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A New Family in the Stable Homotopy Groups of Spheres

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Introduction

Let G_k denote the k-th stable homotopy group $\mathrm{Dir}\lim \pi_{N+k}(S^N)$ of spheres. J. F. Adams [0] and H. Toda [9] discovered a family $\{\alpha_t \in G_{tq-1}, t \geqq 1\}$, $q=2(p-1)$, of elements of order p, for every odd prime p, and later on L. Smith [6] and H. Toda [11] discovered another family $\{\beta_t \in G_{(tp+t-1)q-2}, t \geqq 1\}$ of elements of order p, for every prime $p \geqq 5$. Our main results concern the second family.

THEOREM A. *For every prime $p \geqq 5$ and $t \geqq 1$, there exist $p-1$ elements*

$$\rho_{t,r} \in G_{(tp^2+(t-1)p+r)q-2}, \qquad r = 1, 2, \cdots, p-1,$$

of order p such that

$$\rho_{t,r+s} \in \langle \rho_{t,r}, p, \alpha_s \rangle \qquad \textit{for} \quad r+s \leqq p-1$$

and that the last element $\rho_{t,p-1}$ coincides with the element β_{tp} of L. Smith [6] *and H. Toda* [11]. *Here, $q=2(p-1)$ and $\langle\ ,\ ,\ \rangle$ denotes the stable Toda bracket.*

For $t=1$, this family $\{\rho_{1,r}\}$ coincides with the family $\{\varepsilon_r \in G_{(p^2+r)q-2}, 1 \leqq r \leqq p-1\}$ constructed in [3].

Let M be a Moore space $S^1 \cup_p e^2$ and denote by $\mathscr{A}_k(M)$ the limit group $\mathrm{Dir}\lim [S^{N+k}M, S^N M]$. Let $i: S^1 \to M$ and $\pi: M \to S^2$ be the natural maps and consider the induced homomorphism $\pi_* i^*: \mathscr{A}_k(M) \to G_{k-1}$.

There exists uniquely an element $\alpha \in \mathscr{A}_q(M)$, $q=2(p-1)$, such that $\pi_* i^* \alpha = \alpha_1$, and also there exists a family $\{\beta_{(t)} \in \mathscr{A}_{(tp+t-1)q-1}, t \geqq 1\}$ of $\mathscr{A}_*(M)$ which satisfies $\alpha\beta_{(t)} = \beta_{(t)}\alpha = 0$ and $\beta_{(t)} \in \langle \beta_{(t-1)}, \alpha, \beta_{(1)} \rangle$ [11] (cf. [4]). This family is closely connected with the family $\{\beta_t\}$ via the equality $\pi_* i^* \beta_{(t)} = \beta_t$, and our next results are related to the *α-divisibility* of the elements $\beta_{(tp)}$, $t \geqq 1$.

We constructed in [4] the element ε of $\mathscr{A}_{(p^2+1)q-1}(M)$, which is a generator of the ring $\mathscr{A}_*(M)$. The element $\pi_* i^* \varepsilon$ generates the p-component of $G_{(p^2+1)q-2}$ and there is a relation $\varepsilon\alpha^{p-2} = \alpha^{p-2}\varepsilon = \beta_{(p)}$. Also we defined in [4] a differential D on $\mathscr{A}_*(M)$ of degree $+1$, originally due to P. Hoffman. D is a derivation and the subring $\mathrm{Ker}\, D$ is commutative in graded sense. Our elements α, $\beta_{(t)}$ and ε

belong to this kernel and are commutative to each other.

THEOREM B. *Let p be a prime $\geqq 5$ and set $q=2(p-1)$. Then there exist non zero elements*

$$\rho(t) \in \mathscr{A}_{(tp^2+(t-1)p+1)q-1}(M), \qquad t = 1, 2, \cdots,$$

satisfying the following relations:

(i) $\rho(1) = \varepsilon$,

(ii) $\rho(t)\alpha^{p-2} = \alpha^{p-2}\rho(t) = \beta_{(tp)}$,

(iii) $\rho(t)\alpha^{p-1} = \alpha^{p-1}\rho(t) = 0$,

(iv) $\rho(t+u) \in \langle \rho(t), \alpha^{p-1}, \rho(u) \rangle$,

(v) $\pi_* i^* \rho(t) = \rho_{t,1}$,

(vi) $D(\rho(t)) = 0$,

where $\rho_{t,1}$ is the element in Theorem A.

Let $X(r)=S^2M \cup CS^{rq+2}M$, $q=2(p-1)$, be a mapping cone of an element which represents the element $\alpha^r \in \mathscr{A}_{rq}(M)$, and set $\mathscr{A}_k(X(r))=\mathrm{Dir\,lim}\,[S^{N+k}X(r), S^N X(r)]$. Let $j_r\colon S^2M \to X(r)$ be the inclusion, $k_r\colon X(r) \to S^{rq+3}M$ be the projection, and $A\colon S^q X(r) \to X(r+1)$ and $B\colon X(r+1) \to X(r)$ be the maps naturally defined from α.

There exists uniquely an element $\beta \in \mathscr{A}_{(p+1)q}(X(1))$ such that $k_{1*} j_1^* \beta = \beta_{(1)}$ ([6] and [11]), and this defines the family $\{\beta_{(t)}\}$ by the rule $\beta_{(t)} = k_{1*} j_1^* \beta^t$. Then, the following result enables us to construct our elements $\rho_{t,r}$ and $\rho(t)$.

THEOREM C. *Let p be a prime $\geqq 5$ and $q=2(p-1)$. Then there exist uniquely the elements*

$$R(r) \in \mathscr{A}_{(p^2+p)q}(X(r)), \qquad r = 1, 2, \cdots, p-1,$$

satisfying the following conditions:

(a) $R(1) = \beta^p$;

(b) $AR(r-1) = R(r)A$;

(c) $R(r-1)B = BR(r)$;

(d) $\theta(R(r)) = 0$;

(e) $k_{r*} j_r^* R(r) = \varepsilon\alpha^{p-1-r}$;

where θ is a differential on $\mathscr{A}_*(X(r))$ introduced by H. Toda [11] *(for the definition, see* § 1).

Then, the elements $\rho_{t,r}$ and $\rho(t)$ in Theorems A and B are defined by the compositions:

$$\rho(t)\colon S^{N+at+2}M \xrightarrow{j_{p-1}} S^{N+at}X(p-1) \xrightarrow{R(p-1)} \cdots$$

$$\cdots \xrightarrow{R(p-1)} S^N X(p-1) \xrightarrow{k_{p-1}} S^{N+(p-1)q+3}M,$$

$$\rho_{t,r}\colon S^{N+at+3} \xrightarrow{i} S^{N+at+2}M \xrightarrow{j_{p-r}} S^{N+at}X(p-r) \xrightarrow{R(p-r)} \cdots$$
$$\cdots \xrightarrow{R(p-r)} S^N X(p-r) \xrightarrow{k_{p-r}} S^{N+(p-r)q+3}M \xrightarrow{\pi} S^{N+(p-r)q+5},$$

where $a=(p^2+p)q$.

The following result is useful to prove the non-triviality of our elements.

THEOREM D. *The complex bordism module $\tilde{\Omega}^U_*(Z(r))$ of the mapping cone $Z(r)$ of $R(r)\colon S^{N+a}X(r)\to S^N X(r)$, $a=(p^2+p)q$, is isomorphic, as an Ω^U_*-module, to*

$$\Omega^U_*/(p, [CP(p-1)]^r, [V]^p+[M_r])\cdot\zeta(r), \deg\zeta(r) = N+3,$$
$$\textit{for some} \quad [M_r]\in(p, [CP(p-1)]) \cap \Omega^U_a,$$

where $[V]\in\Omega^U_{2(p^2-1)}$ is the class of the Milnor manifold for the prime p.

This result is closely related to a problem on the realization of Ω^U_*-modules proposed by L. Smith [7].

Our elements in Theorems A–C are constructed starting from the element β. We can also consider a similar procedure of constructing elements with the initial element $R(p-1)$. We obtain the following deeper results than the previous theorems with $t\equiv 0 \bmod p$.

THEOREM A′. *For every prime $p\geqq 5$ and $t\geqq 1$, there exist elements*

$$\rho'_{tp,r}\in G_{(tp^3+tp^2-2p+r+1)q-2}, \qquad 1\leqq r\leqq 2p-2,$$

of order p such that

$$\rho'_{tp,r+s}\in\langle\rho'_{tp,r}, p, \alpha_s\rangle \qquad \textit{for} \quad r+s\leqq 2p-2,$$
$$\rho'_{tp,r} = \rho_{tp,r-p+1} \qquad \textit{for} \quad p\leqq r\leqq 2p-2,$$
$$\rho'_{tp,2p-2} = \beta_{tp^2}.$$

THEOREM B′. *For every prime $p\geqq 5$ and $t\geqq 1$, there exist non zero elements*

$$\rho'(tp)\in\mathscr{A}_{(tp^3+tp^2-2p+2)q-1}(M)$$

such that

$$\rho'(tp)\alpha^{p-1} = \alpha^{p-1}\rho'(tp) = \rho(tp),$$
$$\rho'(tp)\alpha^{2p-3} = \alpha^{2p-3}\rho'(tp) = \beta_{(tp^2)},$$
$$\rho'(tp+up)\in\langle\rho'(tp), \alpha^{2p-2}, \rho'(up)\rangle.$$

THEOREM C′. *There exist elements*

$$R'(r)\in\mathscr{A}_{(p^3+p^2)q}(X(r)) \qquad \textit{for} \quad p\leqq r\leqq 2p-2$$

such that

$$AR'(r-1) = R'(r)A \qquad \text{for} \quad p+1 \leqq r \leqq 2p-2,$$

$$AR(p-1)^p = R'(p)A.$$

THEOREM D′. *Let $Z'(r)$ be the mapping cone of $R(r)^p$ for $1 \leqq r \leqq p-1$ and of $R'(r)$ for $p \leqq r \leqq 2p-2$. Then, as an Ω_*^U-module,*

$$\tilde{\Omega}_*^U(Z'(r)) \approx \Omega_*^U/(p, [CP(p-1)]^r, [V]^{p^2}+[M_r])$$

for some $[M_r] \in (p, [CP(p-1)])$ of dimension $(p^3+p^2)q$.

Our argument of the non-triviality of the elements $\rho_{t,r}$, $\rho'_{tp,r}$, $\rho(t)$, $\rho'(tp)$ is a modification of the technique of $\beta_t \neq 0$ employed by L. Smith [6]. We are concerned with the Hurewicz homomorphism

$$h: \Omega_*^{fr}(Y(r)) \longrightarrow \Omega_*^U(Y(r))$$

for the subcomplex $Y(r)$ of $X(r)$ obtained by removing the top cell of $X(r)$. For $r=1$, $Y(1)$ is the complex $V(1/2)$ considered in [6] and the fact that the image of h for $V(1/2)$ is trivial played an important role in Smith's method. In our proof, a similar fact for $r \geqq 2$ will lead us to the non-triviality.

In § 1 we shall construct the complexes $X(r)$, and in § 2 we shall construct the elements $R(r)$ and prove Theorem C. In § 3, the Ω_*^U-module structures of $X(r)$, $Y(r)$ and $R(r)$ will be studied and Theorem D will be proved. Theorems A and B will be proved in § 4, and the proofs of the lemmas in § 2 will be given in § 5. In § 6, Theorems A′–D′ will be proved, and several relations in $\mathscr{A}_*(M)$ involving the elements $\beta_{(t)}$ and $\rho(t)$ will be obtained in § 7.

§ 1. Z_p-spaces $X(r)$

In this paper, p denotes always a fixed prime integer $\geqq 5$ and we set $q = 2(p-1)$.

For any finite CW-complexes X and Y, we denote by

$$\{X, Y\}_t = \operatorname{Dir\,lim}\,[S^{N+t}X, S^N Y]$$

the t-th stable track group, and set

$$\mathscr{A}_t(X) = \{X, X\}_t,$$

where $S^n X$ is the n-fold suspension of X and $[X, Y]$ is the set of homotopy classes of maps from X to Y.

