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On the Group of Self-Equivalences of the Product of Spheres

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§1. Introduction

The set $\mathscr{E}(X)$ of homotopy classes of self-(homotopy-)equivalences of a based space X forms a group by the composition of maps, and this group is studied by several authors.

The purpose of this note is to study the groups $\mathscr{E}(S^m \times S^n)$ of the products $S^m \times S^n$, where S^k is the k-sphere. These are studied by P. J. Kahn [8] for the case $m=n$, and by A. J. Sieradski [13] for the case $m, n=1, 3, 7$.

In the first, we consider the case $n>m\geqq 2$. Then the wedge $S^m \vee S^n$ is simply connected, and we can apply the results of [10, §§1–2] to the mapping cone $S^m \times S^n = (S^m \vee S^n) \cup e^{m+n}$ of the Whitehead product. Hence, by using the results of W. D. Barcus and M. G. Barratt [3, §4], we have in Theorem 2.6 the exact sequence

$$0 \longrightarrow H_{m,n} \longrightarrow \mathscr{E}(S^m \times S^n) \longrightarrow G_{m,n} \longrightarrow 1,$$

where $H_{m,n}$ is the factor group of $\pi_{m+n}(S^m)+\pi_{m+n}(S^n)$ and $G_{m,n}$ is the subgroup of $\mathscr{E}(S^m \vee S^n)$. In §3, we study some cases that this sequence is split, but the extension of this sequence is not known to us in general. Also, by using the quaternion, we compute $\mathscr{E}(S^m \times S^n)$ for $m=2$, 3 and $n>m$ in Theorems 4.3 and 5.3, and we see that the above sequence is split if $m=2$ and is not split if $m=3$ and $n=5$.

By the same way, we have in Theorem 6.2 the similar exact sequence for the case $n=m\geqq 2$, which is split if n is even. Furthermore, we can determine the group $\mathscr{E}(S^n \times S^n)$ for $n=3$, 7 in Theorem 6.4.

The group $\mathscr{E}(S^1 \times S^n)$ is computed in §§7–8 by the different methods. By attaching i-cells ($i\geqq n+3$) to S^n, we obtain a CW-complex X_{n+1} which kills the r-th homotopy groups of S^n for $r\geqq n+2$, and we see that $\mathscr{E}(S^1 \times S^n)$ is isomorphic to $\mathscr{E}(S^1 \times X_{n+1})$ (Lemma 7.1). Consider the composition

$$f\colon S^1 \times K(Z, n) \longrightarrow K(Z, n) \xrightarrow{f'} K(\pi_{n+1}(S^n), n+2)$$

of the natural projection and the generator f' of $H^{n+2}(Z, n; \pi_{n+1}(S^n))$. Then, it is well known that $S^1 \times X_{n+1}$ is the mapping track E_f of f. Hence, we can apply the results of J. W. Rutter [11] and [10, §5] to $\mathscr{E}(S^1 \times X_{n+1})$, and the group $\mathcal{E}(S^1 \times S^n)$ is determined in Theorem 7.9 for $n \geqq 3$ and in Theorem 8.8 for $n=2$.

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§2. The group $\mathcal{E}(S^m \times S^n)$ for $n>m\geqq 2$

In this note, all (topological) spaces are arcwise connected spaces with base point $*$ and have homotopy types of CW-complexes, and all (continuous) maps and homotopies preserve the base points. For given spaces X and Y, we denote by $[X, Y]$ the set of (based) homotopy classes of maps from X to Y, and by the same letter f a map $f: X \to Y$ and its homotopy class $f \in [X, Y]$. Also, we denote usually by

$$g_*: [X, Y] \longrightarrow [X, Z], \qquad g^*: [Z, X] \longrightarrow [Y, X]$$

the induced maps of a given map $g: Y \to Z$.

The group of homotopy classes of self-homotopy-equivalences of a space X is denoted by

$$\mathcal{E}(X) \quad (\subset [X, X]),$$

whose multiplication is given by the composition of maps.

In the first we consider the group $\mathcal{E}(S^m \vee S^n)$ of the wedge $S^m \vee S^n$ for $n > m \geqq 2$, where S^k is the k-sphere in the real $(k+1)$-space. Let

(2.1) $$i_1: S^m \subset S^m \vee S^n, \qquad i_2: S^n \subset S^m \vee S^n$$

be the inclusion maps and

(2.2) $$\lambda: \pi_n(S^m) \longrightarrow \mathcal{E}(S^m \vee S^n)$$

be the homomorphism given by

(2.3) $$\lambda(\xi) \circ i_1 = i_1, \qquad \lambda(\xi) \circ i_2 = i_1 \circ \xi + i_2$$

for $\xi \in \pi_n(S^m)$, where $\circ$ is the composition of maps and $+$ is the sum in $\pi_n(S^m \vee S^n)$. Then we have the next proposition (cf. [10, §1]).

PROPOSITION 2.4. *For $n > m \geqq 2$, we have the split exact sequence*

$$0 \longrightarrow \pi_n(S^m) \xrightarrow{\lambda} \mathcal{E}(S^m \vee S^n) \longrightarrow Z_2 + Z_2 \longrightarrow 1,$$

and so we have

(2.5) $$\mathcal{E}(S^m \vee S^n) = \{a_{ij}\lambda(\xi) \mid i, j \in Z_2 = \{0, 1\}, \xi \in \pi_n(S^m)\},$$

where $a_{ij}=(-\iota_m)^i \vee (-\iota_n)^j$ ($\iota_k \in \pi_k(S^k)$ *is the class of the identity map*) *with relations*

$$\lambda(\xi)a_{ij} = a_{ij}\lambda((-\iota_m)^i \circ \xi \circ (-\iota_n)^j).$$

The product $S^m \times S^n$ is the mapping cone

$$S^m \times S^n = (S^m \vee S^n) \cup_{[i_1, i_2]} e^{m+n}$$

of the Whitehead product

$$[i_1, i_2]: S^{m+n-1} \longrightarrow S^m \vee S^n$$

of the inclusion maps of (2.1). By the above result and the results of [10, §2], we have the following theorem.

THEOREM 2.6. *Assume* $n>m\geqq 2$. *Then there is an exact sequence*

$$0 \longrightarrow H_{m,n} \xrightarrow{\lambda'} \mathscr{E}(S^m \times S^n) \xrightarrow{\varphi} G_{m,n} \longrightarrow 1. \tag{2.7}$$

The groups $H_{m,n}$ *and* $G_{m,n}$ *are given by*

$$H_{m,n} = \pi_{m+n}(S^n)/[\iota_m, \pi_{n+1}(S^m)] + \pi_{m+n}(S^n)/[\iota_n, \pi_{m+1}(S^n)], \tag{2.8}$$

$$G_{m,n} = \{a_{ij}\lambda(\xi) \mid [\iota_m, \xi] = 0, \xi \in \pi_n(S^m), i, j \in Z_2\} \quad (\subset \mathscr{E}(S^m \vee S^n)), \tag{2.9}$$

and φ *is given by the restriction on* $S^m \vee S^n$.

