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A Note on G($\mathfrak{a}$)-Domains and Hilbert Rings

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In a recent paper [1], we defined the property $J(\Lambda)$ for an integral domain R, which is useful to prove a generalized Hilbert Nullstellensatz. At that time, we restricted ourselves to prime ideals of height one. However, we can readily see that Lemma 1, Lemma 2 and Proposition 1 in Section 1 of [1] are valid, if we replace the set $Ht_1(R)$ of prime ideals of height one (resp. $H_R(D)$) by the set $P(R)$ of non zero prime ideals (resp. $H_R^*(D)$ (see the definition below)). So, in this paper, we define the property $J^*(\mathfrak{a})$ for a cardinal number $\mathfrak{a}$ in place of the property $J(\mathfrak{a})$; here the cardinal number $\mathfrak{a}$ will always be assumed not less than $\aleph_0$, because if $\mathfrak{a}$ is finite, then it is clear that an integral domain R has the property $J^*(\mathfrak{a})$ if and only if R is not a G-domain (see the definition in [4]). Also, by taking account of the fact mentioned above, we define $G(\mathfrak{a})$-domain as a concept against the property $J^*(\mathfrak{a})$, and furthermore by introducing the notion of $G(\mathfrak{a})$-ideal and $H(\mathfrak{a})$-ring similar to G-ideal and Hilbert ring in [4], we can obtain some results generalizing those in [3] and [4].

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1. $G(\mathfrak{a})$-domains

All rings considered are commutative with identity. Let $\mathfrak{a}$ be a cardinal number not less than $\aleph_0$. We say that a polynomial ring over R is an $\mathfrak{a}$-polynomial ring over R if the cardinality of the set of its variables is $\mathfrak{a}$, and we say that an R-algebra A is $\mathfrak{a}$-generated over R if A is an R-homomorphic image of the $\mathfrak{a}$-polynomial ring over R. Call a subset D of an integral domain R a $J(\mathfrak{a})$-subset if D does not contain zero element and if the cardinality of D is not greater than $\mathfrak{a}$. A bit of notation: For an integral domain R, we denote by $P(R)$ the set of non zero prime ideals in R, $Ht_1(R)$ the set of prime ideals of height one, and for a given subset E of R we denote by $H_R^*(E)$ the set of non zero prime ideals in R which contains at least one element of E, $H_R(E)$ the set of prime ideals of height one in R which contains at least one element of E.

DEFINITION. Let R be an integral domain. When $H_R(D)$ is properly contained in $Ht_1(R)$ for any $J(\mathfrak{a})$-subset D of R, then we say that the ring R has the property $J(\mathfrak{a})$. When $H_R^*(D)$ is properly contained in $P(R)$ for any $J(\mathfrak{a})$-subset D of R, then we say that the ring R has the property $J^*(\mathfrak{a})$.

DEFINITION. For an integral domain R, we say that R is a $G(\mathfrak{a})$-domain if and only if R has not the property $J^*(\mathfrak{a})$, namely there exists a $J(\mathfrak{a})$-subset D such that $\mathfrak{p} \cap D \neq \phi$ for any non zero prime ideal $\mathfrak{p}$ in R.

The following propositions follow immediately from definitions.

PROPOSITION 1. *Let R be an integral domain. If any non zero prime ideal in R contains at least a prime ideal of height one, then R has the property $J^*(\mathfrak{a})$ if and only if R has the property $J(\mathfrak{a})$.*

PROPOSITION 2. *Let K be the quotient field of R. Then the following statements are equivalent*:

(a) *R is a $G(\mathfrak{a})$-domain.*

(b) *For some $J(\mathfrak{a})$-subset D of R, we have $K=R[\dots, 1/a,\dots]$, $a \in D$.*

(c) *For some multiplicatively closed subset S of R such that card$(S)\leq\mathfrak{a}$, we have $K=S^{-1}R$.*

(d) *K is $\mathfrak{a}$-generated over R.*

COROLLARY. *If R is a $G(\mathfrak{a})$-domain, then every overring of R is also a $G(\mathfrak{a})$-domain.*

PROPOSITION 3. *If R has the property $J^*(\mathfrak{a})$, then any polynomial ring over R has the property $J^*(\mathfrak{a})$.*

PROOF. Let A be a polynomial ring over R, and E be any $J(\mathfrak{a})$-subset of A. We denote by D the subset of R consisting of non zero coefficients of the elements of E; then D is a $J(\mathfrak{a})$-subset of R. By our assumption, $H_R^*(D)$ is properly contained in $P(R)$. Let $\mathfrak{p}$ be an element of $P(R)$ but not of $H_R^*(D)$. Then $\mathfrak{p}A$ is not an element of $H_A^*(E)$.