Let

$$M = S^1 \cup {}_p e^2$$

be a Moore space of type $(Z_p, 1)$, and

$$i: S^1 \longrightarrow M \quad \text{and} \quad \pi: M \longrightarrow S^2$$

be the natural maps. The group $\mathscr{A}_k(M)$ and the ring $\mathscr{A}_*(M)=\Sigma_k\mathscr{A}_k(M)$ are studied in [12], [11] and [4], and the notations and the results of these works are used in this paper. Let

$$\alpha \in \mathscr{A}_q(M) = Z_p, \qquad q = 2(p-1),$$

be the generator detected by the P^1 operation. The element

$$\alpha_1 = \pi\alpha i \in G_{q-1}$$

is the first element of order p in G_*.

We consider the unstable version of these elements. Let $S^\infty: [S^{N+t}X, S^N Y] \to \{X, Y\}_t$ denote the natural homomorphism. By a well known result of J.-P. Serre, there exists an element $\alpha_1(3) \in \pi_{q+2}(S^3)$ of order p such that $S^\infty\alpha_1(3)=\alpha_1$ and $\alpha_1(3) \notin S\pi_{q+1}(S^2)$. Then we can construct an element

$$a \in [S^{q+2}M, S^2M]$$

of order p satisfying

$$S^\infty a = \alpha \quad \text{and} \quad a \notin S[S^{q+1}M, SM]$$

(cf. [10; pp. 112–113]).

Definition 1.1. *Let $r \geqq 1$. We put $a^{(1)}=a$ and $a^{(r)}=a\circ S^q a\circ\cdots\circ S^{(r-1)q}a$ (composition of r elements) for $r\geqq 2$, and we define complexes $X(r)$ by*

$$X(r) = S^2M \cup_{a^{(r)}} CS^{rq+2}M$$

the mapping cone of the element $a^{(r)} \in [S^{rq+2}M, S^2M]$.

The element $a^{(r)}$ above represents the power $\alpha^r \in \mathscr{A}_{rq}(M)$ of α, and we have a sequence of cofiberings:

$$(1.2) \qquad S^{rq+2}M \xrightarrow{\alpha^r} S^2M \xrightarrow{j_r} X(r) \xrightarrow{k_r} S^{rq+3}M.$$

Convention. We do not distinguish between a map and its stable class, unless otherwise specified.

We then receive the elements

$$i \in \{S^0, M\}_1, \qquad \pi \in \{M, S^0\}_{-2},$$

$$j_r \in \{M, X(r)\}_2, \qquad k_r \in \{X(r), M\}_{-rq-3}.$$

By (1.2), we have the following exact sequences for any finite CW-complex K:

(1.3) $\cdots \to \{K, M\}_{t-rq-2} \xrightarrow{\alpha^r{}_*} \{K, M\}_{t-2} \xrightarrow{j_{r*}} \{K, X(r)\}_t \xrightarrow{k_{r*}} \{K, M\}_{t-rq-3} \to \cdots;$

(1.3)* $\cdots \to \{M, K\}_{t+3} \xrightarrow{\alpha^{r*}} \{M, K\}_{t+rq+3} \xrightarrow{k_r^*} \{X(r), K\}_t \xrightarrow{j_r^*} \{M, K\}_{t+2} \to \cdots.$

By definition, we can construct the elements

$$A \in \{X(r), X(r+1)\}_q \quad \text{and} \quad B \in \{X(r+1), X(r)\}_0$$

such that

$$\text{(1.4)} \qquad \begin{aligned} Aj_r &= j_{r+1}\alpha, & k_r &= k_{r+1}A, \\ j_r &= Bj_{r+1}, & k_rB &= \alpha k_{r+1}. \end{aligned}$$

LEMMA 1.5. *Let $r \geqq 1$ and $s \geqq 1$, and set*

$$\Delta_{r,s}(=\Delta) = j_s k_r \in \{X(r), X(s)\}_{-rq-1}.$$

Then the following is homotopy equivalent to a sequence of cofiberings:

$$S^{sq}X(r) \xrightarrow{A^s} X(r+s) \xrightarrow{B^r} X(s) \xrightarrow{\Delta} S^{sq+1}X(r).$$

PROOF. Put $M_1 = S^2M$, $M_2 = S^{sq+2}M$, $M_3 = S^{(r+s)q+2}M$, $f = a^{(s)}: M_2 \to M_1$ and $g = S^{sq}a^{(r)}: M_3 \to M_2$. Then, $S^{sq}X(r)$, $X(r+s)$ and $X(s)$ are the mapping cones of g, fg and f, respectively, and A^s and B^r are given by $A^s(m_2) = f(m_2) \in M_1$, $A^s((t, m_3)) = (t, m_3) \in CM_3$ and $B^r(m_1) = m_1 \in M_1$, $B^r((t, m_3)) = (t, g(m_3)) \in CM_2$, where $m_i \in M_i$ and $0 \leqq t \leqq 1$ with the identifications $(0, m_3) = g(m_3)$ in $S^{sq}X(r)$, $(0, m_3) = fg(m_3)$ in $X(r+s)$, $(0, m_2) = f(m_2)$ in $X(s)$, and $(1, m_i) = *$ for all cases. Define a homotopy $H_\theta: S^{sq}X(r) \to X(s)$ between $H_0 = B^rA^s$ and $H_1 = *$ by the rule

$$H_\theta(m_2) = (\theta, m_2) \qquad H_\theta((t, m_3)) = (\max\{\theta, t\}, g(m_3)).$$

Then H_θ defines an extension $E: X'(s) \to X(s)$ of B^r, where $X'(s)$ is the mapping cone of A^s. Apparently we can regard $X(s)$ as a subcomplex of $X'(s)$. We see easily that E is a deformation retract and that the composition of the natural shrinking map $X'(s) \to X'(s)/X(r+s) = S^{rq+1}X(r)$ with the inclusion $X(s) \subset X'(s)$ coincides with Δ. Q.E.D.

From the lemma, we have the following exact sequences for any finite CW-complex K:

(1.6) $\cdots \to \{K, X(r)\}_{t-q} \xrightarrow{A_*} \{K, X(r+1)\}_t \xrightarrow{B^r{}_*} \{K, X(1)\}_t \xrightarrow{\Delta_{r*}} \{K, X(r)\}_{t-q-1} \to \cdots;$

(1.6)* $\cdots \to \{X(1), K\}_t \xrightarrow{B^{r*}} \{X(r+1), K\}_t \xrightarrow{A^*} \{X(r), K\}_{t+q} \xrightarrow{\Delta_r^*} \{X(1), K\}_{t-1} \to \cdots;$

where

(1.7) $$\varDelta_r = \varDelta_{1,r} = j_r k_1 \in \{X(1), X(r)\}_{-q-1}$$

and this satisfies

(1.7)′ $$B^s \varDelta_{r+s} = \varDelta_r \quad \textit{for} \quad r, s \geqq 1\,.$$

CONVENTION. We define an n-fold suspension S^nX of a based space X by a smash product $S^n \wedge X$, not by $X \wedge S^n$, and also S^nf by $1 \wedge f$, $1=$the identity map of S^n, for a based map f.

Following H. Toda [11; p. 207], we introduce the following

DEFINITION. *A space X is called a Z_p-space, if there exist two elements $\mu_X \in \{M \wedge X, X\}_{-1}$ and $\phi_X \in \{X, M \wedge X\}_2$ satisfying the equalities*

$$\mu_X\phi_X = 0, \quad \mu_X(i \wedge 1_X) = 1_X, \quad (\pi \wedge 1_X)\phi_X = 1_X,$$

$$(i \wedge 1_X)\mu_X + \phi_X(\pi \wedge 1_X) = 1_{M\wedge X}\,.$$

This condition is equivalent to $p1_X=0$ in $\mathscr{A}_0(X)$ ([11; Lemma 1.2]) and S^nX is a Z_p-space if and only if X is a Z_p-space.

H. Toda then defined an operation

(1.8) $$\theta\colon \{X, Y\}_t \longrightarrow \{X, Y\}_{t+1} \qquad \textit{by} \quad \theta(\gamma) = \mu_Y(1_M \wedge \gamma)\phi_X$$

for Z_p-spaces X and Y, and proved the following results:

PROPOSITION 1.9 ([11; Prop. 2.1, Th. 2.2 and Lemma 2.3]). *Let W, X and Y be Z_p-spaces, and $\gamma \in \{X, Y\}_t$ and $\gamma' \in \{W, X\}_s$ be elements. Then*

(i) *θ is a derivation*: $\theta(\gamma\gamma')=\theta(\gamma)\gamma'+(-1)^t\gamma\theta(\gamma')$.

(ii) *If $\mathscr{A}_1(X) = \mathscr{A}_2(X) = \mathscr{A}_1(Y) = \mathscr{A}_2(Y) = 0$, then θ is a differential*: $\theta\theta(\gamma)=0$.

(iii) *Let $Z=S^nY \cup_f CS^{n+t}X$ be a mapping cone of a representative $f\in [S^{n+t}X, S^nY]$ of γ. Then Z is a Z_p-space if $\theta(\gamma)=0$, and conversely $\theta(\gamma)=0$ if Z is a Z_p-space and $\{Y, X\}_{-t}=\mathscr{A}_1(X)=\mathscr{A}_1(Y)=0$.*

Also he defined

(1.10) $$\lambda_X\colon \mathscr{A}_t(M) \longrightarrow \mathscr{A}_{t+1}(X) \qquad \textit{by} \quad \lambda_X(\xi) = \mu_X(\xi \wedge 1_X)\phi_X$$

for a Z_p-space X, and proved the following

PROPOSITION 1.11 ([11; Cor. 2.5]). *Let X and Y be Z_p-spaces. Then*

$$\lambda_Y(\xi)\gamma = (-1)^{(s+1)t}\gamma\lambda_X(\xi)$$

for any $\xi \in \mathscr{A}_s(M)$ *and* $\gamma \in \{X, Y\}_t$ *with* $\theta(\gamma)=0$.

Now we have defined in the previous paper [4; (1.6)] an operation

$$(1.10)' \qquad D: \mathscr{A}_t(M) \longrightarrow \mathscr{A}_{t+1}(M).$$

This is a special case of (1.10) by the following

PROPOSITION 1.12 (cf. [11; Th. 2.6, Cor. 2.7 and (3.7)]). *The Moore space* M *is a* Z_p*-space and* $\lambda_M = D$. *For* $X=Y=M$, *the operation* θ *of* (1.8) *coincides with* $-D$ *of* (1.10)′. *The element* $\delta = i\pi \in \mathscr{A}_{-1}(M)$ *satisfies* $D(\delta)=1_M$, *and the element* α *satisfies* $D(\alpha)=0$.

From the above results, we obtain easily the following

PROPOSITION 1.13. *Let* $r, s<2p$. *Then* $X(r)$ *is a* Z_p*-space with* $\mathscr{A}_1(X(r))=\mathscr{A}_2(X(r))=0$, *and* $\theta(j_r)=\theta(k_r)=\theta(A)=\theta(B)=0$. *The operations* θ *on* $\mathscr{A}_*(M)$, $\{M, X(r)\}_*$, $\{X(r), M\}_*$ *and* $\{X(r), X(s)\}_*$ *are differentials. For any finite CW-complex* K, *the following groups are linear spaces over* Z_p:

$$\{K, M\}_t, \quad \{M, K\}_t, \quad \{K, X(r)\}_t, \quad \{X(r), K\}_t.$$

PROOF. Since $X(r)$ is a mapping cone of a map which represents α^r, and since $\theta(\alpha^r)=0$ by Propositions 1.9 (i) and 1.12, it follows from Proposition 1.9 (iii) that $X(r)$ is a Z_p-space.

