Prime Ideals of Lie Algebras

Naoki KAWAMOTO

(Received May 18, 1974)

Introduction. The notion of prime ideals plays an important role in the theory of associative algebras. It seems to be interesting for us to know how the corresponding notion behaves itself in Lie algebras. In this paper we shall introduce the notion of prime ideals into Lie algebras which are not necessarily finite-dimensional and investigate their properties.

We give two conditions for ideals to be prime and study the interrelations among prime, semi-prime, irreducible and maximal ideals. We also show that in a Lie algebra satisfying the maximal condition for ideals, any semi-prime ideal is an intersection of finite number of prime ideals and the unique maximal solvable ideal is equal to the intersection of all prime ideals.

The author would like to express his thanks to Professor S. Tôgô for his helpful suggestions and encouragement.

1. Let Φ be a field of arbitrary characteristic. Let L be always a Lie algebra over Φ which is not necessarily finite-dimensional. For any element x of L, $<x^L>$ is the smallest ideal of L containing x [4]. $\mathrm{Rad}_{\mathfrak{S}}(L)$ is the sum of all solvable ideals of L [6]. If L satisfies the maximal (resp. minimal) condition for ideals, we write $L \in \mathrm{Max}-\lhd$ (resp. $\mathrm{Min}-\lhd$) [5].

2. An ideal P of L is called *prime* if $[H, K] \subseteq P$ with H, K ideals of L implies $H \subseteq P$ or $K \subseteq P$.

Let L and L' be Lie algebras and let f: $L \to L'$ be a surjective homomorphism. Then it is easily seen that an ideal P of L containing $\mathrm{Ker} f$ is prime if and only if $f(P)$ is prime in L'.

THEOREM 1. *Let P be an ideal of L. Then the following conditions are equivalent*:

i) *P is prime.*

ii) *If $[a, H] \subseteq P$ for $a \in L$ and an ideal H of L, then either $a \in P$ or $H \subseteq P$.*

iii) *If $[a, <b^L>] \subseteq P$ for a, $b \in L$, then either $a \in P$ or $b \in P$.*

PROOF. i)$\Rightarrow$iii). For each $a \in L$,

$$<a^L> = \sum_{i=0}^{\infty} V_i,$$

where $V_0=(a)$ and $V_i=[\dots[(a), \underbrace{L],\dots, L}_{i}]$. If $[a, <b^L>]\subseteq P$, we assert that $[V_i, <b^L>]\subseteq P$ for all $i\geq 0$. In fact, it is true for $i=0$. Let $i\geq 1$ and assume that the assertion is true for $i-1$. Then

$$\begin{aligned}[V_i, <b^L>] &= [[V_{i-1}, L], <b^L>] \\ &\subseteq [[V_{i-1}, <b^L>], L]+[V_{i-1}, [L, <b^L>]] \\ &\subseteq [P, L]+[V_{i-1}, <b^L>] \subseteq P.\end{aligned}$$

Thus we have the assertion. It follows that

$$[<a^L>, <b^L>] \subseteq P.$$

Since P is prime, either $<a^L> \subseteq P$ or $<b^L> \subseteq P$ and so $a\in P$ or $b\in P$.

iii)$\Rightarrow$ii). Let $a\in L\backslash P$ and let H be an ideal of L such that $[a, H]\subseteq P$. For any $b\in H$, $[a, <b^L>]\subseteq P$ since the ideal $<b^L>$ is contained in H. As $a\notin P$, iii) implies $b\in P$. Hence $H\subseteq P$.

ii)$\Rightarrow$i). Let H, K be ideals of L such that $[H, K]\subseteq P$ and $H\nsubseteq P$. Since $[a, K]\subseteq P$ for any $a\in H\backslash P$, we have $K\subseteq P$ by ii). Therefore P is prime.

This completes the proof.

As in associative rings [3] we say an ideal Q of L to be *semiprime* if the following condition is satisfied: If $H^2\subseteq Q$ for an ideal H of L, then $H\subseteq Q$. Semi-prime in this sense is the same as "primitif" in [7], and the following lemma is noted in [7].

LEMMA 2. *An ideal Q of L is semi-prime if and only if* $\mathrm{Rad}_{\mathfrak{S}}(L/Q)=(0)$.

As in commutative rings [1] we define the irreducibility of ideals as follows: An ideal N of L is said to be *irreducible* if $N=H\cap K$ with H, K ideals of L implies $N=H$ or $N=K$.

LEMMA 3. (1) *Any prime ideal is semi-prime.*

(2) *Any prime ideal is irreducible.*

(3) *Any maximal ideal is irreducible.*

(4) *Among prime, semi-prime, irreducible and maximal ideals, there are no implications besides* (1), (2) *and* (3).

PROOF. (1) and (3) are clear. (2) is immediate since $[H, K]\subseteq H\cap K$ for any ideals H, K of L.

(4) Let L be a 2-dimensional non-abelian Lie algebra, that is, $L=(x, y)$

with $[x, y]=x$. Then the ideals of L are (0), (x) and L. (0) is irreducible but neither prime nor semi-prime, for $(x)^2=(0)$. Apparently (0) is not maximal. (x) is maximal but neither prime nor semi-prime, because $L^2=(x)$. By definition L is prime but not maximal.