PROOF. By the results of [10, §2], we have the exact sequence

$$0 \longrightarrow H_{m,n} \xrightarrow{\lambda'} \mathscr{E}(S^m \times S^n) \xrightarrow{\varphi \times \psi} G'_{m,n} \longrightarrow 1,$$

where $H_{m,n}=\pi_{m+n}(S^m \times S^n)/\mathrm{Im}\,\gamma$ for the homomorphism

$$\gamma = \Gamma(i,f): [S^{m+1} \vee S^{n+1}, S^m \times S^n] \longrightarrow \pi_{m+n}(S^m \times S^n)$$

(i: $S^m \vee S^n \to S^m \times S^n$ is the inclusion, $f=[i_1, i_2]$), and

$$G'_{m,n} = \{(h, \varepsilon) \mid h \in \mathscr{E}(S^m \vee S^n), \varepsilon = \pm\iota \in \mathscr{E}(S^{m+n-1}), h \circ f = f \circ \varepsilon \text{ in } \pi_{m+n-1}(S^m \vee S^n)\}.$$

We see easily that $\Gamma(i,f)$, defined in [10, (2.5)], coincides by definition with the homomorphism

$$\kappa: \pi_{m+1}(X)+\pi_{n+1}(X) \longrightarrow \pi_{m+n}(X)$$

of [3, §8, p. 70] for $X=S^m \times S^n$ and $w=i\circ i_1$, $v=i\circ i_2$. Therefore, by [3, (8.1) (i)] we have

$$\gamma(\eta, \xi) = -[i\circ i_1, \xi]+(-1)^{n+1}[\eta, i\circ i_2]$$

for $\eta \in \pi_{m+1}(S^m \times S^n)$, $\xi \in \pi_{n+1}(S^m \times S^n)$, and we see that $H_{m,n}$ is given by (2.8).

On the other hand, by (2.3) and the definition of the Whitehead product, we have

$$\begin{aligned} a_{ij}\lambda(\xi)\circ f &= [(-1)^i i_1, i_1\circ(-\iota_m)^i\xi+(-1)^j i_2] \\ &= (-1)^i[i_1, i_1\circ(-\iota_m)^i\xi]+(-1)^{i+j}[i_1, i_2]. \end{aligned}$$

By using the direct sum decomposition

$$\pi_{m+n-1}(S^m \vee S^n) \simeq \pi_{m+n-1}(S^m)+\pi_{m+n-1}(S^n)+\pi_{n+m}(S^m \times S^n, S^m \vee S^n),$$

we see easily that

$$a_{ij}\lambda(\xi)\circ f = f\circ\varepsilon \quad \text{if and only if} \quad [\iota_m, \xi] = 0 \quad \text{and} \quad \varepsilon = (-\iota)^{i+j}.$$

Therefore, $G_{m,n}$ of (2.9) is isomorphic to $G'_{m,n}$ by corresponding $a_{ij}\lambda(\xi)\leftrightarrow(a_{ij}\lambda(\xi), (-\iota)^{i+j})$, and the homomorphism $\varphi\times\psi$ corresponds to the restriction φ. q.e.d.

§3. Group extensions in (2.7)

In this section, assume that $n>m\geqq 2$. Let $\xi\in\pi_n(S^m)$ satisfy $[\iota_m, \xi]=0$. Then there is a map F_ξ: $S^m\times S^n\to S^m$ of type (ι_m, ξ) by the definition of the Whitehead product, and we obtain a map

$$\bar{\lambda}(\xi) = (F_\xi, p_2): S^m\times S^n \longrightarrow S^m\times S^n, \tag{3.1}$$

where p_2 is the projection onto the 2nd factor. Consider the elements

$$b_{ij} = (-\iota_m)^i\times(-\iota_n)^j \in \mathscr{E}(S^m\times S^n), \qquad i,j\in Z_2. \tag{3.2}$$

Then we have easily the following lemma by the definition.

LEMMA 3.3. $\varphi(b_{ij}\bar{\lambda}(\xi)) = a_{ij}\lambda(\xi)$,

where φ is the homomorphism in (2.7).

THEOREM 3.4. *Assume that $\bar{\lambda}$ of (3.1) can be chosen so that*

$$\bar{\lambda}(\xi_1)\bar{\lambda}(\xi_2) = \bar{\lambda}(\xi_1+\xi_2), \qquad \bar{\lambda}(\xi_1)b_{ij} = b_{ij}\bar{\lambda}((-\iota_m)^i\circ\xi_1\circ(-\iota_n)^j),$$

for any $\xi_i\in\pi_n(S^m)$ with $[\iota_m, \xi_i]=0$. Then the exact sequence (2.7) is split. Also the action of $G_{m,n}$ on $H_{m,n}$ is given by

$$a_{ij}\lambda(\xi)\cdot(\alpha, \beta) = ((-1)^{i+j}(-\iota_m)^i\circ F_\xi\circ(\alpha, \beta), (-1)^i\beta)$$

for $\alpha \in \pi_{m+n}(S^m)/[\iota_m, \pi_{n+1}(S^m)]$, $\beta \in \pi_{m+n}(S^n)/[\iota_n, \pi_{m+1}(S^n)]$.

PROOF. The former is obtained immediately by Theorem 2.6 and Lemma 3.3. By the definition of $\bar{\lambda}(\xi)$ of (3.1), we have the homotopy commutative diagram

$$\begin{array}{ccccccc}
S^m \times S^n & \xrightarrow{l} & (S^m \times S^n) \vee S^{m+n} & \xrightarrow{1 \vee (\alpha, \beta)} & (S^m \times S^n) \vee (S^m \times S^n) & \xrightarrow{\nabla} & S^m \times S^n \\
\downarrow{\scriptstyle \bar{\lambda}(\xi)} & & \downarrow{\scriptstyle \bar{\lambda}(\xi) \vee 1} & & \downarrow{\scriptstyle \bar{\lambda}(\xi) \vee \bar{\lambda}(\xi)} & & \downarrow{\scriptstyle \bar{\lambda}(\xi)} \\
S^m \times S^n & \xrightarrow{l} & (S^m \times S^n) \vee S^{m+n} & \xrightarrow{1 \vee \bar{\lambda}(\xi) \circ (\alpha, \beta)} & (S^m \times S^n) \vee (S^m \times S^n) & \xrightarrow{\nabla} & S^m \times S^n .
\end{array}$$

The composition of the maps in the upper sequence is $\lambda'(\alpha, \beta)$ by the definition of λ' in [10, § 2], and also the composition of the lower one is $\lambda'(F_{\xi} \circ (\alpha, \beta), \beta)$ by (3.1). These show that

$$\bar{\lambda}(\xi)^{-1} \lambda'(\alpha, \beta) \bar{\lambda}(\xi) = \lambda'(F_{\xi} \circ (\alpha, \beta), \beta).$$

By the same way, we have

$$\begin{aligned} b_{ij}^{-1} \lambda'(\alpha, \beta) b_{ij} &= \lambda'((-\iota_m)^i \circ \alpha \circ (-\iota)^{i+j}, (-\iota_n)^j \circ \beta \circ (-\iota)^{i+j}) \\ &= \lambda'((-1)^{i+j}(-\iota_m)^i \circ \alpha, (-1)^i \beta), \end{aligned}$$

because $(-\iota_n) \circ \beta \equiv -\beta \bmod [\iota_n, \pi_{m+1}(S^n)]$ by [4, Th. 6.7, 6.9]. q.e.d.