Now we have from [12]

$$\begin{aligned}
\mathscr{A}_i(M) &= 0 && \textit{for} \quad i<q-2,\ i\neq -1, 0,\\
\mathscr{A}_i(M) &= Z_p\{\alpha^r\} && \textit{for} \quad i=rq,\ 0\leqq r<2p,\\
\mathscr{A}_i(M) &= 0 && \textit{for} \quad i=rq+1,\ rq+2,\ rq+3,\ 0\leqq r<2p.
\end{aligned}$$

Then by an easy calculation using (1.3) and (1.3)*, we see $\mathscr{A}_1(X(r))=\mathscr{A}_2(X(r))=0$. Hence the operations θ above are differentials by Proposition 1.9 (ii). By (1.2), $S^{rq+3}M$ is a mapping cone of j_r, and so $\theta(j_r)=0$ by Proposition 1.9 (iii). In the same way, we have $\theta(k_r)=0$ and also $\theta(A)=\theta(B)=0$ from Lemma 1.5. For any Z_p-space X, the groups $\{K, X\}_t$ and $\{X, K\}_t$ are linear spaces over Z_p, and in particular the last assertion follows. Q.E.D.

§2. Construction of the elements $R(r)$ of $\mathscr{A}_*(X(r))$

L. Smith [6] and H. Toda [11] considered the spectra $V(k)$ $(k=0, 1, 2)$ which satisfy $\tilde{\Omega}_*^U(V(k)) \approx \Omega_*^U/(p, [V_1], \cdots, [V_k])$ as an Ω_*^U-module, or equivalently, $H^*(V(k); Z_p) \approx E(Q_0, \cdots, Q_k)$ as a module over the Steenrod algebra, where V_j

is the Milnor manifold of dimension $2(p^j-1)$ for the prime p, and Q_j is Milnor's element of degree $2p^j-1$ in the Steenrod algebra mod p.

By definitions and [11; p. 217], the spaces S^nM and $S^nX(1)$ are the $(n+1)$-th and the $(n+3)$-th components of the spectra $\boldsymbol{V}(0)$ and $\boldsymbol{V}(1)$, respectively, and there exists

$$\beta\colon S^{N+(p+1)q}X(1) \longrightarrow S^NX(1) \tag{2.1}$$

for a sufficiently large N such that the mapping cone of β is the $(N+3)$-th component of the spectrum $\boldsymbol{V}(2)$. For this element $\beta \in \mathscr{A}_{(p+1)q}(X(1))$, the elements $\beta_{(s)} \in \mathscr{A}_{(sp+s-1)q-1}(M)$ and $\beta_s \in G_{(sp+s-1)q-2}$ in [4] and [11] are defined by

$$\begin{aligned} &\beta_{(s)} = k_1\beta^s j_1, \quad \beta^s = \beta\cdots\beta \ (s\text{-times composition}), \\ &\beta_s = \pi\beta_{(s)}i. \end{aligned} \tag{2.2}$$

The non-triviality of these elements is proved by L. Smith [6].

We define an element $R(1) \in \mathscr{A}_*(X(1))$ by

$$R(1) = \beta^p \in \mathscr{A}_{(p^2+p)q}(X(1)). \tag{2.3}$$

Then $k_1R(1)j_1 = \beta_{(p)}$ by (2.2). For this element $\beta_{(p)}$ we proved in [4; Prop. 5.2 and 6.2, Remark at the end of § 6] the following result.

Proposition 2.4. *There exists the indecomposable element $\varepsilon \in \mathscr{A}_{(p^2+1)q-1}(M)$ satisfying the following conditions.*

(i) *$\varepsilon_1 = \pi\varepsilon i$ generates the p-primary component of $G_{(p^2+1)q-2}$.*

(ii) *$D(\varepsilon)=0$ for the differential D of* (1.10)$'$.

(iii) $k_1R(1)j_1 = \beta_{(p)} = \varepsilon\alpha^{p-2} = \alpha^{p-2}\varepsilon$.

(iv) $\varepsilon\alpha^{p-1} = \alpha^{p-1}\varepsilon = 0$.

Remark. This element ε is a non zero multiple of that of [4]. The element ε in [4] satisfies $\beta_{(p)} = x\varepsilon\alpha^{p-2}$ for some $x \neq 0 \in Z_p$, and in this paper, we replace ε so that $\beta_{(p)} = \varepsilon\alpha^{p-2}$.

The following lemma is easy. We denote by C_f the mapping cone $Y \cup_f CX$ of $f\colon X \to Y$. Here we identify $(0, x)$ with $f(x)$, and $(1, x)$ with the base point, for any $x \in X$.

Lemma 2.5. (i) *Let $f\colon X \to Y$, $f'\colon X' \to Y'$, $a\colon X \to X'$ and $b\colon Y \to Y'$ be maps such that bf is homotopic to $f'a$. For any homotopy $A_t\colon X \to Y'$ between $A_0 = bf$ and $A_1 = f'a$, we define a map*

$$e = e(A_t) \colon C_f \longrightarrow C_{f'}$$

by $e(y)=b(y)$, $e(t,x)=A_{2t}(x)$ *for* $0 \leqq t \leqq 1/2$, *and* $=(2t-1, a(x))$ *for* $1/2 \leqq t \leqq 1$, *where* $y \in Y \subset C_f$, $(t,x) \in CX \subset C_f$. *Then we have a homotopy commutative diagram of cofiberings*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & C_f & \longrightarrow & SX \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle Sa} \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & C_{f'} & \longrightarrow & SX'. \end{array}$$

(ii) *Further let* $g \colon Y \to Z$, $g' \colon Y' \to Z'$ *and* $c \colon Z \to Z'$ *be maps such that* cg *is homotopic to* $g'b$ *with homotopy* $B_t \colon Y \to Z'$, $B_0 = cg$, $B_1 = g'b$. *Let* $F \colon C_{gf} \to C_g$ *be the map defined by* $F(z) = z \in Z$, *and* $F(t,x) = (t, f(x)) \in CY$, $(t,x) \in CX$, *and* $F' \colon C_{g'f'} \to C_{g'}$ *be the map defined in the same way. Then, we have a homotopy commutative diagram*

$$\begin{array}{ccc} C_{gf} & \xrightarrow{F} & C_g \\ \downarrow{\scriptstyle e(A_t \circ B_t)} & & \downarrow{\scriptstyle e(B_t)} \\ C_{g'f'} & \xrightarrow{F'} & C_{g'}, \end{array}$$

where the homotopy $A_t \circ B_t \colon X \to Z'$ *between* cgf *and* $g'f'a$ *denotes the "composition of two homotopies"* B_t *and* A_t, *and is given by* $B_{2t}f$ *for* $0 \leqq t \leqq 1/2$ *and* $g'A_{2t-1}$ *for* $1/2 \leqq t \leqq 1$.

PROOF. (i) is obvious. Define a homotopy $H_\theta \colon C_{gf} \to C_{g'}$ by

$$H_\theta(z) = c(z), \quad H_\theta(t,x) = B_{2t}f(x) \qquad \text{for} \quad 0 \leqq t \leqq 1/2,$$

$$H_\theta(t,x) = \begin{cases} g'A_{4t-2}(x) & \text{for} \quad 1/2 \leqq t \leqq (2+\theta)/4, \\ ((4t-2-\theta)/(2-\theta), \quad A_\theta(x)) & \text{for} \quad (2+\theta)/4 \leqq t \leqq 1. \end{cases}$$

Then, $H_0 = e(B_t)F$ and $H_1 = F'C$ for some C homotopic to $e(A_t \circ B_t)$. Q.E.D.

The rest of this section is devoted to prove Theorem C.

PROOF OF THEOREM C (FIRST STEP). The condition (c) implies

$$\text{(c)}' \qquad B^{r-1}R(r) = R(1)B^{r-1},$$

and we first prove the theorem for $r \leqq p-2$ under the conditions (a), (b), (c)′, (d) and (e).

LEMMA 2.6. (i) $\{X(1), X(r)\}_{a-2q-1} = 0$ *for* $1 \leqq r \leqq p-4$, $a = (p^2+p)q$.

(ii) $\alpha^* \colon \mathscr{A}_{(p^2+i)q-1}(M) \longrightarrow \mathscr{A}_{(p^2+i+1)q-1}(M)$ *is isomorphic for* $2 \leqq i \leqq p-2$.

This lemma will be proved in § 5.

Now, suppose inductively that there are $R(s)$, $s \leqq r$, satisfying (b) and (c)$'$, for some r. Then, by Lemmas 1.5 and 2.5 (i), we have the relation

$$(*)_{r-1} \qquad R(r-1)\Delta_{r-1} = \Delta_{r-1}R(1) \qquad (\Delta_{r-1} = j_{r-1}k_1).$$

Since $B^{r-1}\Delta_r = \Delta_1$ by (1.7)$'$, we have

$$B_*^{r-1}(R(r)\Delta_r - \Delta_r R(1)) = R(1)\Delta_1 - \Delta_1 R(1).$$

According to [11], our $X(1)$ coincides with $V(1)$ in [11] and our element $\Delta_1 = j_1 k_1$ is equal to δ_1 in [11; p. 219]. By [11; (4.5) (i)] there is a relation $\beta^2\Delta_1 - 2\beta\Delta_1\beta + \Delta_1\beta^2 = 0$, which implies $\beta^p\Delta_1 = \Delta_1\beta^p$. Hence we have $B_*^{r-1}(R(r)\Delta_r - \Delta_r R(1)) = 0$ by (2.3). Applying Lemma 2.6 (i) to the exact sequence (1.6) for $K = X(1)$, we obtain the relation $(*)_r$: $R(r)\Delta_r = \Delta_r R(1)$, if $r \leqq p-3$. Again by Lemmas 1.5 and 2.5 (i), we get an element $R(r+1)$ satisfying (b) and (c)$'$, if $r \leqq p-3$. Since $k_r R(r) j_r \alpha^{r-1} = k_1 R(1) j_1 = \varepsilon\alpha^{p-2}$ by (1.4) and Proposition 2.4 (iii), it follows from Lemma 2.6 (ii) that these elements $R(r)$ satisfy (e). Thus we have

(2.7) *There exist* $R(r) \in \mathscr{A}_a(X(r))$, $r \leqq p-2$, $a = (p^2+p)q$, *satisfying* (a), (b), (c)$'$ *and* (e).

REMARK. Since Lemma 2.6 (i) is not valid for $r = p-3$, the above argument does not hold for $r = p-1$.

Following [4], we define some elements of $\mathscr{A}_*(M)$. Let

$$\delta = i\pi \in \mathscr{A}_{-1}(M).$$

This is a generator of $\mathscr{A}_{-1}(M) = Z_p$ with $D(\delta) = -\theta(\delta) = 1_M$. We put

$$B_s = (\beta_{(1)}\delta)^{p-s}\beta_{(s)} \in \mathscr{A}_{(p^2+s-2)q+2s-3}(M) \qquad \text{for} \quad 1 \leqq s < p,$$

$$B'_s = \left\{ \begin{array}{ll} \beta_{(1)}\delta B_s & \text{for} \quad 1 \leqq s < p \\ \beta_{(2)}\delta\beta_{(p-1)} & \text{for} \quad s = p \end{array} \right\} \in \mathscr{A}_{(p^2+p+s-2)q+2s-5}(M),$$

$$B''_1 = \beta_{(1)}\delta B'_1 \in \mathscr{A}_{(p^2+2p-1)q-5}(M),$$

$$C_s = \alpha\delta B_s \in \mathscr{A}_{(p^2+s-1)q+2s-4}(M) \qquad \text{for} \quad 1 \leqq s < p \quad (C_1 = 0),$$

$$C'_s = \alpha\delta B'_s \in \mathscr{A}_{(p^2+p+s-1)q+2s-6}(M) \quad \text{for} \quad 1 \leqq s \leqq p \quad (C'_1 = 0).$$

Except B_{p-1} these elements belong to $\operatorname{Ker} D = \operatorname{Ker}\theta$. We also denote by

$$Z_p\{\gamma_1, \ldots, \gamma_n\}$$

the linear space over the field Z_p with basis $\gamma_1, \ldots, \gamma_n$.

LEMMA 2.8. *Let* $a=(p^2+p)q$ *and* $R(r)$ *be the elements of* (2.7).

