HIROSHIMA MATH. J.
4 (1974), 619–628

Orbit Method and Nondegenerate Series

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(Received May 14, 1974)

1. If G is a reductive Lie group, then its Plancherel formula ([1], [2], [8]) involves a series of representations for each conjugacy class of Cartan subgroups. These "nondegenerate series" are realized [8] by the action of G on square integrable cohomology of partially holomorphic vector bundles over certain G-orbits on complex flag manifolds. That is similar to their realization by the Kostant-Kirillov orbit method using semisimple orbits. The differences occur when G has noncommutative Cartan subgroups, and also for representations with singular infinitesimal character, i.e. when the semisimple orbit is not regular. Recently Wakimoto [6] used possibly-nonsemisimple orbits to realize the principal series, which is the series for a maximally noncompact Cartan subgroup H, when G is a connected semisimple group and H is commutative (e.g. when G is linear). Here we use our method [8] to extend Wakimoto's procedure and realize all but a few members of every nondegenerate series of unitary representation classes for a reductive group. In the case of regular infinitesimal character there is no essential change from [8]. But in the case of singular infinitesimal character we rely on results of Ozeki and Wakimoto ([4], [6]), using nonsemisimple orbits in an interesting way.

To avoid repetition we assume some acquaintance with [8].

2. G will be a reductive Lie group of the class studied in [8] and [9]. Thus its Lie algebra

(2.1a) $\quad \mathfrak{g}=\mathfrak{c}+\mathfrak{g}_1$ with $\mathfrak{c}$ central and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ semisimple,

we assume

(2.1b) $\quad$ if $g \in G$ then $Ad(g)$ is an inner automorphism on $\mathfrak{g}_C$,

and we suppose that G has a closed normal abelian subgroup Z such that

(2.2a) $\quad Z$ centralizes the identity component G_0 of G,

(2.2b) $\quad ZG_0$ has finite index in G, and

(2.2c) $\quad Z \cap G_0$ is co-compact in the center Z_{G_0} of G_0.

Then the adjoint representation maps G to a closed subgroup $\bar{G}=G/Z_G(G_0)$ of

Research partially supported by N.S.F. Grant GP–16651.

the inner automorphism group $\bar{G}_C = \mathrm{Int}(\mathfrak{g}_C)$, where $Z_G(G_0)$ is the G-centralizer of G_0.

By "Cartan involution" of G we mean an involutive automorphism θ whose fixed point set $K = G^\theta$ is the inverse image (under $G \to \bar{G}$) of a maximal compact subgroup of $\bar{G}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$H = \{g \in G,\ \mathrm{Ad}(g)|_{\mathfrak{h}} \text{ is the identity transformation of } \mathfrak{h}\}$$

denotes the corresponding Cartan subgroup of G, then there is a Cartan involution θ of G with $\theta(H) = H$. This splits

(2.3a) $\quad \mathfrak{h} = \mathfrak{t} + \mathfrak{a} \quad$ where $\mathfrak{t} = \{h \in \mathfrak{h} : \theta(h) = h\}$ and

$$\mathfrak{a} = \{h \in \mathfrak{h} : \theta(h) = -h\} \quad \text{and}$$

(2.3b) $\quad H = T \times A$ where $T = H \cap K$ has Lie algebra $\mathfrak{t}$ and $A = \exp(\mathfrak{a})$,

and the G-centralizer of A splits as

(2.4) $\quad Z_G(A) = M \times A$ where $\theta(M) = M$ and M satisfies (2.1) and (2.2).

Let $\Sigma_{\mathfrak{a}}^+$ be a positive $\mathfrak{a}$-root system on $\mathfrak{g}$ and denote

(2.5) $$\mathfrak{n} = \sum_{\alpha \in \Sigma_{\mathfrak{a}}^+} \mathfrak{g}^\alpha \quad \text{and} \quad N = \exp(\mathfrak{n}).$$

The corresponding "cuspidal parabolic" subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and subgroup $P \subset G$ are given by

(2.6) $$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \quad \text{and} \quad P = MAN.$$

T is a Cartan subgroup of M with $T \cap M_0 = T_0$. The object acting as weight lattice is

(2.7a) $\quad \Lambda_{\mathfrak{t}} = \{\nu \in i\mathfrak{t}^* : \nu \text{ exponentiates to a character } \exp(t) \to e^{\nu(t)} \text{ on } T_0\}$.