COROLLARY 3.5. *Assume that* $n > m \geqq 2$ *and* $[\iota_m, \xi] \neq 0$ *for any nonzero element* $\xi \in \pi_n(S^m)$. *Then we have the split exact sequence*:

$$0 \longrightarrow H_{m,n} \longrightarrow \mathscr{E}(S^m \times S^n) \longrightarrow Z_2 + Z_2 \longrightarrow 0,$$

and the action of $Z_2 + Z_2$ *on* $H_{m,n}$ *are given by*

$$a_{ij} \cdot (\alpha, \beta) = ((-1)^{i+j}(-\iota_m)^i \circ \alpha, (-1)^i \beta).$$

PROOF. It is clear, since $G_{m,n} = \{a_{ij}\} = Z_2 + Z_2$ by the assumption. q.e.d.

EXAMPLE 3.6. *Let* $n - 1 = m \geqq 2$. *Then, we have the exact sequence*

$$0 \longrightarrow H_{m,m+1} \longrightarrow \mathscr{E}(S^m \times S^{m+1}) \longrightarrow G_{m,m+1} \longrightarrow 0,$$

where

$$H_{m,m+1} = \pi_{2m+1}(S^m)/\{[\iota_m, \eta_m \eta_{m+1}]\} + \pi_{2m+1}(S^{m+1})/\{[\iota_{m+1}, \iota_{m+1}]\},$$

$$G_{m,m+1} = \begin{cases} Z_2 + Z_2 + Z_2 & \text{if } m \equiv 3 \bmod 4 \text{ or } m = 2, 6 \\ Z_2 + Z_2 & \text{if } m \not\equiv 3 \bmod 4 \text{ and } m \neq 2, 6, \end{cases}$$

(η_k *is the generator of* $\pi_{k+1}(S^k)$). *Moreover if* $m \not\equiv 3 \bmod 4$ *and* $m \neq 2, 6$, *then the above exact sequence is split with the action given by* $a_{ij} \cdot (\alpha, \beta) = ((-1)^j \alpha, (-1)^i \beta)$.

PROOF. By [5, p. 232] and [6, Lemma 5.1], it is proved that $[\iota_m, \eta_m] \neq 0$ if and only if $m \not\equiv 3 \bmod 4$ and $m \neq 2, 6$. Also $(-\iota_m) \circ \alpha \equiv -\alpha \bmod [\iota_m, \pi_{m+2}(S^m)]$ by [4, Th. 6.7, 6.9]. These results, Theorem 3.4 and Corollary 3.5 show the desired results. q.e.d.

§4. The group $\mathscr{E}(S^2 \times S^n)$ for $n \geqq 3$

In this section, we assume that $n \geqq 3$.

LEMMA 4.1. (i) *The group* $G_{2,n}$ *of* (2.9) *is*

$$G_{2,n} = \{a_{ij}\lambda(\xi) \mid \xi \in \pi_n(S^2), i, j \in Z_2\},$$

and the multiplication is given by

$$a_{ij}\lambda(\xi)a_{i'j'}\lambda(\xi') = a_{i+i',j+j'}\lambda((-1)^{j'}\xi + \xi').$$

(ii) *The group* $H_{2,n}$ *of* (2.8) *is*

$$H_{2,n} = \pi_{n+2}(S^2) + Z_2.$$

PROOF. It is well known that $[\iota_2, \xi] = 0$ for $\xi \in \pi_n(S^2)$ $(n \geqq 3)$. Therefore, $G_{2,n}$ is given as above by Theorem 2.6. It is known that

$$(4.2) \qquad (-\iota_2) \circ \xi = \xi \qquad \text{for} \quad \xi \in \pi_n(S^2),$$

(cf. [12, p. 278]), and we have

$$a_{ij}\lambda(\xi)a_{i'j'}\lambda(\xi') = a_{i+i',j+j'}\lambda((-1)^{j'}\xi + \xi')$$

by Theorem 2.6 and Proposition 2.4. Since $\pi_{n+2}(S^n) = Z_2$, (ii) follows immediately. q.e.d.

Now we have the next theorem by Theorem 3.2.

THEOREM 4.3. *Let* $n \geqq 3$. *Then the exact sequence*

$$0 \longrightarrow H_{2,n} \longrightarrow \mathscr{E}(S^2 \times S^n) \longrightarrow G_{2,n} \longrightarrow 1$$

is split, where $H_{2,n}$ *and* $G_{2,n}$ *are the groups in Lemma* 4.1. *The action of* $G_{2,n}$ *on* $H_{2,n}$ *is given by*

$$a_{ij}\lambda(\xi) \cdot (\alpha, \beta) = ((-1)^{i+j}\alpha + \xi\beta, \beta)$$

for $\xi \in \pi_n(S^2)$, $\alpha \in \pi_{n+2}(S^2)$, $\beta \in \pi_{n+2}(S^n) = Z_2$.

PROOF. Consider the Hopf map $h: S^3 \to S^2$ and a map $F: S^2 \times S^3 \to S^2$ of type (ι_2, h), given by

$$h(q) = qiq^{-1}, \qquad F(p, q) = qpq^{-1},$$

where $q \in S^3$ is a quaternion of norm 1, $p \in S^2$ is a pure quaternion of norm 1, and i is the imaginary unit. Then, we can construct

$$F_\xi = F\circ(\iota_2 \times \xi'): S^2 \times S^n \longrightarrow S^2,$$

$$\bar{\lambda}(\xi) = (F_\xi, p_2): S^2 \times S^n \longrightarrow S^2 \times S^n,$$

for any $\xi \in \pi_n(S^2)$, where $\xi' \in \pi_n(S^3)$ satisfies $h\xi' = \xi$. It is clear that F_ξ is of type (ι_2, ξ). By using the equality

$$\bar{\lambda}(\xi)(p, x) = (\xi'(x)p\xi'(x)^{-1}, x) \qquad \text{for} \quad p \in S^2,\ x \in S^n,$$

we can show that $\bar{\lambda}$ satisfies the assumptions of Theorem 3.4 as follows.