(i) *The groups* $\mathscr{A}_a(X(r))$, $1\leqq r\leqq p-2$, *are equal to*

$$\begin{array}{ll} Z_p\{R(1), j_1B'_3k_1\} & \text{for } r=1, \\ Z_p\{R(2), j_2\delta B'_4\delta k_2\} & \text{for } r=2, \\ Z_p\{R(3), j_3C'_4\delta k_3, j_3\delta C'_4k_3\} & \text{for } r=3,\ p\geqq 7, \\ Z_p\{R(3), j_3C'_4\delta k_3, j_3\delta C'_4k_3, j_3\delta B''_1\delta k_3\} & \text{for } r=3,\ p=5, \\ Z_p\{R(r)\} & \text{for } 4\leqq r\leqq p-2,\ p\geqq 7. \end{array}$$

(ii) $\mathscr{A}_{a+1}(X(r))\cap \operatorname{Ker}\theta\cap \operatorname{Ker}k_{r*}j_r^*\cap \operatorname{Ker}A^*=0$ *for* $2\leqq r\leqq p-2$.

(iii) $\mathscr{A}_a(X(r))\cap \operatorname{Ker}\theta\cap \operatorname{Ker}k_{r*}j_r^*\cap \operatorname{Ker}A^*=0$ *for* $r=2, 3$.

This lemma will be proved in § 5.

PROOF OF THEOREM C (SECOND STEP). By [11; (3.7)], the element β of (2.1) satisfies $\theta(\beta)=0$, and hence we have

$$\theta(R(1))=0$$

by Proposition 1.9 (i) and (2.3). If the element $R(r-1)$ of (2.7) satisfies (d): $\theta(R(r-1))=0$, $r\leqq p-2$, then for the element $R(r)$ of (2.7) the element $\xi=\theta(R(r))$ satisfies $\theta(\xi)=0$, $k_r\xi j_r=-\theta(k_rR(r)j_r)=D(\varepsilon\alpha^{p-1-r})=0$, and $\xi A=\theta(R(r)A)=\theta(AR(r-1))=0$ by Propositions 1.9, 1.12–1.13 and 2.4 (ii). Hence $\xi=0$ by Lemma 2.8 (ii). Therefore by the induction we see that the element $R(r)$ of (2.7) satisfies the condition (d).

Next we prove the uniqueness. If an $R'(2)\in\mathscr{A}_a(X(2))$ satisfies (b), (c)′, (d) and (e), then the difference $\xi=R(2)-R'(2)$ satisfies $\theta(\xi)=0$, $k_2\xi j_2=0$ and $\xi A=0$. So $\xi=0$ and $R'(2)=R(2)$ by Lemma 2.8 (iii). In the same way, $R(3)$ is unique. The uniqueness of $R(r)$, $4\leqq r\leqq p-2$, follows immediately from the last of Lemma 2.8 (i). Thus we have obtained

(2.9) *There exist uniquely the elements* $R(r)$, $r\leqq p-2$, *satisfying the conditions* (a), (b), (c)′, (d) *and* (e).

LEMMA 2.10. *Let* $a=(p^2+p)q$.

(i) $\mathscr{A}_{a+1}(X(2))\cap \operatorname{Ker}\theta\cap \operatorname{Ker}B^*=0$.

(ii) $\mathscr{A}_a(X(2))\cap \operatorname{Ker}\theta=Z_p\{R(2)\}$ *for the element* $R(2)$ *of* (2.9).

The proof of this lemma will be given in § 5.

PROOF OF THEOREM C (THIRD STEP). We consider the condition (c) for $r \geqq 3$. From the condition (b) we have $R(r)A^{r-1} = A^{r-1}R(1)$. From Lemmas 1.5 and 2.5 (i), it follows that there exists $R'(r-1) \in \mathscr{A}_a(X(r-1))$ such that $R'(r-1)B = BR(r)$. We compare this with $R(r-1)$. The element $R'(r-1)$ also satisfies (e), since $k_{r-1}R'(r-1)j_{r-1} = k_{r-1}R'(r-1)Bj_r = k_{r-1}BR(r)j_r = \alpha k_r R(r) j_r = \varepsilon\alpha^{p-r}$. Hence, by Lemma 2.8 (i) we have $R'(r-1) = R(r-1)$ for $5 \leqq r \leqq p-2$, $p \geqq 7$, $R'(3) \equiv R(3) \bmod Z_p\{j_3 C'_4 \delta k_3, j_3 \delta C'_4 k_3\}$ $(+Z_p\{j_3 \delta B''_1 \delta k_3\}$ if $p=5)$ and $R'(2) \equiv R(2) \bmod Z_p \{j_2 \delta B'_4 \delta k_2\}$. By [4; (5.8), (5.13)], $C'_4 \delta\alpha = \delta C'_4 \alpha = 0$ and $B''_1 \delta\alpha = 0$, and so $C'_4 \delta k_3 B = \delta C'_4 k_3 B = \delta B''_1 \delta k_3 B = 0$ by (1.4). Therefore $R(3)B = R'(3)B = BR(4)$, $p \geqq 7$. By Lemma 2.10 (i), we see that $\theta(R'(2)) = 0$, and hence $R'(2) = R(2)$ by Lemma 2.10 (ii). Thus we have proved

(2.11) *The elements* $R(r)$, $r \leqq p-2$, *of* (2.9) *satisfy the condition* (e).

Finally we consider the element $R(p-1)$.

LEMMA 2.12. *Let* $a = (p^2+p)q$.

(i) $k_{p-1*}j^*_{p-1} : \mathscr{A}_a(X(p-1)) \to \mathscr{A}_{(p^2+1)q-1}(M)$ *is monomorphic, and its image is the subgroup* $Z_p\{\varepsilon, C_2\delta, \delta C_2\}$.

(ii) $\mathscr{A}_{a+1}(X(p-1)) \cap \operatorname{Ker}\theta \cap \operatorname{Ker} k_{p-1*}j^*_{p-1} = 0$.

(iii) $k_{1*} : \{X(p-1), X(1)\}_a \longrightarrow \{X(p-1), M\}_{a-q-3}$ *is monomorphic and* $\{X(p-1), M\}_{a-q-3} \cap \operatorname{Ker} j^*_{p-1} \cap \operatorname{Ker}\theta = 0$.

(iv) $j^*_1 : \{X(1), X(p-1)\}_{a+(p-2)q} \longrightarrow \{M, X(p-1)\}_{a+(p-2)q+2}$ *is monomorphic and* $\{M, X(p-1)\}_{a+(p-2)q+2} \cap \operatorname{Ker} k_{p-1*} \cap \operatorname{Ker}\theta = 0$.

This Lemma will be proved in § 5.

PROOF OF THEOREM C (LAST STEP). By Lemma 2.12 (i), there exists an element $R(p-1) \in \mathscr{A}_a(X(p-1))$ such that $k_{p-1}R(p-1)j_{p-1} = \varepsilon$, and this element is unique. By Lemma 2.12 (ii), $\theta(R(p-1)) = 0$ since $\theta(R(p-1))$ belongs to the left side of Lemma 2.12 (ii). Consider the elements $\xi = B^{p-2}R(p-1) - R(1)B^{p-2} \in \{X(p-1), X(1)\}_a$ and $\eta = R(p-1)A^{p-2} - A^{p-2}R(1) \in \{X(1), X(p-1)\}_{a+(p-2)q}$. Then $k_1 \xi j_{p-1} = \alpha^{p-2}k_{p-1}R(p-1)j_{p-1} - k_1 R(1) j_1 = \alpha^{p-2}\varepsilon - \varepsilon\alpha^{p-2} = 0$ and $\theta(k_1\xi) = 0$. So we have $\xi = 0$ by Lemma 2.12 (iii). By a similar way using Lemma 2.12 (iv) instead of (iii), we also have $\eta = 0$. Thus, we obtain

$$(*) \qquad B^{p-2}R(p-1) = R(1)B^{p-2},$$

$$(*)' \qquad R(p-1)A^{p-2} = A^{p-2}R(1).$$

Applying Lemma 2.5 (i) to the cofibering in Lemma 1.5, we can construct an element $R'(p-2)\in\mathscr{A}_a(X(p-2))$ with $R(p-1)A=AR'(p-2)$ from $(*)$. This element satisfies $k_{p-2}R'(p-2)j_{p-2}=\varepsilon\alpha$, and hence $R'(p-2)=R(p-2)$ for $p\geqq 7$ and $R'(3)\equiv R(3) \bmod Q=Z_p\{j_3C'_4\delta k_3,\ j_3\delta C'_4k_3,\ j_3\delta B''_1\delta k_3\}$ for $p=5$ by Lemma 2.8 (i). We therefore have $R(p-1)A=AR'(p-2)=AR(p-2)$ because $A_*Q=0$. In the same way, $(*)'$ leads to $BR(p-1)=R(p-2)B$. Thus, we have obtained

(2.13) *There exists uniquely the element* $R(p-1)$ *satisfying the conditions* (b)–(e).

By (2.9), (2.11) and (2.13), we complete the proof of Theorem C. Q.E.D.

REMARK. Theorem C does not hold for $r=p$. There exists no $R(p)\in\mathscr{A}_{(p^2+p)q}(X(p))$, since the existence of an $R(p)$ yields the decomposition $\varepsilon=(k_pR(p)j_p)\alpha$ while ε is indecomposable.

REMARK 2.14. Let η_1 and η_2 be the elements of $\mathscr{A}_a(X(p-1))$ such that $k_{p-1}\eta_1j_{p-1}=C_2\delta$ and $k_{p-1}\eta_2j_{p-1}=\delta C_2$. By Lemma 2.12 (i), these are unique and we have

$$\mathscr{A}_a(X(p-1)) = Z_p\{R(p-1), \eta_1, \eta_2\}. \tag{2.15}$$

According to [4], the conditions (i)–(iv) of Proposition 2.4 do not uniquely determine the element ε. It is determined up to the subgroup $Z_p\{C_2\delta-\delta C_2\}$. The element $R(p-1)$ satisfying (b)–(d) of Theorem C is determined up to the subgroup $Z_p\{\eta_1-\eta_2\}$, and any different choices of $R(p-1)$ and ε are related with each other by the equality

$$k_{p-1}(R(p-1)+x(\eta_1-\eta_2))j_{p-1} = \varepsilon+x(C_2\delta-\delta C_2), \quad x\in Z_p.$$

§3. Complex bordism modules

The complex bordism ring Ω^U_* consists of all bordism classes of stably almost complex manifolds. It is a polynomial ring over Z having one polynomial-generator $x_i\in\Omega^U_{2i}$ for each even degree $2i>0$. If $i+1=p^s$ for a prime p, then one may choose x_i such that all integral Chern numbers of x_i are divisible by p, and a manifold M^{2i} representing such x_i is called a Milnor manifold (for the prime p). The complex projective space $P=CP(p-1)$ is a Milnor manifold of dimension $2(p-1)$, and we denote by V a Milnor manifold of dimension $2(p^2-1)$ for the prime p. (Cf. [8; pp. 128–130] and [6]).

For a finite CW-complex X, the reduced complex bordism module of X is denoted by $\tilde{\Omega}^U_*(X)$. It is a module over the ring Ω^U_*, and the functor $\tilde{\Omega}^U_*(\cdot)$ forms a reduced homology theory. Denote by $S\colon\tilde{\Omega}^U_i(X)\to\tilde{\Omega}^U_{i+1}(SX)$ the suspension isomorphism.

For $X=M$, we see easily that

$$\tilde{\Omega}_*^U(M) = \Omega_*^U/(p)\cdot\mu, \qquad \mu \in \tilde{\Omega}_1^U(M),$$

as an Ω_*^U-module, where (p) stands for the ideal generated by $p \in \Omega_0^U = Z$. L. Smith [6; Th. 1.5] has proved the following result.

LEMMA 3.1. *The Ω_*^U-homomorphism*

$$\alpha_*: \tilde{\Omega}_*^U(S^{q+2}M) \longrightarrow \tilde{\Omega}_*^U(S^2M)$$

is given by $\alpha_*(S^{q+2}\mu)=[P]S^2\mu$, $[P]=[CP(p-1)]$.

Then $(\alpha^r)_*$ sends $S^{rq+2}\mu$ to $[P]^r S^2\mu$, and it is a monomorphism since $[P]^r$ is not a zero divisor in $\Omega_*^U/(p)$. So we obtain the following result.