We replace G by a $\boldsymbol{Z}_2$-extension if necessary so that, for all H and all choices $\Sigma_{\mathfrak{t}}^+$ of positive $\mathfrak{t}_C$-root system on $\mathfrak{m}_C$,

(2.7b) $$\rho_{\mathfrak{t}} = \frac{1}{2} \sum_{\varphi \in \Sigma_{\mathfrak{t}}^+} \varphi \text{ is contained in } \Lambda_{\mathfrak{t}}.$$

The relative discrete series $(M_0)\hat{}_{\mathrm{disc}}$ of unitary representation classes of M_0 is parameterized by

(2.8a) $\quad \Lambda_{\mathfrak{t}}'' = \{\nu \in \Lambda_{\mathfrak{t}} : \nu \text{ is } \mathfrak{m}\text{-regular, i.e. } \langle \nu, \varphi \rangle \neq 0 \text{ for all } \varphi \in \Sigma_{\mathfrak{t}}^+\}$

as follows. If $\nu \in \Lambda_{\mathfrak{t}}''$ denote

(2.8b) $$s_M(\nu) = |\{\text{compact } \varphi \in \Sigma_t^+ : \langle \nu, \varphi \rangle < 0\}| + \\ + |\{\text{noncompact } \varphi \in \Sigma_t^+ : \langle \nu, \varphi \rangle > 0\}| .$$

Then the class $[\eta_\nu] \in (M_0)^{\hat{}}_{\text{disc}}$ for $\nu \in \Lambda_t''$ is the one whose distribution character satisfies

(2.9) $$\Psi_{\eta_\nu}|_{T_0 \cap M''} = (-1)^{s_M(\nu)} \{ \prod_{\varphi \in \Sigma_t^+} (e^{\varphi/2} - e^{-\varphi/2}) \}^{-1} \sum_{W(M_0, T_0)} \det(w) e^{w\nu}$$

where M'' is the regular set and $W(M_0, T_0)$ is the Weyl group. The relative discrete series of

(2.10) $$M^\dagger = \{m \in M : \text{Ad}(m) \text{ is an inner automorphism of } M_0\} \\ = Z_M(M_0) M_0$$

consists of the $[\chi \otimes \eta_\nu]$ where $[\chi] \in Z_M(M_0)^{\hat{}}$ and $[\eta_\nu] \in (M_0)^{\hat{}}_{\text{disc}}$ both restrict to the same unitary character on $Z_{M_0} = Z_M(M_0) \cap M_0$. The relative discrete series $\hat{M}_{\text{disc}}$ of M consists of the classes

(2.11) $$[\eta_{\chi,\nu}] = [\text{Ind}_{M^\dagger \uparrow M}(\chi \otimes \eta_\nu)] \text{ where } [\chi \otimes \eta_\nu] \in (M^\dagger)^{\hat{}}_{\text{disc}}.$$

Finally, the H-series of unitary representation classes of G consists of the

(2.12) $$[\pi_{\chi,\nu,\sigma}] = [\text{Ind}_{P \uparrow G}(\eta_{\chi,\nu} \otimes e^{i\sigma})], \ [\eta_{\chi,\nu}] \in \hat{M}_{\text{disc}} \text{ and } \sigma \in \mathfrak{a}^* .$$

This series depends only on the conjugacy class of H in G, and not on the choice of $\Sigma_{\mathfrak{a}}^+$. The Plancherel measure on $\hat{G}$ is concentrated on the union of the various H-series.

3. Fix a semisimple element $x \in \mathfrak{g}$. Then x is contained in some Cartan subalgebras of $\mathfrak{g}$, and we choose

(3.1) $\mathfrak{h}$: maximally split among the Cartans of $\mathfrak{g}$ that contain x.

With $\mathfrak{h}$ fixed, we choose θ and obtain the splitting (2.3) and (2.4). Now choose

(3.2a) $\Sigma_{\mathfrak{a}}^+$: any positive $\mathfrak{a}$-root system on $\mathfrak{g}$, and

(3.2b) $\Sigma_{\mathfrak{t}}^+$: positive $\mathfrak{t}_C$-root system on $\mathfrak{m}_C$ with $\varphi(ix) \geqslant 0$ for $\varphi \in \Sigma_{\mathfrak{t}}^+$.