$$\begin{aligned}\bar{\lambda}(\xi_1)\bar{\lambda}(\xi_2)(p, x) &= (\xi_1'(x)\xi_2'(x)p\xi_2'(x)^{-1}\xi_1'(x)^{-1}, x)\\ &= \bar{\lambda}(\xi_1 + \xi_2)(p, x),\\ \bar{\lambda}(\xi)b_{ij}(p, x) &= (\xi'(y)p^{-i}\xi'(y)^{-1}, y) \qquad (y = (-\iota_n)^j(x))\\ &= b_{ij}\bar{\lambda}(\xi\circ(-\iota_n)^j)(p, x)\\ &= b_{ij}\bar{\lambda}((-\iota_2)^i\circ\xi\circ(-\iota_n)^j)(p, x) \quad \text{by (4.2).}\end{aligned}$$

Also, it is easy to see that

$$F\circ(h \times \iota_3) = h\circ m\circ T,$$

where $m: S^3 \times S^3 \to S^3$ is the multiplication of S^3 and $T: S^3 \times S^3 \to S^3 \times S^3$ is the switching map. Therefore, for any $\alpha = h\alpha' \in \pi_{n+2}(S^2)$ and $\beta \in \pi_{n+2}(S^2)$, we have

$$\begin{aligned}F_\xi(\alpha, \beta) &= F\circ(h \times \iota_3)\circ(\alpha', \xi'\beta)\\ &= h(\xi'\beta + \alpha') = \alpha + \xi\beta.\end{aligned}$$

These show the desired results by Theorem 3.4. q.e.d.

§5. The group $\mathscr{E}(S^3 \times S^n)$ for $n \geqq 4$

In this section, we study the case $m = 3$.

For any $\xi \in \pi_n(S^3)$, we have $[\iota_3, \xi] = 0$ and we can define maps

$$E_{\xi}: S^3 \times S^n \longrightarrow S^3, \qquad \bar{\lambda}(\xi): S^3 \times S^n \longrightarrow S^3 \times S^n$$

by $E_{\xi}(x, y)=x\xi(y)$, $\bar{\lambda}(\xi)(x, y)=(x\xi(y), y)$. By Theorem 2.6, we have the exact sequence

$$0 \longrightarrow \pi_{n+3}(S^3)+\pi_{n+3}(S^n) \xrightarrow{\lambda'} \mathscr{E}(S^3 \times S^n) \xrightarrow{\varphi} G_{3,n} \longrightarrow 1, \tag{5.1}$$

where

$$G_{3,n} = \{a_{ij}\lambda(\xi) \mid \xi \in \pi_n(S^3),\, i, j \in Z_2\}.$$

Since $\bar{\lambda}(\xi)$ is of type (ι_3, ξ), we have

$$\varphi(b_{ij}\bar{\lambda}(\xi)) = a_{ij}\lambda(\xi) \qquad \text{for} \quad \xi \in \pi_n(S^3), \tag{5.2}$$

where b_{ij} are the elements of (3.2).

THEOREM 5.3. *Let* $n \geqq 4$. *Then we have*

$$\mathscr{E}(S^3 \times S^n) = \{b_{ij}\bar{\lambda}(\xi)\lambda'(\alpha, \beta) \mid \alpha \in \pi_{n+3}(S^3), \beta \in \pi_{n+3}(S^n), \xi \in \pi_n(S^3),$$
$$i, j \in Z_2\}.$$

The group structure of $\mathscr{E}(S^3 \times S^n)$ *is given as follows.*

(i) $\lambda'(\alpha_1, \beta_1)\lambda'(\alpha_2, \beta_2) = \lambda'(\alpha_1+\alpha_2, \beta_1+\beta_2)$,

(ii) $\bar{\lambda}(\xi_1)\bar{\lambda}(\xi_2) = \bar{\lambda}(\xi_1+\xi_2)$,

(iii) $b_{ij}b_{i'j'} = b_{i+i',j+j'}, \qquad b_{00} = 1$;

(iv) $\bar{\lambda}(\xi)b_{01} = b_{01}\bar{\lambda}(-\xi)$,

(v) $\bar{\lambda}(\xi)b_{10} = b_{10}\bar{\lambda}(-\xi)\lambda'(\omega_3 S^3\xi, 0)$;

(vi) $\lambda'(\alpha, \beta)b_{01} = b_{01}\lambda'(-\alpha, -(-\iota_n)\circ\beta)$,

(vii) $\lambda'(\alpha, \beta)b_{10} = b_{10}\lambda'(\alpha, -\beta)$,

(viii) $\lambda'(\alpha, \beta)\bar{\lambda}(\xi) = \bar{\lambda}(\xi)\lambda'(\alpha-\xi\beta, \beta)$.

Here, $S^3: \pi_n(S^3) \to \pi_{n+3}(S^6)$ *is the suspension homomorphism. Also* ω_3 *is a generator of* $\pi_6(S^3)=Z_{12}$ *given by*

$$\pi^*(\omega_3) = \phi, \tag{5.4}$$

where $\phi: S^3 \times S^3 \to S^3$ *is the commutator map*: $\phi(p, q)=pqp^{-1}q^{-1}$, *and* $\pi: S^3 \times S^3 \to (S^3 \times S^3)/(S^3 \vee S^3)=S^6$ *is the collapsing map, (cf. e.g.* [2, p. 173]).

To prove the theorem, we use the next two lemmas.

Lemma 5.5. *Let $p_1: S^3 \times S^n \to S^3$ and $p_2: S^3 \times S^n \to S^n$ be the projections. Then we have*

$$p_1 \cdot \xi p_2 = \pi^*(\omega_3 S^3 \xi) \cdot \xi p_2 \cdot p_1 = \xi p_2 \cdot p_1 \cdot \pi^*(\omega_3 S^3 \xi),$$

where $\pi: S^3 \times S^n \to (S^3 \times S^n)/(S^3 \vee S^n) = S^{n+3}$ is the collapsing map.

Proof. It is easy to see that

$$p_1 \cdot \xi p_2 \cdot p_1^{-1} \cdot \xi p_2^{-1} = \phi \circ (\iota_3 \times \xi)$$
$$= \omega_3 \circ \pi \circ (\iota_3 \times \xi) = \omega_3 \circ S^3 \xi \circ \pi,$$

by (5.4), and we have the first equality. Therefore we have the desired results, since ϕ is homotopic to the map $S^3 \times S^3 \to S^3$ given by $(p, q) \to p^{-1} q^{-1} pq$.

q.e.d.

Lemma 5.6. *For the monomorphism λ' in (5.1), we have*

$$\lambda'(\alpha, 0) f = ((p_1 f) \cdot (\alpha \pi f), p_2 f)$$

for any $\alpha \in \pi_{n+3}(S^3)$ and $f: S^3 \times S^n \to S^3 \times S^n$.