PROPOSITION 3.2. $\tilde{\Omega}_*^U(X(r))=\Omega_*^U/(p, [P]^r)\cdot\xi(r)$ *as an Ω_*^U-module, where* $\xi(r)=j_{r*}(S^2\mu) \in \tilde{\Omega}_3^U(X(r))$ *and* $(p, [P]^r)$ *stands for the ideal generated by p and* $[P]^r$.

Now our element β of (2.1) coincides with the element $\tilde{\psi}$ constructed by L. Smith [6; Th. 4.10]. He also has proved in [6; Prop. 4.11] the following result.

LEMMA 3.3. *The Ω_*^U-homomorphism*

$$\beta_*: \tilde{\Omega}_*^U(S^{N+(p+1)q}X(1)) \longrightarrow \tilde{\Omega}_*^U(S^N X(1))$$

is given by $\beta_*(S^{N+(p+1)q}\xi(1))=[V]S^N\xi(1)$ *where $[V]$ is the class represented by the Milnor manifold V of dimension* $(p+1)q$.

PROPOSITION 3.4. *Let $1 \leqq r \leqq p-1$ and set $a=(p^2+p)q$. Then the Ω_*^U-homomorphism*

$$R(r)_*: \tilde{\Omega}_*^U(S^{N+a}X(r)) \longrightarrow \tilde{\Omega}_*^U(S^N X(r))$$

satisfies $R(r)_*(S^{N+a}\xi(r))=([V]^p+[M_r])S^N\xi(r)$ *for some* $[M_r] \in (p, [P]) \cap \Omega_a^U$.

PROOF. Consider the Ω_*^U-homomorphism

$$A_*: \tilde{\Omega}_*^U(S^q X(r)) \longrightarrow \tilde{\Omega}_*^U(X(r+1)).$$

Then $A_*(S^q\xi(r))=A_*j_{r*}(S^{q+2}\mu)$ by Proposition 3.2, and $A_*j_{r*}(S^{q+2}\mu)=j_{r+1*}\alpha_*(S^{q+2}\mu)=[P]j_{r+1*}(S^2\mu)=[P]\xi(r+1)$ by (1.4) and Lemma 3.1. Hence we have

$$A_*(S^q\xi(r)) = [P]\xi(r+1). \tag{3.5}$$

By Lemma 3.3 and (2.3), the proposition holds obviously for $r=1$. If the proposition holds for some $r-1$, then we have

$$\begin{aligned}[P]R(r)_*(S^{N+a}\xi(r) &= R(r)_*([P]S^{N+a}\xi(r)) \\ &= R(r)_*A_*(S^{N+a+q}\xi(r-1)) && \text{by (3.5)} \\ &= A_*R(r-1)_*(S^{N+a+q}\xi(r-1)) && \text{by Theorem C(b)} \\ &= A_*([V]^p+[M_{r-1}])S^{N+q}\xi(r-1) \\ &= [P]([V]^p+[M_{r-1}])S^N\xi(r) && \text{by (3.5).}\end{aligned}$$

By Proposition 3.2, the kernel of the left translation

$$[P]\cdot : \tilde{\Omega}^U_*(X(r)) \longrightarrow \tilde{\Omega}^U_*(X(r))$$

is the submodule generated by $[P]^{r-1}\xi(r)$, and hence

$$R(r)_*(S^{N+a}\xi(r)) = ([V]^p+[M_{r-1}]+[N][P]^{r-1})S^N\xi(r)$$

for some $[N]\in\Omega^U_*$. By putting $[M_r]=[M_{r-1}]+[N][P]^{r-1}$, the proposition holds for r. Q.E.D.

PROOF OF THEOREM D. The theorem is a direct consequence of Proposition 3.4. Q.E.D.

We consider the Hurewicz homomorphism

$$h\colon \{S^0, X\}_* = \tilde{\Omega}^{fr}_*(X) \longrightarrow \tilde{\Omega}^U_*(X),$$

which is induced by the inclusion $\boldsymbol{S}\subset \boldsymbol{MU}$ of spectra. We are concerned with the image of h for the $(rq+4)$-skeleton

$$Y(r) = S^3 \cup_p e^4 \cup_{\alpha_r} e^{rq+4}$$

of $X(r)$. Let

$$Y(r) \xrightarrow{l_r} X(r) \xrightarrow{\pi k_r} S^{rq+5}, \tag{3.6}$$

$$S^2M \xrightarrow{j'_r} Y(r) \xrightarrow{k'_r} S^{rq+4} \tag{3.7}$$

be the cofiberings, where l_r and j'_r are the inclusions. Denote the boundary homomorphisms for (3.6) and (3.7) by

$$\partial\colon \tilde{\Omega}^U_{i+1}(S^{rq+5}) \longrightarrow \tilde{\Omega}^U_i(Y(r)),$$

$$\partial'\colon \tilde{\Omega}^U_{i+1}(S^{rq+4}) \longrightarrow \tilde{\Omega}^U_i(S^2M).$$

Then, applying $\tilde{\Omega}^U_*(\ \ )$ to (3.6)–(3.7) and using Lemma 3.1 and Proposition 3.2, we obtain easily the following

PROPOSITION 3.8. *Let* $\eta(r)=j'_{r*}(S^2\mu)\in\tilde{\Omega}_3^U(Y(r))$ *and* $\tau(r)=\partial\sigma_{rq+5}\in\tilde{\Omega}_{rq+4}^U(Y(r))$, *where* $\sigma_i\in\tilde{\Omega}_i^U(S^i)$ *is the canonical generator. Then we have the direct sum decomposition*

$$\tilde{\Omega}_*^U(Y(r)) = \Omega_*^U/(p, [P]^r)\cdot\eta(r)+\Omega_*^U\cdot\tau(r).$$

Also the following relations hold:

$$l_{r*}\eta(r) = \xi(r), \qquad (\pi k_r)^*\xi(r) = 0,$$

$$k'_{r*}\tau(r) = p\sigma_{rq+4}, \qquad \partial'\sigma_{rq+4} = [P]^r S^2\mu.$$

Now let γ be any element of $\{S^0, Y(r)\}_n$. We consider its image $h(\gamma)\in\tilde{\Omega}_n^U(Y(r))$ by h. If n is even, then $h(\gamma)$ lies in $\Omega_*^U\tau(r)$ and its k'_{r*}-image lies in the image of $h\colon G_*\to\Omega_*^U$, which is trivial for $*\neq 0$ and is isomorphic for $*=0$ [8; p. 133]. Since k'_{r*} is monomorphic for even n by Proposition 3.8, it follows that $h(\gamma)=0$ if $n\neq rq+4$ and $h(\gamma)$ is an integer multiple of $\tau(r)$ if $n=rq+4$. Next, if n is odd and $n<rq+4$, $j'_{r*}\colon\{S^0, M\}_{n-2}\to\{S^0, Y(r)\}_n$ is epimorphic. Also $j'_{r*}\colon\Omega_{n-2}^U(M)\to\tilde{\Omega}_n^U(Y(r))$ is epimorphic by Proposition 3.8. Using the known results [1; Th. 3.1 and Cor. 3.3] for the image of $h\colon\{S^0, M\}_*\to\tilde{\Omega}_*^U(M)$, we see that $h(\gamma)=0$ if $n\not\equiv 3 \bmod q$ and $h(\gamma)$ is a multiple of $[P]^j\eta(r)$ if $n=jq+3$. Finally, if n is odd and $n\geqq rq+4$, then $h(\gamma)=0$ is proved as follows. Let $K=S^N Y(r)\cup_\gamma e^{N+n+1}$ be the mapping cone of γ. Consider the annihilator ideal

$$A(\kappa) = \{x\in\Omega_*^U \mid x\kappa = 0 \quad\text{in}\quad \tilde{\Omega}_*^U(K)\}$$

of the canonical class $\kappa\in\tilde{\Omega}_{N+3}^U(K)$, which is the h-image of the inclusion $S^{N+3}\subset K$. By Proposition 3.8 and an easy calculation, we have

$$A(\kappa) = (p, [P]^r, [M]),$$

for $[M]\in\Omega_{n-3}^U$ determined by $h(\gamma)=[M]\eta(r)$. We apply the discussion in the proof of Theorem 4.3 of [2] to our K*). We consider the case $X(g,f)=K$, $t=1$, $[V_g]=[P]^r$, $\varepsilon=0$ and $[V]=[M]$ in the proof of this theorem. Then, the proof implies that $[M]$ lies in $(p, [P]^r)$. Hence we have $h(\gamma)\in(p, [P]^r)\eta(r)=0$ by Proposition 3.8.

From the above discussions, we have proved

PROPOSITION 3.9. *The image of the Hurewicz homomorphism*

$$h\colon\{S^0, Y(r)\}_*\longrightarrow\tilde{\Omega}_*^U(Y(r))$$

*) The author is indebted to Prof. Larry Smith for this discussion.

is additively generated by the following elements:

$$[P]^j\eta(r) \qquad (0 \leqq j < r), \quad \tau(r).$$

§4. Proof of Theorems A and B

Definition 4.1. *Let* $t \geqq 1$ *and* $1 \leqq r \leqq p-1$. *Then we define* $\rho(t, r) = k_{p-r}R(p-r)^t j_{p-r} \in \mathscr{A}_{(tp^2+(t-1)p+r)q-1}(M)$.

For $t=1$ and for $r=p-1$, we see immediately

$$\rho(1, r) = \varepsilon\alpha^{p-1-r}, \tag{4.2}$$

$$\rho(t, p-1) = \beta_{(tp)}, \tag{4.3}$$

by (e) of Theorem C and by (2.2)–(2.3), respectively. By (1.4) and (b) of Theorem C,

$$\begin{aligned}\rho(t, r-1)\alpha &= k_{p-r+1}R(p-r+1)^t A j_{p-r} \\ &= k_{p-r+1}AR(p-r)^t j_{p-r} = \rho(t, r),\end{aligned}$$

and in the same way $\alpha\rho(t, r-1) = \rho(t, r)$ by (c) of Theorem C.

$$\rho(t, r) = \alpha\rho(t, r-1) = \rho(t, r-1)\alpha. \tag{4.4}$$

$$\rho(t, r) = \alpha^{r-1}\rho(t, 1) = \rho(t, 1)\alpha^{r-1}. \tag{4.4$'$}$$

We have also

$$\alpha^{p-r}\rho(t, r) = \rho(t, r)\alpha^{p-r} = 0, \tag{4.5}$$

since $\alpha^{p-r}k_{p-r} = j_{p-r}\alpha^{p-r} = 0$. Then, the stable Toda bracket $\langle\rho(t, r), \alpha^{p-r}, \rho(s, r)\rangle$ is well defined, while $\rho(t+s, r)$ belongs to this bracket because $\rho(t+s, r) = (k_{p-r}R(p-r)^t)(R(p-r)^s j_{p-r})$ and $j_{p-r}^*(k_{p-r}R(p-r)^t) = \rho(t, r)$, $k_{p-r*}(R(p-r)^s j_{p-r}) = \rho(s, r)$. Thus, we obtain

$$\rho(t+s, r) \in \langle\rho(t, r), \alpha^{p-r}, \rho(s, r)\rangle. \tag{4.6}$$

We have proved $\theta(j_{p-r}) = \theta(k_{p-r}) = 0$ in Proposition 1.13 and $\theta(R(p-r)) = 0$ in Theorem C(d). Hence we have

$$D(\rho(t, r)) = 0, \tag{4.7}$$

by repeating Proposition 1.9 (i) and by Proposition 1.12.

Definition 4.8. *Let* $t \geqq 1$ *and* $1 \leqq r \leqq p-1$. *Then we define*

$$\rho(t) = \rho(t, 1) \in \mathscr{A}_{(tp^2+(t-1)p+1)q-1}(M),$$

$$\rho_{t,r} = \pi\rho(t, r)i \in G_{(tp^2+(t-1)p+r)q-2}.$$

PROOF OF THEOREM A. Obviously $p\rho_{t,r}=0$. By (4.4)$'$, $\rho_{t,r+s}$ is equal to $\pi\rho(t, r)\alpha^s i$, which lies in $\langle \rho_{t,r}, p, \alpha_s \rangle$ because $i^*(\pi\rho(t, r))=\rho_{t,r}$ and $\pi_*(\alpha^s i)=\alpha_s$. By (4.3) and (2.2), we have $\rho_{t,p-1}=\beta_{tp}$.