These specify a positive $\mathfrak{h}_C$-root system Σ^+ on $\mathfrak{g}_C$ such that

(3.2c) $$\Sigma_{\mathfrak{a}}^+ = \{\gamma|_{\mathfrak{a}} : \gamma \in \Sigma^+ \text{ and } \gamma_{\mathfrak{a}}| \neq 0\} \text{ and} \\ \Sigma_{\mathfrak{t}}^+ = \{\gamma|_{\mathfrak{t}} : \gamma \in \Sigma^+ \text{ and } \gamma|_{\mathfrak{a}} = 0\} .$$

Evidently the centralizer of x in $\mathfrak{g}$ is

(3.3) $$\mathfrak{g}^x = \mathfrak{g} \cap \mathfrak{g}_C^x \text{ where } \mathfrak{g}_C^x = \mathfrak{h}_C + \sum_{\gamma\in\Sigma^+,\gamma(x)=0}(\mathfrak{g}_C^\gamma + \mathfrak{g}_C^{-\gamma}).$$

Ozeki and Wakimoto [4, Lemma 4.4 and its proof] proved

(3.4a) if $\varphi \in \Sigma_t^+$ with $\varphi(x) = 0$, and if $\gamma \in \Sigma^+$ with $\varphi = \gamma|_t$, then $\mathfrak{g}_C^\gamma \subset \mathfrak{k}_C$

where $\mathfrak{k} = \mathfrak{g}^\theta$, Lie algebra of $K = G^\theta$. In other words

(3.4b) $\mathfrak{u} = \mathfrak{g}^x \cap \mathfrak{m}$ is contained in $\mathfrak{k}$.

This says

(3.5) $$e = \sum e_\alpha,\ 0 \neq e_\alpha \in \mathfrak{g}^x \cap \mathfrak{g}^\alpha, \text{ is regular-nilpotent in } \mathfrak{g}^x$$

where the sum runs over {simple $\alpha \in \Sigma_a^+ : \alpha = \gamma|_a$ with $\gamma(x) = 0$}. Now, according to Wakimoto [6, Theorem 3.6],

(3.6) $$\mathfrak{q} = (\mathfrak{t}_C + \sum_{\varphi\in\Sigma_t,\varphi(ix)\geqq 0} \mathfrak{g}_C^\varphi) + \mathfrak{a}_C + \mathfrak{n}_C$$

is a complex polarization of $\mathfrak{g}$ for $x+e$. If τ denotes complex conjugation of $\mathfrak{g}_C$ over $\mathfrak{g}$ then we note

(3.7) $$\mathfrak{q} + \tau\mathfrak{q} = \mathfrak{m}_C + \mathfrak{a}_C + \mathfrak{n}_C = \mathfrak{p}_C \quad\text{and}\quad \mathfrak{q} \cap \tau\mathfrak{q} = \mathfrak{u}_C + \mathfrak{a}_C + \mathfrak{n}_C.$$

In case x is regular, $\mathfrak{g}^x = \mathfrak{h}$, so $e = 0$ and $\mathfrak{q} \cap \tau\mathfrak{q} = \mathfrak{t}_C + \mathfrak{a}_C + \mathfrak{n}_C$.

Lemma 3.8. *The polarization $\mathfrak{q}$ for $x+e$ is* $\mathrm{Ad}(G^{x+e})$*-invariant.*

Proof. We may replace G by $\mathrm{Ad}(G) = G/Z_G(G_0) = \bar{G}$ for the proof, thus assuming $G \subset \mathrm{Int}(\mathfrak{g}_C) = G_C$.

Since x is semisimple, e is nilpotent, and $[x, e] = 0$, the centralizers satisfy $G^{x+e} = G^x \cap G^e = (G^x)^e$.