Proof. The desired equality follows from

$$p_1 \lambda'(\alpha, 0) = p_1 \cdot \alpha \pi, \qquad p_2 \lambda'(\alpha, 0) = p_2,$$

which are seen by the definition: $\lambda'(\alpha, 0) = \nabla \circ (1 \vee (\alpha, 0)) \circ l$. q.e.d.

Remark. *If $n \geqq 5$*, we see easily by definition that

$$\lambda'(\alpha, \beta) = (p_1 \cdot \alpha \pi, p_2 + \beta \pi)$$

where $+$ is the sum in the cohomotopy group $[S^3 \times S^n, S^n]$.

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. By (5.1) and (5.2), it is sufficient to prove the relations (i)–(viii). (i)–(iii) are seen easily.

(iv) $\bar{\lambda}(\xi) b_{01} = (p_1 \cdot \xi p_2, p_2)(\iota_3 \times (-\iota_n))$

$$= (p_1 \cdot (-\xi p_2), (-\iota_n) \circ p_2) = b_{01} \bar{\lambda}(-\xi).$$

(viii) $\bar{\lambda}(\xi) \lambda'(\alpha, \beta) \bar{\lambda}(-\xi) = \lambda'(\bar{\lambda}(\xi) \circ (\alpha, \beta))$

$$= \lambda'((p_1 \cdot \xi p_2, p_2) \circ (\alpha, \beta)) = \lambda'(\alpha + \xi \beta, \beta).$$

(v) $\bar{\lambda}(\xi) b_{10} = (p_1 \cdot \xi p_2, p_2)(-\iota_3 \times \iota_n)$

$$= ((-p_1)\cdot \xi p_2, p_2) = b_{10}((-\xi p_2)\cdot p_1, p_2)$$

$$= b_{10}(p_1\cdot(-\xi p_2)\cdot \pi^*(\omega_3 S^3\xi), p_2) \quad \text{by Lemma 5.5}$$

$$= b_{10}\lambda'(\omega_3 S^3\xi, 0)\bar{\lambda}(-\xi) \quad \text{by Lemma 5.6 and } \pi\bar{\lambda}(-\xi)=\pi$$

$$= b_{10}\bar{\lambda}(-\xi)\lambda'(\omega_3 S^3\xi, 0) \quad \text{by (viii).}$$

(vi) $b_{01}\lambda'(\alpha, \beta)b_{01} = \lambda'(b_{01}(\alpha, \beta)(-\iota)) = \lambda'(-\alpha, -(-\iota_n)\beta)$.

(vii) is similar. q.e.d.

COROLLARY 5.8. *If $\omega_{3*}S^3: \pi_n(S^3)\to\pi_{n+3}(S^3)$ is 0-map, then the exact sequence* (5.1) *is split, where the multiplication of $G_{3,n}$ is given in Theorem* 3.4.

COROLLARY 5.9. *Assume that there is an element $\xi\in\pi_n(S^3)$ such that*

$$2\alpha+\xi\beta+\omega_3 S^3\xi \neq 0 \quad \textit{for any} \quad \alpha\in\pi_{n+3}(S^3),\ \beta\in\pi_{n+3}(S^n).$$

Then the sequence (5.1) *is not split.*

PROOF. It follows from Proposition 2.4 that $(a_{10}\lambda(\xi))^2=1$. On the other hand, using the relations in Theorem 5.3, we have

$$(b_{10}\bar{\lambda}(\xi)\lambda'(\alpha, \beta))^2 = b_{10}\bar{\lambda}(\xi)b_{10}\lambda'(\alpha, -\beta)\bar{\lambda}(\xi)\lambda'(\alpha, \beta) \quad \text{by (vii)}$$

$$= \bar{\lambda}(-\xi)\lambda'(\omega_3 S^3\xi, 0)\lambda'(\alpha, -\beta)\lambda'(\alpha+\xi\beta, \beta)\bar{\lambda}(\xi) \quad \text{by (v), (viii)}$$

$$= \lambda'(2\alpha+\xi\beta+\omega_3 S^3\xi, 0) \quad \text{by (i), (viii), (ii).}$$

The last element is not zero by the assumption, and we have the corollary. q.e.d.

EXAMPLE 5.10. *The next exact sequence is not split.*

$$0 \longrightarrow Z_{24}+Z_2 \longrightarrow \mathscr{E}(S^3\times S^5) \longrightarrow Z_2+Z_2+Z_2 \longrightarrow 0.$$

PROOF. For the element $\eta_3^2\in\pi_5(S^3)$, we have

$$2\alpha+\eta_3^2\beta+\omega_3 S^3\eta_3^2 \neq 0 \qquad \text{for any} \quad \alpha\in\pi_8(S^3),\ \beta\in\pi_8(S^5),$$

by [14, Prop. 5.3, 5.6, 5.9], and so the desired results by the above corollary. q.e.d.

§6. The group $\mathscr{E}(S^n\times S^n)$

Let $GL(2, Z)$ be the group of integral 2×2 matrices having integral inverse matrices, with the usual multiplication. Then, it is easy to see that there is an isomorphism

$$\chi: GL(2, Z) \longrightarrow \mathscr{E}(S^n{}_n \vee S^n) \tag{6.1}$$

given by

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \nabla((i_1a+i_2b) \vee (i_1c+i_2d)),$$

where i_j: $S^n \to S^n \vee S^n$ is the inclusion to the j-th factor, ∇ is the folding map, and $k \in Z$ means the map of degree k.

The following theorem is proved essentially by P. J. Kahn [8, §2.3].[1)]

THEOREM 6.2. *The following sequence is exact*:

$$0 \longrightarrow H_{n,n} \xrightarrow{\lambda'} (S^n \times S^n) \longrightarrow G_{n,n} \longrightarrow 1,$$

where

$$H_{n,n} = \pi_{2n}(S^n)/\{[\iota_n, \iota_n]\} + \pi_{2n}(S^n)/\{[\iota_n, \iota_n]\};$$

$$G_{n,n} = \begin{cases} GL(2, Z) & \text{if } n = 1, 3, 7, \\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, Z), ab \equiv cd \equiv 0 \bmod 2\right\} & \text{if } n \text{ is odd and } \neq 1, 3, 7, \\ \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\} & \text{if } n \text{ is even.} \end{cases}$$

PROOF. By the same way as Theorem 2.6, it is sufficient to show that the group $G'_{n,n}$ in the proof of Theorem 2.6 is isomorphic to the group $G_{n,n}$ in the theorem.

It follows immediately that

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}[i_1, i_2] = [i_1a+i_2b, i_1c+i_2d]$$

$$= ac[i_1, i_1]+(ad+(-1)^n bc)[i_1, i_2]+bd[i_2, i_2].$$

On the other hand, it is well-known that $[\iota_n, \iota_n]=0$ if $n=1, 3, 7$, and the order of $[\iota_n, \iota_n]$ is 2 if n is odd and $n \neq 1, 3, 7$, and is infinite if n is even (cf. e.g. [7, p. 336]). Therefore, we have the desired results by studying the conditions that the last element is equal to $[i_1, i_2]\circ\varepsilon$. q.e.d.