Let S_1, S_2 and S_3 be finite-dimensional simple Lie algebras. Let $L=S_1 \oplus S_2 \oplus S_3$. Then the ideals containing S_1 properly are $S_1 \oplus S_2$, $S_1 \oplus S_3$ and L. Therefore S_1 is semi-prime. Since

$$[S_1 \oplus S_2, S_1 \oplus S_3] \subseteq S_1 = (S_1 \oplus S_2) \cap (S_1 \oplus S_3),$$

S_1 is neither prime nor irreducible. S_1 is obviously not maximal.

This completes the proof.

PROPOSITION 4. *Let P be an ideal of L and let $P \neq L$.*

(1) *P is prime if and only if P is irreducible and semi-prime.*

(2) *Let $L \in \mathrm{Min}-\triangleleft$. Then P is prime if and only if there is the smallest ideal M containing P properly and such that M/P is not abelian.*

PROOF. (1) Let P be irreducible and semi-prime, and let H, K be ideals of L satisfying $[H, K] \subseteq P$. If we put $N=(H+P) \cap (K+P)$, then

$$N^2 \subseteq [H+P, K+P] \subseteq P.$$

Hence $N \subseteq P$ and

$$P=(H+P) \cap (K+P).$$

Then $P=H+P$ or $P=K+P$, that is, $H \subseteq P$ or $K \subseteq P$. Therefore P is prime. The converse is shown in Lemma 3.

(2) Since $L \in \mathrm{Min}-\triangleleft$, there is a minimal ideal which contains P properly. If such ideals M_1, M_2 are distinct, then

$$[M_1, M_2] \subseteq M_1 \cap M_2 = P.$$

If P is prime, then $M_1 \subseteq P$ or $M_2 \subseteq P$, which is a contradiction. Therefore there exists a unique minimal ideal M. M/P is not abelian because $(M/P)^2=(0)$ implies $M \subseteq P$.

Conversely, let M be the ideal satisfying the condition. Assume that P is not prime. Then there exist ideals H, K of L such that

$$H \nsubseteq P,\ K \nsubseteq P \text{ and } [H, K] \subseteq P.$$

$H+P$ and $K+P$ contain P properly and therefore $M \subseteq H+P$, $M \subseteq K+P$. Hence

$$M^2 \subseteq [H+P, K+P] \subseteq P.$$

that is, $(M/P)^2=(0)$, which is a contradiction. Therefore P is prime.

This completes the proof.

THEOREM 5. *Let M be a maximal ideal of L. Then the following conditions are equivalent*:

i) *M is prime.*

ii) *M is semi-prime.*

iii) $\dim L/M>1$.

PROOF. i)$\Rightarrow$ii). This is obvious.

ii)$\Rightarrow$iii). If $\dim L/M=1$, then L/M is abelian and $L^2\subseteq M$. Since M is semi-prime, this implies $L\subseteq M$, which contradicts the maximality of M.

iii)$\Rightarrow$i). If M is not prime, then there exist ideals H, K of L satisfying $H \nsubseteq M$, $K\nsubseteq M$ and $[H,K]\subseteq M$. M is maximal, whence $H+M=K+M=L$ and

$$L^2 = [H+M, K+M] \subseteq M.$$

L/M is then abelian. Furthermore it has no proper ideal by maximality of M. Therefore the dimension of L/M must be 1.

This completes the proof.

COROLLARY 6. (1) *Let L be a perfect Lie algebra. Then a maximal ideal of L is prime.*

(2) *Let $L=\bigoplus_{\lambda\in\Lambda} L_\lambda$ where L_λ, $\lambda\in\Lambda$, are simple. Let H be an ideal of L and let $H\neq L$. Then H is prime if and only if H is maximal, and if and only if H is irreducible.*

PROOF. (1) Let M be a maximal ideal of L. Assume that M is not semi-prime, then there exists an ideal H such that $H^2\subseteq M$ and $H\nsubseteq M$. Since M is maximal, $H+M=L$ and

$$L^2 = (H+M)^2 \subseteq M,$$

which is a contradiction. Hence by Theorem 5 M is prime.

(2) Since L is perfect, a maximal ideal of L is prime by (1) and a prime ideal is irreducible by Lemma 2. Let H be an irreducible ideal which is different from L. Then

$$L = H \oplus \Big(\bigoplus_{\lambda\in\Lambda'} L_\lambda\Big) \qquad (\Lambda' \subseteq \Lambda).$$

If Λ' has more than two elements, there are L_{λ_1}, L_{λ_2} $(\lambda_1, \lambda_2\in\Lambda')$ such that $H\neq H\oplus L_{\lambda_1}$, $H\neq H\oplus L_{\lambda_2}$, and therefore

$$H = (H \oplus L_{\lambda_1}) \cap (H \oplus L_{\lambda_2}).$$

This contradicts the irreducibility of H. Therefore Λ' has only one element, and H is maximal. This completes the proof.