Observe that $\mathfrak{q} \cap \mathfrak{g}_C^x = \mathfrak{p}^x$, which is a minimal parabolic subalgebra of $\mathfrak{g}^x$. It follows ([3]; see [5]) that $\mathfrak{q} \cap \mathfrak{g}^x$ is an invariant polarization of $\mathfrak{g}^x$ for e. Writing P, P_C and Q for the parabolic subgroups with respective Lie algebras $\mathfrak{p}$, $\mathfrak{p}_C$ and $\mathfrak{q}$, $G^{x+e} = (G^x)^e \subset P^x \subset P_C^x = G_C \cap Q \subset Q$. Thus G^{x+e} normalizes Q. Q.E.D.

4. We briefly recall the orbit method as it would apply to G. Let $y \in \mathfrak{g}$ corresponding to the linear functional $y^*: z \to \langle y, z\rangle$ on $\mathfrak{g}$, and let $\mathfrak{q}$ be a G^y-invariant polarization of $\mathfrak{g}$ for y. Then one has groups

$E = G^y \cdot E_0$ where E_0 is the analytic group for $\mathfrak{e} = (\mathfrak{q} + \tau\mathfrak{q}) \cap \mathfrak{g}$,

$D = G^y \cdot D_0$ where D_0 is the analytic group for $\mathfrak{d} = (\mathfrak{q} \cap \tau\mathfrak{q}) \cap \mathfrak{g}$.

Suppose that y is integral in the sense that

$$\hat{D}_y = \{\text{unitary characters } \xi \text{ on } D: d\xi(z) = i<y, z> \text{ for } z \in \mathfrak{d}\}$$

is not empty. Every $\xi \in \hat{D}_y$ specifies a G-homogeneous complex line bundle.

$$\mathscr{L}_\xi \to G/D \text{ associated to } \xi \otimes e^\rho \text{ where } \rho(z) = \frac{1}{2} \operatorname{trace}_{\mathfrak{g}/\mathfrak{e}} \operatorname{ad}(z)$$

which is holomorphic over every fibre of $G/D \to G/E$. One looks for a corresponding Hodge-Dolbeault theory which will produce Hilbert spaces $H_2^{0,s}(\mathscr{L}_\xi)$ that are square integrable cohomology groups for the cochain complex $\{A^{0,s}(\mathscr{L}_\xi); \bar{\partial}\}$ where

$A^{0,s}(\mathscr{L}_\xi)$: C^∞ objects that are $\mathscr{L}_\xi$-valued $(0, s)$-forms on each gE/D,
$\bar{\partial}$: operator whose every $\mathscr{L}_\xi|_{gE/D}$-restriction is the usual $\bar{\partial}$ there.
If this is done correctly, the natural action of G is
$\pi_{y,\mathfrak{q},\xi,s}$: unitary representation of G on $H_2^{0,s}(\mathscr{L}_\xi)$.
In fact we will modify this general pattern as in [6] and [8], enlarging D and E to contain Cartan subgroups of G. Then the results of [8] will apply directly.

5. We describe our modification of the orbit method as applied to the element $y = x + e \in \mathfrak{g}$ of § 3, and we prove the lemma that allows one to apply the results of [8].

Retain the setup and notation of § 3. Using (3.7) and Lemma 3.8, we consider the groups E and D of § 4 for $y = x + e$, but we replace them by their respective finite extensions

(5.1a) $P^\dagger = M^\dagger AN$ where $M^\dagger = Z_M(M_0)M_0$ as in (2.10), and

(5.1b) $L = UAN$ where $U = G^x \cap M^\dagger$ is in K by (3.4b).

Notice that $P^\dagger = EH_0 = TE_0$ and $L = HD_0 = TD_0$.

Recall $\bar{G} = G/Z_G(G_0) \subset \operatorname{Int}(\mathfrak{g}_C) = \bar{G}_C$. Using the terminology ([7], [8]) of real group orbits on complex flags,

Lemma 5.2. *Let $\bar{Q}$ denote the parabolic subgroup of $\bar{G}_C$ with Lie algebra $\bar{\mathfrak{q}} = ad_{\mathfrak{g}_C}(\mathfrak{q})$, and let X be the complex flag manifold $\bar{G}_C/\bar{Q}$. Then there is a measurable integrable orbit $Y = G(x_0) \subset X$ such that $P^\dagger$ is the G-normalizer of the holomorphic arc component of Y through x_0 and L is the isotropy subgroup of G at x_0.*