COROLLARY 6.3. (P. J. Kahn [8, Th. 4]) *If n is even, then the sequence in Theorem 6.2 is split. Also the action of $G_{n,n}$ on $H_{n,n}$ is given by*

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1) It seems to the author that the consideration for the case $n=3,7$ is neglected and that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of [8, p. 34] should be $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot(\xi_1, \xi_2) = (\xi_2, \xi_1), \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot(\xi_1, \xi_2) = (-\xi_2, \xi_1).$$

In the rest of this section, assume that $n=3, 7$.

For any $N=(n_{ij}) \in GL(2, Z)$, we define the element $\bar{\lambda}(N) \in \mathscr{E}(S^n \times S^n)$ by

$$\bar{\lambda}(N)(p, q) = (p^{n_{11}}q^{n_{12}}, p^{n_{21}}q^{n_{22}}) \qquad \text{for} \quad p, q \in S^n,$$

where the multiplication is the one of quaternions or Cayley numbers. Then we have the following theorem.

THEOREM 6.4. *Let $n=3, 7$. Then*

$$\mathscr{E}(S^n \times S^n) = \{\lambda'(\alpha, \beta)\bar{\lambda}(N) \mid \alpha, \beta \in \pi_{2n}(S^n), N \in GL(2, Z)\},$$

and the multiplication is given as follows:

(i) $\lambda'(\alpha, \beta)\lambda'(\alpha', \beta') = \lambda'(\alpha+\alpha', \beta+\beta')$,

(ii) $\bar{\lambda}(N)\lambda'(\alpha, \beta) = \lambda'(|N|(n_{11}\alpha+n_{12}\beta), |N|(n_{21}\alpha+n_{22}\beta))\bar{\lambda}(N)$,

(iii) $\bar{\lambda}(N)\bar{\lambda}(M) = \lambda'(a_1\omega_n, a_2\omega_n)\bar{\lambda}(NM)$,

$a_i = -|NM|(n_{i1}n_{i2}m_{12}m_{21}+\binom{n_{i1}}{2}m_{11}m_{12}+\binom{n_{i2}}{2}m_{21}m_{22})(i=1, 2)$, *where $N=(n_{ij})$, $M=(m_{ij})$, and $|N|$ means the determinant of N, and ω_n is a generator of $\pi_{2n}(S^n)=Z_{12}$ or Z_{120}.*

Before we prove this theorem, we show the next two lemmas.

LEMMA 6.5. $\alpha\pi\cdot p_i = p_i\cdot\alpha\pi$ *for $\alpha \in \pi_6(S^3)$, $i=1, 2$.*

PROOF. By the commutative diagram

$$\begin{array}{ccccccc} S^3\times S^3 & \xrightarrow{(\pi, p_i)} & S^6\times S^3 & \xrightarrow{\alpha\times 1} & S^3\times S^3 & \xrightarrow{\phi} & S^3 \\ & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \| \\ & & S^9 & \xrightarrow{S^3\alpha} & S^6 & \xrightarrow{\omega_3} & S^3 \end{array}$$

we have $\alpha\pi\cdot p_i\cdot\alpha\pi^{-1}\cdot p_i^{-1} = (\alpha\pi, p_i)^*\phi = 0$. q.e.d.

LEMMA 6.6. $r(mp_1\cdot np_2) = rmp_1\cdot rnp_2\cdot(-\binom{r}{2}mn\omega_3\pi)$.

PROOF. This lemma follows from $p_2\cdot p_1 = p_1\cdot p_2\cdot(-\omega_3\pi)$ (Lemma 5.5) and Lemma 6.5. q.e.d.

PROOF OF THEOREM 6.4. We prove the theorem for $n=3$, and the theorem for $n=7$ is proved by the same way.

By Theorem 6.2, we have the exact sequence

$$0 \longrightarrow \pi_6(S^3)+\pi_6(S^3) \xrightarrow{\lambda'} \mathscr{E}(S^3\times S^3) \xrightarrow{\varphi} GL(2, Z) \longrightarrow 1\,.$$

We notice that $\lambda'(\alpha, \beta)=(p_1\cdot\alpha\pi, p_2\cdot\beta\pi)$ for $\alpha, \beta \in \pi_6(S^3)$, and we have the desired results by Lemmas 6.5, 6.6. q.e.d.

§7. The group $\mathscr{E}(S^1\times S^n)$ for $n\geqq 3$

In the rest of this paper, we consider the groups $\mathscr{E}(S^1\times S^n)$ $(n\geqq 2)$. For these groups, we cannot use the methods in §2 since $S^1\vee S^n$ is not simply connected.

By attaching i-cells $(i\geqq n+2)$ to a given CW-complex X, we obtain a CW-complex X_n which kills r-th homotopy groups of X for $r>n$:

$$\pi_r(X_n)=0 \quad (r>n), \qquad i_{n*}\colon \pi_r(X)\simeq \pi_r(X_n) \quad (r\leqq n),$$

($i_n\colon X\to X_n$ is the inclusion).

Lemma 7.1. *If X is an n-dimensional CW-complex. Then we have iso morphisms*

$$\mathscr{E}(X)\simeq \mathscr{E}(X_n), \qquad \mathscr{E}(S^1\times X)\simeq \mathscr{E}(S^1\times X_{n+1})\,.$$

Proof. It is easy to see that the induced maps

$$i_n^*\colon [X_n, X_n] \longrightarrow [X, X_n], \qquad i_{n*}\colon [X, X] \longrightarrow [X, X_n]$$

are bijective by the elementary homotopy theory. Therefore

$$i_{n*}^{-1} i_n^*\colon [X_n, X_n] \longrightarrow [X, X]$$

is bijective, and we have the first isomorphism.

It is obvious that $S^1\times X_{n+1}$ is obtained from $S^1\times X$ by attaching i-cells $(i\geqq n+3)$ and kills the r-th homotopy groups of $S^1\times X$ for $r>n+1$. Therefore, we have the second isomorphism from the above result. q.e.d.

Remark. The first isomorphism in the above lemma is shown in [1, Lemma 5.1] under the additional assumption that X is 1-connected.

Now, consider the case $X=S^n$ for $n\geqq 3$. Then, it is well known that X_n and X_{n+1} are embeddable in the sequence of the induced fiberings

$$\begin{array}{ccccccc} \Omega A & \xrightarrow{i'} & E_{f'} & \xrightarrow{p'} & K(Z, n) & \xrightarrow{f'} & A \\ \| & & \| & & \| & & \| \\ K(Z_2, n+1) & & X_{n+1} & & X_n & & K(Z_2, n+2) \end{array} \qquad (7.2)$$

of the generator f' of $H^{n+2}(Z, n; Z_2)=Z_2$ (cf. e.g. [9, p. 140]). Therefore, we have the sequence of the induced fiberings

$$\Omega A \xrightarrow{i} S^1 \times X_{n+1} (= E_f) \xrightarrow{p} S^1 \times X_n \xrightarrow{f} A \tag{7.3}$$

of $f=f' \circ p_2$ such that $p=\iota_1 \times p'$, $i=(*, i')$.