PROPOSITION 4. *Let $R\subset A$ be integral domains. Then the following statements hold.*

(a) *If A is algebraic over R and R is a $G(\mathfrak{a})$-domain, then A is a $G(\mathfrak{a})$-domain.*

(b) *If A is $\mathfrak{a}$-generated over R and A is a $G(\mathfrak{a})$-domain, then R is a $G(\mathfrak{a})$-domain.*

(c) *In particular, if A is algebraic over R and A is $\mathfrak{a}$-generated over R, then R is a $G(\mathfrak{a})$-domain if and only if A is a $G(\mathfrak{a})$-domain.*

PROOF. Let K and L be the quotient fields of R and A respectively.

(a) By Proposition 2, $K=R[\dots, a_i,\dots]$, $i\in I$, where card$(I)\leq\mathfrak{a}$. Then $A[\dots, a_i,\dots]$, $i\in I$, is algebraic over K, and hence is itself a field, therefore necessarily equal to L.

(b) Let $U=\{t\}$ be a subset of A such that $\dots, t,\dots$ are algebraically independent over R and A is algebraic over $R[\dots, t, \dots]$, $t \in U$. If $R[\dots, t, \dots]$ is a $G(\mathfrak{a})$-domain, then R is a $G(\mathfrak{a})$-domain by Proposition 3. Therefore we may assume that A is algebraic over R. By our assumption, $L = A[\dots, c_i, \dots]$, $i \in I$, and $A = R[\dots, d_j, \dots]$, $j \in J$, where $\operatorname{card}(I)$, $\operatorname{card}(J) \leq \mathfrak{a}$. The elements c_i, d_j are algebraic over R and consequently satisfy equations with coefficients in R, say

$$a_i c_i^m + \cdots = 0$$

$$b_j d_j^n + \cdots = 0.$$

Since $L = R[\dots, c_i, \dots, d_j, \dots]$ is integral over $R[\dots, a_i^{-1}, \dots, b_j^{-1}, \dots]$ and L is a field, $R[\dots, a_i^{-1}, \dots, b_j^{-1}, \dots]$ is necessarily equal to K.

Proposition 5. *Let $R \subset A$ be integral domains. If A is integral over R, then the following statements are equivalent*:

(a) *R has the property $J^*(\mathfrak{a})$.*

(b) *A has the property $J^*(\mathfrak{a})$.*

Proof. (b)⇒(a) follows from (a) of Proposition 4.

(a)⇒(b). Let $E = \{a_i; i \in I\}$ be a $J(\mathfrak{a})$-subset of A, and $a_i^{n_i} + \cdots + d_i = 0$ be the smallest degree equation of a_i over R. Clearly $D = \{d_i; i \in I\}$ is a $J(\mathfrak{a})$-subset of R, and so by our assumption, we can choose a non zero prime ideal $\mathfrak{p}$ of R which is not in $H_R^*(D)$. Let $\mathfrak{P}$ be a prime ideal of A lying over $\mathfrak{p}$. Then clearly $\mathfrak{P}$ is not an element of $H_A^*(E)$.

Proposition 6. *R is a $G(\mathfrak{a})$-domain if and only if there exists a maximal ideal $\mathfrak{m}$ in the $\mathfrak{a}$-polynomial ring over R with contracts in R to zero ideal.*

Proof. Let K be the quotient field of R. Suppose R is a $G(\mathfrak{a})$-domain. By Proposition 2, K is of the form $R[\dots, a_i, \dots]$, $i \in I$, where $\operatorname{card}(I) = \mathfrak{a}$. Let φ be an R-homomorphism of $R[\dots, X_i, \dots]$, $i \in I$, onto K such that $\varphi(X_i) = a_i$, and $\mathfrak{m}$ be the $\operatorname{Ker}(\varphi)$. Then $\mathfrak{m}$ is a maximal ideal in $R[\dots, X_i, \dots]$, $i \in I$, and $\mathfrak{m} \cap R = 0$. Conversely, suppose that there exists a maximal ideal $\mathfrak{m}$ in the $\mathfrak{a}$-polynomial ring A over R such that $\mathfrak{m} \cap R = 0$. Since $A/\mathfrak{m}$ is $\mathfrak{a}$-generated over R and a field is a $G(\mathfrak{a})$-domain, R is a $G(\mathfrak{a})$-domain by (b) of Proposition 4.