Proof. Let $\Pi_\mathfrak{t}$ be the simple $\mathfrak{t}_C$-root system on $\mathfrak{m}_C$ corresponding to $\Sigma_\mathfrak{t}^+$ (3.2b) and let Π be the simple $\mathfrak{h}_C$-root system on $\mathfrak{g}_C$ corresponding to Σ^+ (3.2c). Define

$$\Phi_{\mathfrak{t}} = \{\varphi \in \Pi_{\mathfrak{t}} : \varphi(x) = 0\} \quad \text{and} \quad \Phi = \Phi_{\mathfrak{t}} \cup (\Pi \backslash \Pi_{\mathfrak{t}}) \subset \Pi .$$

Using this data, the construction [8, 6.7.6] gives our algebra $\mathfrak{q}$ and so the assertions follow directly from [8, Proposition 6.7.4] and [8, Corollary 6.7.7].

Q.E.D.

6. We examine the representations of L that give the bundles to which we apply our variation on the orbit method. Those are the elements of

(6.1) $\hat{L}_{x+e} = \{[\lambda] \in \hat{L}:$ for $l \in \mathfrak{l}$, $d\lambda(l)$ is multiplication by $i<x+e, l>\}$

Since $\mathfrak{l} = \mathfrak{u} + \mathfrak{a} + \mathfrak{n} \subset \mathfrak{p}$ and $e \in \mathfrak{n} = \mathfrak{p}^{\perp} \subset \mathfrak{l}^{\perp}$,
if $u \in \mathfrak{u}$, $a \in \mathfrak{a}$ and $n \in \mathfrak{n}$ then $<x+e, u+a+n> = <x, u> + <x, a>$.
Thus we define

(6.2a) $\sigma_x \in \mathfrak{a}^*$ by the property $\sigma_x(a) = <x, a>$ for all $a \in \mathfrak{a}$,

(6.2b) $\nu_x \in \mathfrak{u}^*$ by the property $\nu_x(u) = i<x, u>$ for all $u \in \mathfrak{u}$.

Then of course

(6.3) $\hat{U}_x = \{[\mu] \in \hat{U}: d\mu(u)$ is multiplication by $\nu_x(u)\}$

is nonempty just when ν_x integrates to a character

(6.4) $e^{\nu_x} \in \hat{U}_0$ given by $e^{\nu_x}(\exp u) = e^{\nu_x(u)}$ for $u \in \mathfrak{u}$.

LEMMA 6.5. *$U = Z_M(M_0)U_0$ and $U_0 = U \cap M_0$, so $\hat{U}_x = \{[\chi \otimes e^{\nu_x}]: [\chi] \in Z_M(M_0)^{\wedge}$ and $\chi|_{Z_M(M_0) \cap U_0}$ is a multiple of $e^{\nu_x}\}$.*

PROOF. Recall (5.1). As $x \in \mathfrak{m} + \mathfrak{a}$ we have $Z_M(M_0) \subset G^x$ so $Z_M(M_0) \subset G^x \cap M^{\dagger} = U$. The holomorphic arc component mentioned in Lemma 5.2 is $P^{\dagger}(x_0) \cong P^{\dagger}/L = M^{\dagger}/U = M_0/U \cap M_0$. Since $G(x_0)$ is of flag type [7, Theorem 9.2 (ii)], its holomorphic arc components are simply connected [7, Theorem 5.4]. Thus $U_0 = U \cap M_0$ and it follows that $U = Z_M(M_0)U_0$. Q.E.D.

If $[\lambda] \in \hat{L}_{x+e}$, then $d\lambda(\mathfrak{n}) = 0$, so λ annihilates N, and thus λ is a representation of $UA = U \times A$ lifted to L. Now (6.2), (6.3), (6.4) and Lemma 6.5 give us

PROPOSITION 6.6. *$\hat{L}_{x+e}$ is nonempty just when $e^{\nu_x} \in \hat{U}_0$ is defined, and $\hat{L}_{x+e} = \{[\mu \otimes e^{i\sigma_x}]: [\mu] \in \hat{U}_x\}$.*