LEMMA 7.4. *The two induced maps*

$$i_*: [\Omega A, \Omega A] \longrightarrow [\Omega A, E_f], \quad p^*: [S^1 \times X_n, S^1 \times X_n] \longrightarrow [E_f, S^1 \times X_n]$$

are both bijective.

PROOF. Since $i'=p_2 \circ i$, we have

$$i'_* = p_{2*} i_*: [\Omega A, \Omega A] \xrightarrow{i_*} [\Omega A, E_f] \xrightarrow{p_{2*}} [\Omega A, X_{n+1}].$$

Using the homotopy exact sequence of the fibering (X_{n+1}, p', X_n) in (7.2), we see easily that i'_* is bijective. Also p_{2*} is bijective since $E_f=S^1 \times X_{n+1}$. Therefore i_* is bijective.

It is easy to see that p^* is equal to

$$H^1(S^1)+H^n(K(Z, n)) \xrightarrow{\iota_1^*+p'^*} H^1(S^1)+H^n(X_{n+1}),$$

which is isomorphic. q.e.d.

By applying [10, Prop. 5.6] for f in (7.3),

LEMMA 7.5. *We have the exact sequence*

$$i^{*-1}(0) \xrightarrow{\kappa} \mathscr{E}(S^1 \times X_{n+1}) \xrightarrow{\varphi \times \psi} \mathscr{E}(S^1 \times X_n) \times \mathscr{E}(\Omega A)$$

of homomorphisms, where $i^*: [S^1 \times X_{n+1}, \Omega A] \to [\Omega A, \Omega A]$ *and* $i^{*-1}(0)$ *is a group with an unusual multiplication* $\oplus$.

On this sequence, we have the following three lemmas.

LEMMA 7.6. $\operatorname{Im}(\varphi \times \psi) = \mathscr{E}(S^1 \times X_n) = Z_2 + Z_2.$

PROOF. It is clear that $\mathscr{E}(A)=\mathscr{E}(\Omega A)=1$, since $A=K(Z_2, n+2)$. Therefore $f \circ \xi \simeq f$ for any $\xi \in \mathscr{E}(S^1 \times X_n)$, and there is $h \in \mathscr{E}(S^1 \times X_{n+1})$ such that $p \circ h \simeq \xi \circ p$, i.e., $(\varphi \times \psi)(h)=(\xi, 1)$. This shows the first equality. Since $X_n=K(Z, n)$ we see that $\mathscr{E}(X_n)=Z_2$ and $[S^1 \wedge X_n, X_n]=0$, and so the second equality by [10, Example 5.10]. q.e.d.

LEMMA 7.7. $i^{*-1}(0) = Z_2.$

PROOF. By using the Serre cohomology sequence, we have

$$H^n(X_{n+1}; Z_2) = Z_2, \qquad H^{n+1}(X_{n+1}; Z_2) = 0.$$

Therefore, we see that $[S^1 \times X_{n+1}, \Omega A] = H^n(X_{n+1}; Z_2) + H^{n+1}(X_{n+1}; Z_2) = Z_2$, and $i^* = (*, i')^*$ is equal to 0. q.e.d.

LEMMA 7.8. *κ is monomorphic.*

PROOF. By the results of J. W. Rutter [11, Cor. 1.3.2], $\mathrm{Ker}\,\kappa$ is equal to the image of the homomorphism $\Delta: [S^1 \times X_{n+1}, \Omega(S^1 \times X_n)] \to [S^1 \times X_{n+1}, \Omega A]$. The left hand side is equal to $H^1(S^1 \wedge (S^1 \times X_{n+1})) + H^n(S^1 \wedge (S^1 \times X_{n+1})) = 0$, and so we have the lemma. q.e.d.

By the above results, we obtain the following

THEOREM 7.9. $\mathscr{E}(S^1 \times S^n) = Z_2 + Z_2 + Z_2$ *for* $n \geqq 3$.

PROOF. By Lemmas 7.1, 7.5–7.8, we have the exact sequence

$$0 \longrightarrow Z_2 \longrightarrow \mathscr{E}(S^1 \times S^n) \longrightarrow Z_2 + Z_2 \longrightarrow 0.$$

Consider the elements $b_{ij} = (-\iota_1)^i \times (-\iota_n)^j \in \mathscr{E}(S^1 \times S^n)$. Then, by the definition of the isomorphism $\mathscr{E}(S^1 \times S^n) \simeq \mathscr{E}(S^1 \times X_{n+1})$ in Lemma 7.1 and the epimorphism $\varphi: \mathscr{E}(S^1 \times X_{n+1}) \to \mathscr{E}(S^1 \times X_n) = Z_2 + Z_2$, it is easy to see that the subgroup $\{b_{ij} | i, j \in Z_2\} \subset \mathscr{E}(S^1 \times S^n)$ is mapped isomorphically onto $Z_2 + Z_2$. q.e.d.

§8. The groups $\mathscr{E}(S^1 \times S^2)$ and $\mathscr{E}(S^1 \times CP^n)$

By the similar way in §7, we consider the groups $\mathscr{E}(S^1 \times S^2)$ and $\mathscr{E}(S^1 \times CP^n)$ $(n \geqq 1)$ more generally, where CP^n is the complex n-dimensional projective space.

Let Y_{2n+1} be the CW-complex obtained from CP^n by attaching i-cells $(i \geqq 2n+3)$ so that Y_{2n+1} kills the r-th homotopy group of CP^n for $r > 2n+1$. Then we have the following lemma by Lemma 7.1.

LEMMA 8.1. $\qquad \mathscr{E}(S^1 \times CP^n) \simeq \mathscr{E}(S^1 \times Y_{2n+1})$.

It is well known that Y_{2n+1} is embeddable in the sequence of the induced fiberings

$$\begin{array}{ccccccc} \Omega B & \xrightarrow{i'} & Y_{2n+1} & \xrightarrow{p'} & K(Z, 2) & \xrightarrow{f'} & K(Z, 2n+2) \\ & & \| & & \| & & \| \\ & & Y & & K & & B \end{array} \tag{8.2}$$

of the generator f' of $H^{2n+2}(K)$. Therefore, we have the sequence of the induced fiberings

$$\Omega B \xrightarrow{i} S^1 \times Y \xrightarrow{p} S^1 \times K \xrightarrow{f} B \tag{8.3}$$

of $f=f'\circ p_2$ such that $p=\iota_1\times p'$, $i=(*, i')$. Then, Lemma 7.4 holds similarly for (8.3) and we have the following lemma by the similar way as Lemma 7.5.