Consider the element

$$\gamma = R(p-r)^t j_{p-r} i \in \{S^0, X(p-r)\}_{at+3}, \quad a=(p^2+p)q,$$

and the commutative triangle:

$$\begin{tikzcd}
S^{N+at+3} \wedge S^{2N'} \arrow[rr, "\gamma\wedge h"] \arrow[dr, "j_{p-r}i\wedge h"'] & & S^N X(p-r) \wedge MU(N') \\
& S^{N+at} X(p-r) \wedge MU(N'), \arrow[ur, "R(p-r)^t\wedge 1"'] &
\end{tikzcd}$$

where $h: S^{2N'} \subset MU(N')$ is the inclusion, which defines the Hurewicz homomorphism $h: \tilde{\Omega}^{fr}_*(\cdot) \to \tilde{\Omega}^U_*(\cdot)$. Then, $\gamma \wedge h$ represents the Hurewicz-image $h(\gamma) \in \tilde{\Omega}^U_{N+at+3}(S^N X(p-r))$ of γ, and $j_{p-r} i \wedge h$ represents the element $S^N\xi(p-r) \in \tilde{\Omega}^U_{N+3}(S^N X(p-r))$ in Proposition 3.2. We have therefore by Proposition 3.4

$$\begin{aligned} h(\gamma) &= ([V]^p + [M_{p-r}])^t S^N \xi(p-r) \\ &\equiv [V]^{pt} S^N \xi(p-r) \qquad \mathrm{mod}\,(p, [P]) S^N \xi(p-r). \end{aligned}$$

Now, assume that $\rho_{t,r}=\pi k_{p-r}\gamma=0$. Then by using (3.7), there exists an element $\gamma' \in \{S^0, Y(p-r)\}_{at+3}$ such that $l_{p-r*}(\gamma')=\gamma$. By Proposition 3.9, we have $h(\gamma')=0$. Thus, $h(\gamma)=l_{p-r*}h(\gamma')=0$. This is a contradiction. Therefore $\rho_{t,r}\neq 0$ and the proof is complete. Q.E.D.

PROOF OF THEOREM B. (i) is obvious. (ii) follows immediately from (4.3) and (4.4)$'$. (iii) and (iv) are restatements of (4.5) and (4.6), respectively. (v) follows immediately from Definition 4.8. Since $\rho_{t,1}\neq 0$ by Theorem A, $\rho(t)$ is non zero. (vi) is a restatement of (4.7) for $r=1$. Q.E.D.

REMARK. We have noticed in Remark 2.14 that the element $R=R(p-1)$ is determined modulo $\eta=\eta_1-\eta_2$ if we do not fix ε of Proposition 2.4. By the definition of η and by the results on $\mathscr{A}_*(X(1))$ of [11], we have $\eta=A^{p-2}\xi B^{p-2}$ for the element $\xi=(\beta^2\alpha''-\alpha''\beta^2)\beta'^{p-2} \in \mathscr{A}_*(X(1))$, where $\alpha'' \in \mathscr{A}_{q-2}(X(1))$ and $\beta' \in \mathscr{A}_{pq-2}(X(1))$ are the generators of the ring $\mathscr{A}_*(X(1))$ given in [11]. We also have $\xi\beta^{tp}=\beta^{tp}\xi=(\beta^{tp+2}\alpha''-\alpha''\beta^{tp+2})\beta'^{p-2}$ by using the results of [11].

Hence, $\eta^2=0$, $R^t\eta=\eta R^t$ and $k_{p-1}R^t\eta j_{p-1}=\delta\alpha\delta(\beta_{(1)}\delta)^{p-2}\beta_{(tp+2)}-\alpha\delta(\beta_{(1)}\delta)^{p-2}\cdot\beta_{(tp+2)}\delta$, while the last element is trivial by [5; Corollary 2]. Thus, we see that any choice of ε does not change the elements $\rho(t)$ and $\rho_{t,r}$ (except the initial elements $\rho(1)=\varepsilon$ and $\rho_{1,1}=\varepsilon_1$).

§5. Auxiliary calculations

In this section, we shall prove Lemmas 2.6, 2.8, 2.10 and 2.12 in §2. The calculations in this section are based on the results in [4; Th. 0.1] for the ring $\mathscr{A}_*(M)$.

Let $\mathscr{A}'_k(M)$ be the subgroup of $\mathscr{A}_k(M)$ generated by the following elements of degree k:

$\delta^a B_s \delta^b$	of degree	$(p^2+s-2)q+2s-3-a-b$,	$2\leqq s<p$,
$\delta^a C_s \delta^b$	of degree	$(p^2+s-1)q+2s-4-a-b$,	$2\leqq s<p$,
$\delta^a B'_s \delta^b$	of degree	$(p^2+p+s-2)q+2s-5-a-b$,	$1\leqq s\leqq p$,
$\delta^a C'_s \delta^b$	of degree	$(p^2+p+s-1)q+2s-6-a-b$,	$2\leqq s\leqq p$,
$\delta^a B''_1 \delta^b$	of degree	$(p^2+2p-1)q-5-a-b$,	
$\delta^a(\beta_{(1)}\delta)^r\bar{\varepsilon}\delta^b$	of degree	$(p^2+rp+1)q-2r-2-a-b$,	$r=0, 1$,
$\delta^a\varepsilon\alpha^i\delta^b$	of degree	$(p^2+i+1)q-1-a-b$,	$0\leqq i\leqq p-2$,
$\delta^a\varepsilon\alpha^i\delta\alpha\delta^b$	of degree	$(p^2+i+2)q-2-a-b$,	$0\leqq i\leqq p-4$,
$\varepsilon\alpha^{p-2}\delta\alpha\delta^a$	of degree	$(p^2+p)q-2-a$,	
$\bar{\varphi}\delta^a$	of degree	$(p^2+p)q-3-a$.	

Here a, $b=0$ or 1 and we use the notations of elements appeared ahead of Lemma 2.8.

Also let $A(\alpha, \delta)$ be the subring of $\mathscr{A}_*(M)$ generated by the elements α and δ, and let $A_k(\alpha, \delta)=A(\alpha, \delta)\cap\mathscr{A}_k(M)$. Then by [4; Th. 0.1], we have the direct sum decomposition:

$$\mathscr{A}_k(M)=\mathscr{A}'_k(M)+A_k(\alpha,\delta) \qquad \text{for} \quad (p^2-1)q\leqq k\leqq(p^2+2p)q-4.$$

By [4; Th. 4.1], the homomorphisms $\alpha^*\colon A_k(\alpha,\delta)\to A_{k+q}(\alpha,\delta)$ and $\alpha_*\colon A_k(\alpha,\delta)\to A_{k+q}(\alpha,\delta)$ are isomorphic if $k\geqq 0$. Hence, by (1.3)–(1.3)* for $K=M$, we obtain the following

LEMMA 5.1. (i) *For $(p^2+r-1)q+2\leqq k\leqq(p^2+2p)q-2$, the following sequence is exact*:

$$\mathscr{A}'_{k-rq-2}(M)\xrightarrow{\alpha^r{}_*}\mathscr{A}'_{k-2}(M)\xrightarrow{j_{r*}}\{M, X(r)\}_k\xrightarrow{k_{r*}}\mathscr{A}'_{k-rq-3}(M)\xrightarrow{\alpha^r{}_*}\mathscr{A}'_{k-3}(M).$$

(ii) *For $(p^2-1)q-2\leqq k\leqq(p^2+2p-r)q-7$, the following is exact*:

$$\mathscr{A}'_{k+3}(M)\xrightarrow{\alpha^{r*}}\mathscr{A}'_{k+rq+3}(M)\xrightarrow{k_r^*}\{X(r), M\}_k\xrightarrow{j_r^*}\mathscr{A}'_{k+2}(M)\xrightarrow{\alpha^{r*}}\mathscr{A}'_{k+rq+2}(M).$$

Also, there exist the following exact sequences:

$$(5.2)\quad \cdots \to \{X(r), M\}_{k-sq-2} \xrightarrow{\alpha^s_*} \{X(r), M\}_{k-2} \xrightarrow{j_{s*}} \{X(r), X(s)\}_k \xrightarrow{k_{s*}} \{X(r), M\}_{k-sq-3} \to \cdots;$$

$$(5.2)^*\quad \cdots \to \{M, X(s)\}_{k+3} \xrightarrow{\alpha^{r*}} \{M, X(s)\}_{k+rq+3} \xrightarrow{k_r^*} \{X(r), X(s)\}_k \xrightarrow{j_r^*} \{M, X(s)\}_{k+2} \to \cdots$$

In the following, we put

$$a = (p^2+p)q$$

and $Z_p\{\gamma_1,\dots, \gamma_n\}$ stands for the linear space over Z_p with basis $\gamma_1,\dots, \gamma_n$.

Proof of Lemma 2.6. (i) Since $\mathscr{A}'_{a-q}(M)=0$, $\mathscr{A}'_{a-2q-1}(M)=Z_p\{\varepsilon\alpha^{p-3}\}$ and $\mathscr{A}'_{a-q-1}(M)=Z_p\{\varepsilon\alpha^{p-2}\}$, it follows from Lemma 5.1 (ii) that $\{X(1), M\}_{a-2q-3}=0$. We have $\mathscr{A}'_{a-(r+1)q-1}(M)=Z_p\{\varepsilon\alpha^{p-2-r}\}=\alpha^*\mathscr{A}'_{a-(r+2)q-1}(M)$ for $0\leqq r\leqq p-3$. Also $\mathscr{A}'_{a-(r-2)q-2}(M)\cap \operatorname{Ker}\alpha^*=0$ for $1\leqq r\leqq p-4$. Hence, $\{X(1), M\}_{a-(r+2)q-4}=0$ for $1\leqq r\leqq p-4$ by using Lemma 5.1 (ii). From these results on $\{X(1), M\}_*$, we obtain $\{X(1), X(r)\}_{a-2q-1}=0$ for $1\leqq r\leqq p-4$ by using (5.2).

(ii) This is obvious, since $\mathscr{A}'_{(p^2+i)q-1}(M)=Z_p\{\varepsilon\alpha^{i-1}\}$ for $2\leqq i\leqq p-1$.
Q.E.D.

Proof of Lemma 2.8. (i) Since $\mathscr{A}'_{a-rq-1}(M)=Z_p\{\varepsilon\alpha^{p-1-r}\}=\operatorname{Ker}\alpha^{r*}$ for $1\leqq r\leqq p-2$ and $\mathscr{A}'_a(M)=0$, we have $\{X(r), M\}_{a-rq-3}=Z_p$, generated by an element ξ_r satisfying $j_r^*\xi_r=\varepsilon\alpha^{p-1-r}$, by using Lemma 5.1 (ii). The element $R(r)$ of (2.7) satisfies $k_rR(r)j_r=\varepsilon\alpha^{p-1-r}$, and hence we can take $\xi_r=k_rR(r)$ and so $k_{r*}\colon \mathscr{A}_a(X(r))\to\{X(r), M\}_{a-rq-3}$ is epimorphic. Since $\mathscr{A}_{a+1}(M)=0$ and $\mathscr{A}'_a(M)=0$, $k_r^*\colon \mathscr{A}_{a+rq+1}(M)\to\{X(r), M\}_{a-2}$ is isomorphic by Lemma 5.1 (ii). Hence we have $\{X(r), M\}_{a-2}=Z_p\{B'_3k_1\}$ for $r=1$, $=Z_p\{\delta B'_4\delta k_2\}$ for $r=2$, $=Z_p\{C'_4\delta k_3, \delta C'_4k_3\}$ $(+Z_p\{\delta B''_1\delta k_3\}$ if $p=5)$ for $r=3$, and $=0$ for $4\leqq r\leqq p-1$, from the results of $\mathscr{A}_{a+rq+1}(M)$.