Since $Z_M(M_0)$ has a co-compact central subgroup, $Z_M(M_0)^{\wedge}$ consists of finite dimensional classes. If H is commutative, so is $T = \{m \in M: \mathrm{Ad}(m)|_{\mathfrak{t}}$ is the identity on $\mathfrak{t}\}$, which evidently contains $Z_M(M_0)$, so further $Z_M(M_0)^{\wedge}$ consists of 1-dimensional classes. Thus

COROLLARY 6.7. *The representation classes in $\hat{L}_{x+e}$ are finite dimensional. If H is commutative,* e.g. *if G is a connected linear group, then $\hat{L}_{x+e}$ consists of unitary characters.*

7. We produce the bundle, the cohomologies and the representations corresponding to a class $[\lambda]=[\mu\otimes e^{i\sigma_x}]=[\chi\otimes e^{\nu_x}\otimes e^{i\sigma_x}]\in\hat{L}_{x+e}$. Retain the notation of §§ 3, 5 and 6.

Let $\rho_{\mathfrak{a}}=\frac{1}{2}\sum(\dim\mathfrak{g}^{\alpha})\alpha$ where the sum runs over $\Sigma_{\mathfrak{a}}^{+}$. Then $\mathfrak{l}$ acts on $\mathfrak{g}/\mathfrak{l}$ with trace $-2\rho_{\mathfrak{a}}$. Now consider the G-homogeneous complex vector bundle

(7.1) $\quad \mathscr{U}_{\lambda}=\mathscr{U}_{\mu,\sigma_x}\to G/L$ associated to $\lambda\otimes e^{\rho_{\mathfrak{a}}}=\chi\otimes e^{\nu_x}\otimes e^{\rho_{\mathfrak{a}}+i\sigma_x}$.

Every fibre of $G/L\to G/P^{\dagger}$ has a complex structure specified by

(7.2a) $\quad \mathfrak{q}/\mathfrak{l}_C$ is the holomorphic tangent space to $S=P^{\dagger}/L$ at $1\cdot L$

and, viewing gS as the fibre of $G/L\to G/P^{\dagger}$ over $gP^{\dagger}$,

(7.2b) $\quad$ if $g,g'\in G$ then $g\colon g'S\to(gg')S$ is holomorphic.

Just as in [8, Lemma 8.1.5], now

(7.3a) $\quad$ each $\mathscr{U}_{\mu,\sigma_x}|_{gS}$ is an $\mathrm{Ad}(g)P^{\dagger}$-homogeneous holomorphic bundle

in such a way that

(7.3b) $\quad$ if $g,g'\in G$ then $g\colon\mathscr{U}_{\mu,\sigma_x}|_{g'S}\to\mathscr{U}_{\mu,\sigma_x}|_{gg'S}$ is holomorphic.

It also defines a G-homogeneous vector bundle

(7.4) $\quad \mathscr{T}\to G/L$ such that $\mathscr{T}|_{gS}$ is the holomorphic tangent bundle of gS.

We now have G-homogeneous bundles $\mathscr{U}_{\mu,\sigma_x}\otimes\Lambda^r(\mathscr{T}^*)\otimes\Lambda^s(\bar{\mathscr{T}}^*)$, $0\leqslant r$, $s\leqslant n=\dim_C S$, whose sections are the "$\mathscr{U}_{\mu,\sigma_x}$-valued partial (r,s)-forms on G/L." The $\bar{\partial}$-operators of the $\mathscr{U}_{\mu,\sigma_x}|_{gS}$ fit together to give first order operators on the spaces of C^{∞} $\mathscr{U}_{\mu,\sigma_x}$-valued partial (r,s)-forms, which we denote

$$\bar{\partial}\colon A^{r,s}(\mathscr{U}_{\mu,\sigma_x})\to A^{r,s+1}(\mathscr{U}_{\mu,\sigma_x}). \tag{7.5}$$

The representations $\pi_{x+e,\mathfrak{q},\lambda,s}$ of G are supposed to be unitary representations of G on square integrable cohomology spaces of the complex $\{A^{0,s}(\mathscr{U}_{\mu,\sigma_x});\bar{\partial}\}$.