LEMMA 8.4. *We have the exact sequence*

$$i^{*-1}(0) \xrightarrow{\kappa} \mathscr{E}(S^1\times Y) \xrightarrow{\varphi\times\psi} \mathscr{E}(S^1\times K)\times\mathscr{E}(\Omega B)$$

of homomorphisms, where $i^*: [S^1\times Y, \Omega B]\to[\Omega B, \Omega B]$ *and* $i^{*-1}(0)$ *is a group with a multiplication* $\oplus$.

In this lemma, we have the following three lemmas.

LEMMA 8.5. *By the natural projection* $\mathscr{E}(S^1\times K)\times\mathscr{E}(\Omega B)\to\mathscr{E}(S^1\times K)$, $\mathrm{Im}(\varphi\times\psi)$ *is isomorphic to*

$$\mathrm{Im}\,\varphi = \mathscr{E}(S^1\times K) = Z_2+Z_2 .$$

PROOF. By the definition of $\varphi\times\psi$ in [10, p. 26], $\mathrm{Im}(\varphi\times\psi)$ is the set of $(h, \varepsilon)\in\mathscr{E}(S^1\times K)\times\mathscr{E}(\Omega B)$ such that the following diagram is homotopy commutative for some $h_1\in\mathscr{E}(S^1\times Y)$:

$$(*)\qquad \begin{array}{ccccc} \Omega B & \xrightarrow{i} & S^1\times Y & \xrightarrow{p} & S^1\times K \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h} \\ \Omega B & \xrightarrow{i} & S^1\times Y & \xrightarrow{p} & S^1\times K \end{array}$$

Then, we have the right commutative square in the following diagram:

$$(**)\qquad \begin{array}{ccccc} H^{2n+2}(B) & \underset{\cong}{\xleftarrow{\tau_1}} & H^{2n+1}(\Omega B) & \underset{\cong}{\xrightarrow{\tau}} & H^{2n+2}(S^1\times K) \\ \downarrow{\scriptstyle\varepsilon^*} & & \downarrow{\scriptstyle\varepsilon^*} & & \downarrow{\scriptstyle h^*} \\ H^{2n+2}(B) & \underset{\cong}{\xleftarrow{\tau_1}} & H^{2n+1}(\Omega B) & \underset{\cong}{\xrightarrow{\tau}} & H^{2n+2}(S^1\times K) \end{array}$$

where τ and τ_1 are the transgressions. Since the left square in (**) is clearly commutative and $f^*=\tau\circ\tau_1^{-1}$, we see that $h^*f^*=f^*\varepsilon^*$. These show that

$$\mathrm{Im}(\varphi\times\psi) = \{(h, \varepsilon)\in\mathscr{E}(S^1\times K)\times\mathscr{E}(B)\,|\,f\circ h = \varepsilon\circ f\} .$$

Furthermore, for any $h\in\mathscr{E}(S^1\times K)$, there is a unique element $\varepsilon\in\mathscr{E}(B)$ such that $h^*f^*=f^*\varepsilon^*$. Therefore we have $\mathrm{Im}(\varphi\times\psi)$ is isomorphic to $\mathrm{Im}\,\varphi=\mathscr{E}(S^1\times K)$, which is Z_2+Z_2 by the second equality of Lemma 7.6. q.e.d.

LEMMA 8.6. $$i^{*-1}(0) = [S^1\times Y, \Omega B] = Z .$$

PROOF. In the cohomology exact sequence

$$[S^1\times K, \Omega B] \xrightarrow{p^*} [S^1\times Y, \Omega B] \xrightarrow{i^*} [\Omega B, \Omega B]$$

of the fibering (8.3), we see that $i^*=0$ by the same way as Lemma 7.7. Also, the multiplication $\oplus$ of $i^{*-1}(0)=\operatorname{Im} p^*$ in Lemma 8.4 coincides with the usual multiplication $+$, by [10, Lemma 5.4 (ii)]. q.e.d.

LEMMA 8.7. *κ in Lemma 8.4 is monomorphic.*

PROOF. By the results of J. W. Rutter [11, Cor. 1.3.2, Th. 1.4.3], $\operatorname{Ker}\kappa$ is equal to the image of

$$(\Omega f)_*\colon [S^1\times Y, \Omega(S^1\times K)] \longrightarrow [S^1\times Y, \Omega B].$$

Since $B=K(Z, 2n+2)$, Ωf is homotopic to the constant map, and we have the lemma. q.e.d.

THEOREM 8.8. *Let $n\geqq 1$. Then we have the split exact sequence*

$$0\longrightarrow Z \xrightarrow{\kappa} \mathscr{E}(S^1\times CP^n) \longrightarrow Z_2+Z_2 \longrightarrow 0,$$

where the action of Z_2+Z_2 on Z is given by

$$((-1)^i, (-1)^j)\cdot m = (-1)^{i+j}m, \qquad \textit{for} \quad m\in Z,\ i, j\in Z_2.$$

PROOF. By Lemmas 8.1–8.7, we have the above exact sequence. Consider the elements $b_{ij}=(-\iota_1)^i\times(-\iota)^j\in\mathscr{E}(S^1\times CP^n)=\mathscr{E}(S^1\times Y)$, where $-\iota$ is the generator of $\mathscr{E}(CP^n)=Z_2$. It is easy to see that the subgroup $Z_2+Z_2=\{b_{ij}| i, j\in Z_2\}\subset\mathscr{E}(S^1\times CP^n)$ is mapped isomorphically onto Z_2+Z_2 of the right hand side. Therefore the above sequence is split.

To study the action, we consider the diagram

$$\begin{array}{ccccccc}
S^1\times Y & \xrightarrow{\Delta} & S^1\times Y\times S^1\times Y & \xrightarrow{1\times m} & S^1\times Y\times K(Z, 2n+1) & \xrightarrow{k} & S^1\times Y\\
\downarrow{\scriptstyle b_{ij}} & & \downarrow{\scriptstyle b_{ij}\times b_{ij}} & & \downarrow{\scriptstyle b_{ij}\times\varepsilon} & & \downarrow{\scriptstyle b_{ij}}\\
S^1\times Y & \xrightarrow{\Delta} & S^1\times Y\times S^1\times Y & \xrightarrow{1\times m'} & S^1\times Y\times K(Z, 2n+1) & \xrightarrow{k} & S^1\times Y,
\end{array}$$

where Δ is the diagonal map, k is the multiplication, and the compositions of the maps in the horizontal sequences are equal to $\kappa(m)$ and $\kappa(m')$ respectively by the definition of κ (cf. [10, (5.2)]). It is easy to see that the above diagram is commutative for $\varepsilon=(-1)^{j(n-1)}$ and $m'=(-1)^{i+j}m$ and we have the desired results. q.e.d.

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