3. Let H be an ideal of L. We denote by $r(H)$ the intersection of all the prime ideals of L containing H. We write r_L for $r(0)$, the intersection of all the prime ideals of L, and call it the *prime radical* of L.

If $L \in \mathrm{Max}-\lhd$ and H is an ideal of L, then there exist a finite number of prime ideals P_i $(i=1,\dots,n)$ such that

$$r(H) = \bigcap_{i=1}^{n} P_i .$$

This can be proved in the same way as for nonassociative rings in [2]. Let P be a prime ideal of L and let $H_1,\dots, H_n$ be ideals of L such that $\bigcap_{i=1}^{n} H_i \subseteq P$. Then $H_i \subseteq P$ for some i, because

$$[\dots[H_1, H_2],\dots, H_n] \subseteq \bigcap_{i=1}^{n} H_i \subseteq P .$$

From this fact it is easily seen that the above expression of $r(H)$ is unique whenever $P_i \nsubseteq P_j$ $(i \neq j)$. The following may be pointed out: If $P_1,\dots, P_n$ are prime ideals of L and H is an ideal of L such that $H \subseteq \bigcup_{i=1}^{n} P_i$, then $H \subseteq P_i$ for some i.

We here show that there is an intimate connection between the prime radical and $\mathrm{Rad}_{\mathfrak{S}}(L)$.

THEOREM 7. *$\mathrm{Rad}_{\mathfrak{S}}(L)$ is contained in* r_L. *If $L \in \mathrm{Max}-\lhd$, then* $\mathrm{Rad}_{\mathfrak{S}}(L)$ *equals* r_L.

PROOF. Let H be a solvable ideal of L. Then there is an integer $n \geq 0$ such that $H^{(n)} = (0)$. For any prime ideal P of L we have $H \subseteq P$ since $H^{(n)} = (0) \subseteq P$. Therefore $\mathrm{Rad}_{\mathfrak{S}}(L) \subseteq \mathrm{r}_L$.

If $L \in \mathrm{Max}-\lhd$, then $\mathrm{Rad}_{\mathfrak{S}}(L)$ is the unique maximal solvable ideal of L. Assume that r_L is not solvable. Let $\mathfrak{C}$ be a collection of ideals H such that $\mathrm{r}_L^{(n)} \nsubseteq H$ for all integers $n \geq 0$. $\mathfrak{C}$ is not empty because $(0) \in \mathfrak{C}$. Hence $\mathfrak{C}$ has a maximal element P. We claim that P is prime. If there are ideals H, K of L such that

$$H \nsubseteq P, \quad K \nsubseteq P \quad \text{and} \quad [H, K] \subseteq P,$$

then $H+P$, $K+P \notin \mathfrak{C}$ by definition of P. Hence

$$r_L^{(n)} \subseteq H+P, \qquad r_L^{(m)} \subseteq K+P$$

for some integers n, $m \geq 0$. Let $k = \max\{n, m\}$. Then

$$\mathrm{r}_L^{(k+1)} \subseteq [H+P, K+P] \subseteq P.$$

But this contradicts $P \in \mathfrak{C}$. Hence P is prime and $\mathrm{r}_L \nsubseteq P$, which contradicts the definition of r_L. Therefore r_L is solvable and $r_L \subseteq \mathrm{Rad}_{\mathfrak{S}}(L)$, which completes the proof.

We finally give characterizations of semi-prime ideals when $L \in \mathrm{Max}-\triangleleft$.

COROLLARY 8. *Let $L \in \mathrm{Max}-\triangleleft$ and Q be an ideal of L. Then the following statements are equivalent*:

i) *Q is semi-prime.*

ii) $Q = \mathrm{r}(Q)$.

iii) *Q is a finite intersection of prime ideals of L.*

PROOF. i)$\Rightarrow$ii). If Q is semi-prime, $\mathrm{Rad}_{\mathfrak{S}}(L/Q)=(0)$ by Lemma 2. By Theorem 7 the intersection of all prime ideals of L/Q equals (0). Therefore Q equals the intersection of all prime ideals of L containing Q.

ii)$\Rightarrow$iii) follows from the fact stated in the beginning fo this section. An intersection of semi-prime ideals is easily seen to be always semi-prime, whence we have iii)$\Rightarrow$i) by Lemma 2.

References

[1] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Massachusetts, 1969.

[2] E. A. Behrens, Zur additiven Idealtheorie in nichtassoziativen Ringen, Math. Z., **64** (1956), 169–182.

[3] N. H. McCoy, The Theory of Rings, Macmillian, New York, 1964.

[4] I. N. Stewart, Lie Algebras, Lecture Notes in Mathematics 127, Springer, Berlin-Heidelberg-New York, 1970.

[5] I. N. Stewart, Infinite-dimensional Lie algebras in the spirit of infinite group theory, Compositio Math., **22** (1970), 313–331.

[6] S. Tôgô, Radicals of infinite dimensional Lie algebras, Hiroshima Math J., **2** (1972), 179–203.

[7] F. H. Vasilescu, Radical d'une algèbre de Lie de dimension infinie, C. R. Acad. Sc. Paris, Série A, **274** (1972), 536–538.

Department of Mathematics,
Faculty of Science,
Hiroshima University