Comparing our spaces, bundles and complex structures with those of [8, § 8], we identify G/L with the orbit $Y=G(x_0)\subset X$ of Lemma 5.2 and the fibres gS of $G/L\to G/P^{\dagger}$ with the holomorphic arc components of Y, with complex structures on the gS induced by X and partial holomorphic structure on $\mathcal{U}_{\mu,\sigma_x}$ the same as that of [8, Lemma 8.1.5]. Thus, square integrable cohomology spaces of the cochain complex $\{A^{0,s}(\mathcal{U}_{\mu,\sigma_x});\ \bar\partial\}$ are provided by the Hilbert spaces

$$(7.6)\qquad H_2^{0,s}(\mathcal{U}_{\mu,\sigma_x}):\ \begin{cases}\mathcal{U}_{\mu,\sigma_x}\text{-valued square integrable partially}\\ \text{harmonic } (0,s)\text{-forms on } G/L \text{ as in [8, § 8. 1].}\end{cases}$$

on which G has a natural action [8, 8.1.10],

$$(7.7)\qquad \pi^s_{\mu,\sigma_x}:\ \text{unitary representation of } G \text{ on } H_2^{0,s}(\mathcal{U}_{\mu,\sigma_x}).$$

Now the desired $\pi_{x+e,\mathfrak{q},\lambda,s}$ for our modification of the orbit method, are just the π^s_{μ,σ_x} of [8, § 8.1].

8. We recall the main result of [8], which more or less identifies the $\pi_{x+e,\mathfrak{q},\lambda,s}$ $=\pi^s_{\mu,\sigma_x}$ in terms of the H-series classes described above in § 2.

Let $x\in\mathfrak{g}$ and retain the notation of §§ 3 through 7. Suppose that e^{ν_x} exists. As $\varphi(ix)\geqslant 0$ and $<\varphi,\rho_{\mathfrak{t}}>\ >0$ for all $\varphi\in\Sigma_{\mathfrak{t}}^+$, we have

$$(8.1)\qquad \begin{aligned}&\nu_x+\rho_{\mathfrak{t}}\in\Lambda_{\mathfrak{t}}'' \text{ with}\\ &s_M(\nu_x+\rho_{\mathfrak{t}}) = |\{\varphi\in\Sigma_{\mathfrak{t}}^+:\quad \varphi \text{ is noncompact}\}|\,.\end{aligned}$$

Since $\nu_x+\rho_{\mathfrak{t}}\in\Lambda_{\mathfrak{t}}''$, [8, Theorem 8.3.4] applies. It says that the sum ${}^H\pi^s_{\mu,\sigma_x}$ of the H-series constituents of π^s_{μ,σ_x} is the (discrete) direct sum of the irreducible subrepresentations of π^s_{μ,σ_x}, that it has a well-defined distribution character $\Theta({}^H\pi^s_{\mu,\sigma_x})$ and that the alternating sum of those characters is an H-series character

$$(8.2)\qquad \sum_{s\geqslant 0}(-1)^s\Theta({}^H\pi^s_{\mu,\sigma_x}) = (-1)^{|\Sigma_{\mathfrak{t}}^+|+s_M(\nu_x+\rho_{\mathfrak{t}})}\Theta(\pi_{\chi,\nu_x+\rho_{\mathfrak{t}},\sigma_x})$$

Further, $[\mathfrak{m},\mathfrak{m}]$ determines a constant $b_H\geqslant 0$ such that

$$(8.3)\qquad \begin{cases}\text{if } |<\nu_x+\rho_{\mathfrak{t}},\varphi>|>b_H \text{ for all } \varphi\in\Sigma_{\mathfrak{t}}^+\\ \text{then } H_2^{0,s}(\mathcal{U}_{\mu,\sigma_x}) = 0 \text{ for } s\neq s_M(\nu_x+\rho_{\mathfrak{t}}) \text{ and}\\ [\pi^{s_M(\nu_x+\rho_{\mathfrak{t}})}_{\mu,\sigma_x}] = [\pi_{\chi,\nu_x+\rho_{\mathfrak{t}},\sigma_x}]\,.\end{cases}$$