Since $\mathscr{A}_{a+1}(M)=0$ and $\mathscr{A}'_{a-rq}(M)$ is equal to 0 for $1\leqq r\leqq p-3$ and to $Z_p\{\delta C_3\delta\}=\operatorname{Ker}\alpha_r^*$ for $r=p-2$, the group $\{X(r), M\}_{a-rq-2}$ is equal to 0 for $1\leqq r\leqq p-3$ and to $Z_p\{\xi\}$ for $r=p-2$ by Lemma 5.1 (ii), where ξ satisfies $j_{p-2}^*\xi=\delta C_3\delta$. Then the element $\alpha_*^{p-2}\xi$ belongs to $k_{p-2}^*\langle\alpha^{p-2}, \delta C_3\delta, \alpha^{p-2}\rangle$ by the definition of the Toda bracket. As $\langle\alpha^{p-2}, \delta C_3\delta, \alpha^{p-2}\rangle\subset\langle\alpha^{p-3}, \alpha\delta C_3\delta, \alpha^{p-2}\rangle=\langle\alpha^{p-3}, 0, \alpha^{p-2}\rangle=0 \bmod \text{zero}$, we see $\alpha_*^{p-2}\xi=0$. Therefore, $\alpha_*^{p-2}\{X(r), M\}_{a-rq-2}=0$ for $1\leqq r\leqq p-2$. Thus, the desired results follow from (5.2).

(ii) In the same way as (i), we can calculate the group $\mathscr{A}_{a+1}(X(r))$ as follows:

$$(5.3)\qquad \mathscr{A}_{a+1}(X(r)) = \begin{cases} Z_p\{j_2B'_4\delta k_2, j_2\delta B'_4k_2\} & \textit{for } r=2,\\ Z_p\{j_3C'_4k_3\} & \textit{for } r=3,\ p\geqq 7,\\ Z_p\{j_3C'_4k_3, j_3B''_1\delta k_3, j_3\delta B''_1k_3, \xi\} & \text{for } r=3,\ p=5,\\ 0 & \textit{for } 4\leqq r\leqq p-2,\ p\geqq 7,\end{cases}$$

where ξ *satisfies* $k_3\xi j_3=\delta C_3\delta$.

We have also

$$(5.4)\qquad \mathscr{A}_{a+2}(X(r)) \cap \operatorname{Im} k_r^*j_{r*} = \begin{cases} Z_p\{j_2B'_4k_2\} & \textit{for } r=2,\\ Z_p\{j_3B''_1k_3, j_3\delta B'_p\delta k_3\} & \textit{for } r=3,\ p=5,\\ 0 & \textit{for } 3\leqq r\leqq p-2,\ p\geqq 7.\end{cases}$$

Apparently, all elements in (5.3) except ξ lie in the kernel of $k_*^rj_{r*}$. Since $D(B'_4)=D(C'_4)=D(B''_1)=0$, we have the following values of $\theta\colon \mathscr{A}_{a+1}(X(r))\to \mathscr{A}_{a+2}(X(r))$ by Propositions 1.9, 1.12 and 1.13:

$$\theta(j_2B'_4\delta k_2)=j_2B'_4k_2,\quad \theta(j_2\delta B'_4k_2) = -j_2B'_4k_2,$$

$$\theta(j_3C'_4k_3)=0,\quad \theta(j_3B''_1\delta k_3)=j_3B''_1k_3,\quad \theta(j_3\delta B''_1k_3) = -j_3B''_1k_3.$$

Therefore by (5.4), we see that

$$\mathscr{A}_{a+1}(X(r)) \cap \operatorname{Ker}\theta \cap \operatorname{Ker} k_{r*}j_r^* = \begin{cases} Z_p\{j_2(B'_4\delta+\delta B'_4)k_2\} & \text{for } r=2,\\ Z_p\{j_3(B''_1\delta+\delta B''_1)k_3\} & \text{for } r=3,\ p=5,\\ 0 & \textit{for } 3\leqq r\leqq p-2,\ p\geqq 7.\end{cases}$$

By (1.4), $A^*j_2(B'_4\delta+\delta B'_4)k_2=j_2(B'_4\delta+\delta B'_4)k_1$ and $A^*j_3(B''_1\delta+\delta B''_1)k_3=j_3(B''_1\delta+\delta B''_1)k_2$, which are non zero since $k_{r-1}^*j_{r*}\colon \mathscr{A}_{a+rq+2}(M)\to\{X(r-1), X(r)\}_{a+q+1}$ is monomorphic for $r=2, 3$. Thus, (ii) is obtained.

(iii) This is proved similarly as (ii) by using (i) and (5.3) instead of (5.3) and (5.4). Q.E.D.

PROOF OF LEMMA 2.10. (i) As is seen in the proof of Lemma 2.8 (ii), $\mathscr{A}_{a+1}(X(2))\cap\operatorname{Ker}\theta=Z_p\{j_2(B'_4\delta+\delta B'_4)k_2\}$. By (1.4), $B^*(j_2\delta B'_4k_2)=j_2\delta B'_4\alpha k_3=0$ and $B^*(j_2B'_4\delta k_2)=j_2B'_4\delta\alpha k_3=j_2C'_4k_3$, which is non zero. This shows (i).

(ii) By Lemma 2.8 (i), this follows from the relations $\theta(R(2))=0$ and $\theta(j_2\delta B'_4\delta k_2)=-j_2B'_4\delta k_2-j_2\delta B'_4k_2\neq 0$. Q.E.D.

PROOF OF LEMMA 2.12. (i) Since $\mathscr{A}'_a(M)=0$ and $\mathscr{A}_{a+(p-1)q+1}(M)=0$, $\{X(p-1), M\}_{a-2}=0$ by Lemma 5.1 (ii). By a similar calculation, we see that $j_{p-1}^*\colon \{X(p-1), M\}_{a-(p-1)q-3}\to \mathscr{A}'_{a-(p-1)q-1}(M)=Z_p\{\varepsilon, C_2\delta, \delta C_2\}$ is isomorphic.

Also, $k_{p-1*}\colon A_a(X(p-1))\to\{X(p-1), M\}_{a-(p-1)q-3}$ is isomorphic, and (i) is proved.

(ii) Similarly as Lemma 2.8 (ii), we obtain

$$\mathscr{A}_{a+1}(X(p-1))=\begin{cases} Z_p\{\xi\} & \textit{for}\quad p\geqq 7,\\ Z_p\{\xi, j_{p-1}\delta C'_p\delta k_{p-1}\} & \textit{for}\quad p=5,\end{cases}$$

where ξ satisfies $k_{p-1}\xi j_{p-1}=C_2$. Also, $\theta(j_{p-1}\delta C'_p\delta k_{p-1})=-j_{p-1}C'_p\delta k_{p-1}+j_{p-1}\delta C'_p k_{p-1}\neq 0$. This shows (ii).

(iii) As is seen in (i), $\{X(p-1), M\}_{a-2}=0$. So, k_{1*} is monomorphic by (5.2). Since $\mathscr{A}'_{a+(p-2)q}(M)=Z_p\{\delta C'_4\delta\}=\operatorname{Coker}\alpha^{p-1*}$, we see from Lemma 5.1 (ii) that

$$\{X(p-1), M\}_{a-q-3}\cap \operatorname{Ker} j^*_{p-1} = Z_p\{\delta C'_4\delta k_{p-1}\}.$$

Since $\theta(\delta C'_4\delta k_{p-1})=-C'_4\delta k_{p-1}+\delta C'_4 k_{p-1}\neq 0$, we obtain (iii).

(iv) This is proved in the same way as (iii). Q.E.D.

At the end of this section, we prepare some lemmas which are applied in the next two sections. These lemmas are proved by an argument similar to the previous calculations, and the proof is omitted.

LEMMA 5.5. *Let* $b=(p^2+p-1)q-1$ $(=a-q-1)$.

(i) *The image of*

$$k_{p-2*}j^*_{p-2}\colon \mathscr{A}_b(X(p-2)) \longrightarrow \mathscr{A}_{(p^2+1)q-2}(M)$$

is the subgroup $Z_p\{\bar{\varepsilon}-\varepsilon\delta-\delta\varepsilon, \delta C_2\delta\}$.

(ii) $\mathscr{A}_{b+1}(X(p-2))\cap \operatorname{Ker} k_{p-2*}j^*_{p-2}\cap \operatorname{Ker}\theta = 0$.

(iii) $k_{p-2*}j^*_1\colon \{X(1), X(p-2)\}_{b+(p-3)q} \longrightarrow \mathscr{A}_{b-q-1}(M)$ *is monomorphic.*

LEMMA 5.6. *Let* $R=R(p-1)$ *and* $\Delta=\Delta_{p-1,p-1}=j_{p-1}k_{p-1}$. *Let* η_1 *and* η_2 *be the elements of* (2.15). *Then we have*

$$\mathscr{A}_{(p^2+1)q-1}(X(p-1)) = Z_p\{R\Delta, \Delta R, \eta_1\Delta, \Delta\eta_1, \eta_2\Delta, \Delta\eta_2\},$$

$$\mathscr{A}_{(p^2-p+2)q-2}(X(p-1))\cap \operatorname{Im} k^*_{p-1}j_{p-1*} = Z_p\{\Delta R\Delta, \Delta\eta_1\Delta, \Delta\eta_2\Delta\},$$

and hence

$$\mathscr{A}_{(p^2+1)q-1}(X(p-1))\cap \operatorname{Ker}\Delta_*\cap \operatorname{Ker}\Delta^* = 0.$$

§6. Proof of Theorems A′ – D′

From now on we denote simply by

$$R \quad \text{and} \quad \Delta$$

the elements $R(p-1)$ and $\Delta_{p-1,p-1}=j_{p-1}k_{p-1}$ of $\mathscr{A}_*(X(p-1))$, respectively.

LEMMA 6.1. $$\lambda_{X(p-1)}(\varepsilon\delta) = R\Delta-\Delta R.$$

PROOF. Put $\xi=\lambda_X(\varepsilon\delta)-R\Delta+\Delta R$, $X=X(p-1)$, and denote simply by $j=j_{p-1}$ and $k=k_{p-1}$. Then, $\Delta^2=jkjk=0$ by $kj=0$ and $\Delta R\Delta=j\varepsilon k$ by Theorem C(e). So we have

$$\begin{aligned} \xi\Delta &= \lambda_X(\varepsilon\delta)jk+j\varepsilon k \\ &= j\lambda_M(\varepsilon\delta)k+j\varepsilon k && \text{by } \theta(j)=0 \text{ and Proposition 1.11} \\ &= jD(\varepsilon\delta)k+j\varepsilon k && \text{by Proposition 1.12} \\ &= -j\varepsilon k+j\varepsilon k=0 && \text{by Propositions 1.12 and 2.4.} \end{aligned}$$

Similarly we have $\Delta\xi=0$.

Therefore ξ lies in the group $\mathscr{A}_{(p^2+1)q-1}(X)\cap\operatorname{Ker}\Delta_*\cap\operatorname{Ker}\Delta^*$, which is trivial by Lemma 5.6. Thus $\xi=0$. Q.E.D.

THEOREM 6.2. *For $R=R(p-1)$ and $\Delta=j_{p-1}k_{p-1}$, the following formula holds in $\mathscr{A}_*(X(p-1))$:*

$$R^2\Delta-2R\Delta R+\Delta R^2 = 0.$$

PROOF. By Proposition 1.12, the element $\lambda_{X(p-1)}(\varepsilon\delta)$ commutes with any element in $\mathscr{A}_*(X(p-1))\cap\operatorname{Ker}\theta$, i.e.,

(6.3) $(R\Delta-\Delta R)\xi=(-1)^{\deg\xi}\xi(R\Delta-\Delta R)$ *for any* $\xi\in\mathscr{A}_*(X(p-1))\cap\operatorname{Ker}\theta$.

Then the theorem is a restatement of (6.3) for $\xi=R$. Q.E.D.

Repeating Theorem 6.2 and the relation $\Delta^2=0$, we obtain

COROLLARY 6.4. *The following relations hold.*

(i) $$R^r\Delta R^s = sR^{r+s-1}\Delta R+(1-s)R^{r+s}\Delta = rR\Delta R^{r+s-1}+(1-r)\Delta R^{r+s}.$$

(ii) $$R^r\Delta R^s\Delta = \Delta R^s\Delta R^r = sR^{r+s-1}\Delta R\Delta = s\Delta R\Delta R^{r+s-1}.$$

(iii) *Any monomial on R and Δ involving three or more Δ's is zero.*

PROOF OF THEOREM C′. By Corollary 6.4(i) and Theorem C(b), we have

(*) $R^p\Delta = \Delta R^p$ and $R(p-r)^pA = AR(p-r-1)^p$, $1\leqq r\leqq p-2$, $R=R(p-1)$.