In other words, $[\pi_{\chi,\nu_x+\rho_{\mathfrak{t}},\sigma_x}]$ always is a subrepresentaion of the $[\pi_{x+e,\mathfrak{q},\lambda,s}]$, $[\lambda]=[\chi\otimes e^{\nu_x}\otimes e^{i\sigma_x}]\in\hat{L}_{x+e}$, obtained from our variation on the orbit method. And if $<\nu_x+\rho_{\mathfrak{t}},\varphi>\ >b_H$ for all $\varphi\in\Sigma_{\mathfrak{t}}^+$, then

$$(8.4)\qquad [\pi_{x+e,\mathfrak{q},\lambda,s_M}] = [\pi_{\chi,\nu_x+\rho_{\mathfrak{t}},\sigma_x}] \text{ where}$$

$$s_M = |\{\varphi \in \Sigma_{\mathfrak{t}}^+ : \quad \varphi \text{ is noncompact}\}|.$$

9. We reformulate the discussion of § 8, realizing the various nondegenerate series of G by the modified orbit method.

THEOREM 9.1. *Let H be a Cartan subgroup of G and $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$ an H-series representation class such that*

(9.2) *if φ is a noncompact $\mathfrak{t}_C$-root of $\mathfrak{m}_C$ then $<\varphi, \nu> \neq 0$.*

Define $x \in \mathfrak{h}$ by $\nu = \nu_x$ and $\sigma = \sigma_x$, that is

(9.3) *$\nu(t) = i<x, t>$ for $t \in \mathfrak{t}$ and $\sigma(a) = <x, a>$ for $a \in \mathfrak{a}$.*

Then $\mathfrak{h}$ is maximally split among the Cartan subalgebras of $\mathfrak{g}$ that contain x. Let e be a regular-nilpotent element of $\mathfrak{g}^x$ and consider the representations

$$\pi_{x+e,\mathfrak{q},\lambda,s}, \qquad [\lambda] = [\chi \otimes e^{\nu} \otimes e^{i\sigma}] \in \hat{L}_{x+e},$$

of §§ 6 and 7.

1. *$[\pi_{\chi,\nu+\rho\mathfrak{t},\sigma}]$ is implicitly realized on the orbit of $x+e$ as a subrepresentation of an $[\pi_{x+e,\mathfrak{q},\lambda,s}]$, $0 \leqslant s \leqslant \frac{1}{2} \dim_R M^{\dagger}/U$.*

2. *If the roots are ordered as in (3.2), and if for every $\varphi \in \Sigma_{\mathfrak{t}}^+$ the nonnegative number $<\nu+\rho_{\mathfrak{t}}, \varphi>$ is $> b_H$, then $[\pi_{\chi,\nu+\rho\mathfrak{t},\sigma}]$ is explicitly realized on the orbit of $x+e$ by*

(9.4) $$[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}] = [\pi_{x+e,\mathfrak{q},\lambda,s_M}] \text{ where}$$

$$s_M = |\{\varphi \in \Sigma_{\mathfrak{t}}^+ : \quad \varphi \text{ is noncompact}\}|.$$

In the case of the principal series, every $\mathfrak{t}_C$-root of $\mathfrak{m}_C$ is compact, so (9.2) is automatic and $s_M = 0$. Also, there $b_H = 0$. Thus we recover Wakimoto's result [6, Theorem 6.6] as the case where G is a connected semisimple Lie group and H is commutative in

COROLLARY 9.5. *Let $[\pi_{\chi,\nu+\rho\mathfrak{t},\sigma}]$ be a principal series representation class of G, that is an H-series class where H is a maximally split Cartan subgroup of G. Define $x \in \mathfrak{h}$ by (9.3), let e be a regular-nilpotent element of $\mathfrak{g}^x$, and suppose that the roots are ordered as in (3.2). Then $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$ is realized on the orbit of $x+e$ as the representation $[\pi_{x+e,\mathfrak{q},\chi\otimes e^{\nu}\otimes e^{i\sigma},0}]$ of G on square integrable partially holomorphic sections of $\mathscr{U}_{\chi\otimes e^{\nu},\sigma} \to G/L$.*

Finally we note that if H is not maximally split, i.e. if the H-series is not the principal series, then $\Sigma_{\mathfrak{t}}^{+}$ does contain a noncompact root, so the H-series classes $[\pi_{\chi,\rho_{\mathfrak{t}},\sigma}]$ do not satisfy (9.2) and thus are not realized by the procedure of Theorem 9.1.

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