For each element in $(*)$, we take a map representing it and denote this map by the same letter. So, we interpret that the symbol $=$ in $(*)$ means "is homotopic to". And we choose a homotopy for each equality in $(*)$. Then we obtain the following homotopy commutative diagrams:

$$(D_r)\qquad \begin{array}{ccccc} S^{N+a-(p-r-1)q-1}X(p-r-1) & \xrightarrow{S^aA} & S^{N+a-(p-r)q-1}X(p-r) & \xrightarrow{S^a(\Delta A^{r-1})} & S^{N+a}X(p-1) \\ \downarrow{\scriptstyle R(p-r-1)^p} & & \downarrow{\scriptstyle R(p-r)^p} & & \downarrow{\scriptstyle R^p} \\ S^{N-(p-r-1)q-1}X(p-r-1) & \xrightarrow{A} & S^{N-(p-r)q-1}X(p-r) & \xrightarrow{\Delta A^{r-1}} & S^NX(p-1) \end{array}$$

for $r=1, 2,\cdots, p-2$, $a=(p^3+p^2)q$ and a large N such that the homotopy of the right square of (D_{r+1}) is obtained by composing two homotopies of (D_r).

Applying Lemma 2.5 (i)–(ii) to (D_r), we obtain elements $R'(2p-r)$, $r=2, 3,\cdots, p$, such that the following diagrams (E_r) for $1\leqq r\leqq p-2$ and (F_r) for $1\leqq r\leqq p-1$ are homotopy commutative:

$$(E_r)\qquad \begin{array}{ccc} S^aC_{\Delta A^r} & \xrightarrow{S^a\bar{A}} & S^aC_{\Delta A^{r-1}} \\ \downarrow{\scriptstyle R'(2p-r-2)} & & \downarrow{\scriptstyle R'(2p-r-1)} \\ C_{\Delta A^r} & \xrightarrow{\bar{A}} & C_{\Delta A^{r-1}}, \end{array}$$

$$(F_r)\qquad \begin{array}{ccccc} S^{N+a}X(p-1) & \xrightarrow{S^ai} & S^aC_{\Delta A^{r-1}} & \xrightarrow{S^aj} & S^{N+a-(p-r)q}X(p-r) \\ \downarrow{\scriptstyle R^p} & & \downarrow{\scriptstyle R'(2p-r-1)} & & \downarrow{\scriptstyle R(p-r)^p} \\ S^NX(p-1) & \xrightarrow{i} & C_{\Delta A^{r-1}} & \xrightarrow{j} & S^{N-(p-r)q}X(p-r)\,. \end{array}$$

By Lemma 1.5, we may replace, up to homotopy, $C_{\Delta A^r}$, $C_{\Delta A^{r-1}}$, $\bar{A}$, i and j by $S^{N-(p-r-1)q}X(2p-r-2)$, $S^{N-(p-r)q}X(2p-r-1)$, A, A^{p-r} and B^{p-1}, respectively. Thus, we obtain the elements $R'(r)$, $p\leqq r\leqq 2p-2$, satisfying $AR'(r-1)=R'(r)A$ for $p+1\leqq r\leqq 2p-2$, $AR(p-1)^p=R'(p)A$ and the following relations

$$(6.5)\qquad B^{p-1}R'(r) = R(r-p+1)^pB^{p-1} \qquad \textit{for} \quad p\leqq r\leqq 2p-2\,.$$

Q.E.D.

Proof of Theorem D′. We consider

$$R'(r)^*\colon \tilde{\Omega}^U_*(S^{N+a}X(r)) \longrightarrow \tilde{\Omega}^U_*(S^NX(r)), \quad 1\leqq r\leqq 2p-2,$$

where $R'(r)=R(r)^p$ for $1\leqq r\leqq p-1$, $a=(p^3+p^2)q$. By using (3.5) and Theorem C′, we can prove inductively that

$$(6.6)\qquad R'(r)_*(S^{N+a}\xi(r)) = ([V]^{p^2}+[M_r])\cdot\xi(r)$$

in the same way as Proposition 3.4. This shows the theorem. Q.E.D.

PROOF OF THEOREMS A′–B′. Similarly as Definition 4.8, we define $\rho'_{tp,r}$ and $\rho'(tp)$ by

$$\rho'_{tp,r} = \begin{cases} \pi k_{2p-r-1}R'(2p-r-1)^t j_{2p-r-1} i & \text{for} \quad 1 \leqq r \leqq p-1 \\ \pi k_{2p-r-1}R(2p-r-1)^{tp} j_{2p-r-1} i & \text{for} \quad p \leqq r \leqq 2p-2, \end{cases}$$

$$\rho'(tp) = k_{2p-2}R'(2p-2)^t j_{2p-2}.$$

Then, a similar discussion to Theorem A using (6.6) instead of Proposition 3.4 leads to $\rho'_{tp,r} \neq 0$, and hence $\rho'(tp) \neq 0$. Also it is easy to see from Theorem C′ and (6.5) that these elements satisfy the desired relations. Q.E.D.

§7. Some relations in $\mathscr{A}_*(M)$

PROPOSITION 7.1. *The following relations hold.*

(i) $\rho(t)\rho(u) = 0 \quad$ *if* $\quad t+u \not\equiv 0 \bmod p$.

(ii) $(t+u)\rho(t)\beta_{(s+up)} = t\rho(t+u)\beta_{(s)}$.

PROOF. (i) By Corollary 6.4,

$$(t+u)R^t \Delta R^u = tR^{t+u}\Delta + u\Delta R^{t+u}.$$

Since $k\Delta = \Delta j = 0$, we have $(t+u)\rho(t)\rho(u) = 0$.
(ii) Since $\Delta A^{p-2} = jk_1$ and $(t+u)kR^t\Delta R^u = tkR^{t+u}\Delta$, we have

$$\begin{aligned} (t+u)\rho(t)\beta_{(s+up)} &= (t+u)kR^t jk_1 R(1)^u \beta^s j_1 && \text{by Theorem C(a)} \\ &= (t+u)kR^t \Delta R^u A^{p-2}\beta^s j_1 && \text{by Theorem C(b)} \\ &= tkR^{t+u}\Delta A^{p-2}\beta^s j_1 = t\rho(t+u)\beta_{(s)}. \end{aligned}$$

Q.E.D.

REMARK 7.2. By using the recent results [5; Corollary 2], we can further discuss the triviality of the product $\rho(t)\beta_{(s)}$. In fact, $\rho(t)\beta_{(s)} = t\rho(1)\beta_{(s+tp-p)}$ by Proposition 7.1 (i). By Proposition 6.9 of [4], $\rho(1)\beta_{(s)}$ is a multiple of $\alpha\delta(\beta_{(1)}\delta)^{p-1}\beta_{(s+1)}$, which is trivial if $s \geqq p$.

In the following proposition, we say that the element ξ is *divisible by* η, if there are elements ζ and ζ' such that $\xi = \eta\zeta = \zeta'\eta$. Further if $\zeta = \zeta'$, we say that ξ is *strictly divisible by* η.

PROPOSITION 7.3. (i) *The element* $\beta_{(tp)}$ *is strictly divisible by* α^{p-2}, *and the element* $\beta_{(tp^2)}$ *is strictly divisible by* α^{2p-3}.

(ii) *The element* $\rho(tp)$ *is strictly divisible by* α^{p-1}, *and the element* $\rho(r)\rho(s)$

is divisible by α^{p-1}.

(iii) *The element* $\beta_{(r)}\beta_{(s)}$ *is strictly divisible by* α^{p-3}.

REMARK. By Proposition 7.1 (i), the second statement of (ii) is efficient for $r+s\equiv 0 \bmod p$. Also, by [11; Th. 5.1], so is the statement (iii). The statement (i) may be considered as a result corresponding to the following fact on the p-divisibility of the α-family $\{\alpha_r\}$ in $G_*[0]$ (cf. [4; §4]).

The element $\alpha_{rp}\in G_{rpq-1}$ *is divisible by* p, *and the element* $\alpha_{rp^2}\in G_{rp^2q-1}$ *is divisible by* p^2.

PROOF OF PROPOSITION 7.3. (i) and the first of (ii) are easy consequences of Theorems B and B'. Since $j_*\rho(r)\rho(s)=\Delta R^r\Delta R^s j=R^s\Delta R^r\Delta j=0$ by Corollary 6.4, it follows from (1.3) that $\rho(r)\rho(s)=\alpha^{p-1}\xi$ for some ξ. Similarly $k^*\rho(r)\rho(s)=0$ and $\rho(r)\rho(s)=\xi'\alpha^{p-1}$, and the second half of (ii) is proved.

By [11; (5.1)(iv) and (5.4)(i)], it suffices to show (iii) for the case $r=1$, $s=tp-1$. For $t=1$, we have proved in [4; (6.11)] that $\beta_{(1)}\beta_{(p-1)}=\eta\alpha^{p-3}=\alpha^{p-3}\eta$ for some non zero multiple η of the element $\bar{\varepsilon}-\varepsilon\delta-\delta\varepsilon\in\mathscr{A}_{(p^2+1)q-2}(M)$. Then by Lemma 5.5 we obtain the following result similarly as Theorem C.

(7.4) *There exists an element*

$$S\in\mathscr{A}_{(p^2+p-1)q-1}(X(p-2))$$

satisfying the relations $SA^{p-3}=-A^{p-3}(\beta^{p-1}\Delta_1\beta)$, $\theta(S)=0$ *and* $k_{p-2}Sj_{p-2}=\eta$, *where* $\Delta_1=j_1k_1$.

We put

$$\sigma(t)=k_{p-2}R(p-2)^{t-1}Sj_{p-2}.$$

Then, by (1.4), (7.4), Theorem C and (2.2)–(2.3), we have

$$D(\sigma(t))=-\theta(\sigma(t))=0,$$

$$\sigma(t)\alpha^{p-3}=-\beta_{(tp-1)}\beta_{(1)}.$$

The subring $\operatorname{Ker} D$ of $\mathscr{A}_*(M)$ is commutative [4; (1.11)]. Hence $\beta_{(1)}\beta_{(tp-1)}=\sigma(t)\alpha^{p-3}=\alpha^{p-3}\sigma(t)$ as desired. Q.E.D.

PROPOSITION 7.5. $\alpha\delta\beta_{(tp)}\delta=\delta\alpha\delta\beta_{(tp)}$ *for* $t\geqq 1$.

PROOF. To prove the proposition, we in troduce an element $\alpha''\in\mathscr{A}_{q-2}(X(1))$ due to H. Toda [11]. This is a generator of $\mathscr{A}_{q-2}(X(1))=Z_p$ and satisfies $\alpha''j_1=-j_1\delta\alpha\delta$ and $k_1\alpha''=-\delta\alpha\delta k_1$ [11; Lemma 3.1, Th. 3.5, (5.6)]. He also has proved the relation $\beta^r\alpha''\beta^s=s\beta^{r+s-1}\alpha''\beta+(1-s)\beta^{r+s}\alpha''$ [11; Prop. 4.7]. Thus, we have $\beta^{tp}\alpha''=\alpha''\beta^{tp}$ and so $\alpha\delta\beta_{(tp)}\delta=\beta_{(tp)}\delta\alpha\delta=k_1\beta^{tp}j_1\delta\alpha\delta=-k_1\beta^{tp}\alpha''j_1=k_1\alpha''\beta^{tp}j_1=\delta\alpha\delta k_1\beta^{tp}j_1=\delta\alpha\delta\beta_{(tp)}$. Q.E.D.

COROLLARY 7.6. *The element* $\alpha_1\beta_{tp}=\alpha_1\rho_{t,p-1}\in G_*$ *is divisible by* p.

NOTE. L. Smith has obtained independently the same results as our Theorems A and D. His results will be appeared in a paper entitled "On realizing complex bordism modules IV. Applications to the stable homotopy groups of spheres".

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