2. $H(\mathfrak{a})$-rings

Kaplansky defines G-ideals and Hilbert rings in [4] as follows: A prime ideal $\mathfrak{p}$ in a ring R is a G-ideal if $R/\mathfrak{p}$ is a G-domain. A ring R is a Hilbert ring if every G-ideal in R is maximal.

So we shall define $G(\mathfrak{a})$-ideals and $H(\mathfrak{a})$-rings after Kaplansky's definitions.

Definition. Let $\mathfrak{p}$ be a prime ideal in a ring R. We say that $\mathfrak{p}$ is a $G(\mathfrak{a})$-ideal if $R/\mathfrak{p}$ is a $G(\mathfrak{a})$-domain.

A ring R is an $H(\mathfrak{a})$-ring if every $G(\mathfrak{a})$-ideal in R is a maximal ideal.

REMARK. (a) A homomorphic image of an $H(\mathfrak{a})$-ring is an $H(\mathfrak{a})$-ring.

(b) An $H(\mathfrak{a})$-ring is a Hilbert ring, because G-domain is a $G(\mathfrak{a})$-domain.

(c) Let k be a field with cardinality $\leq \aleph_0$. Then $k[X]$ is a Hilbert ring but not an $H(\aleph_0)$-ring.

(d) Let R be a unique factorization domain. If $\operatorname{card}(Ht_1(R)) > \mathfrak{a}$, then R has the property $J^*(\mathfrak{a})$.

COROLLARY to Proposition 6. *A prime ideal* $\mathfrak{p}$ *in a ring* R *is a* $G(\mathfrak{a})$*-ideal if and only if it is a contraction of some maximal ideal in the* $\mathfrak{a}$*-polynomial ring over* R.

PROPOSITION 7. *Let* k *be a field, and* I *be a non empty set. If* $\operatorname{card}(k) > \mathfrak{a}$ *and* $\operatorname{card}(I) \leq \mathfrak{a}$*, then* $A = k[\dots, X_i, \dots]$, $i \in I$*, is an* $H(\mathfrak{a})$*-ring.*

PROOF. Let $\mathfrak{p}$ be a non maximal prime ideal in A, and let $U = \{t_j; j \in J\}$ be a subset of $A/\mathfrak{p}$ such that $\dots, t_j, \dots$ are algebraically independent over k and $A/\mathfrak{p}$ is algebraic over $k[\dots, t_j, \dots]$, $j \in J$. Note that U is not empty because $\mathfrak{p}$ is not maximal. The ring $k[\dots, t_j, \dots]$ has the property $J^*(\mathfrak{a})$ by (d) of Remark; therefore $A/\mathfrak{p}$ has the property $J^*(\mathfrak{a})$ by Proposition 4.

THEOREM 1. *Let* k *be a field. Then the following statements are equivalent.*

(a) $\operatorname{card}(k) > \mathfrak{a}$.

(b) $k[X]$ *has the property* $J^*(\mathfrak{a})$.

(c) $k[X]$ *is an* $H(\mathfrak{a})$*-ring.*

(d) *If* I *is a non empty set such that* $\operatorname{card}(I) \leq \mathfrak{a}$*, then* $k[\dots, X_i, \dots]$, $i \in I$*, has the property* $J^*(\mathfrak{a})$.

(e) *If* I *is a non empty set such that* $\operatorname{card}(I) \leq \mathfrak{a}$*, then* $k[\dots, X_i, \dots]$, $i \in I$*, is an* $H(\mathfrak{a})$*-ring.*

(f) *If* I *is a set such that* $\operatorname{card}(I) = \mathfrak{a}$*, then* $k[\dots, X_i, \dots]$, $i \in I$*, is a Hilbert ring.*

PROOF. (a)$\Rightarrow$(b) and (b)$\Rightarrow$(d) follow from Proposition 3 and (d) of the preceding remark.

(d)$\Rightarrow$(a). If we assume that $\operatorname{card}(k) \leq \mathfrak{a}$, then $\operatorname{card}(k[\dots, X_i, \dots]) \leq \mathfrak{a}$; therefore $k[\dots, X_i, \dots]$ clearly has not the property $J^*(\mathfrak{a})$.

(a)$\Rightarrow$(c) and (a)$\Rightarrow$(e) follow from Proposition 7.

(c)$\Rightarrow$(b) and (e)$\Rightarrow$(d) are trivial.

The equivalence of (a) and (f) is proved in Proposition 2 of [1].

PROPOSITION 8. *Let* $R \subset A$ *be rings such that* A *is integral over* R. *Then* R *is an* $H(\mathfrak{a})$*-ring if and only if* A *is an* $H(\mathfrak{a})$*-ring.*

This follows immediately from Proposition 5.

In [3], O. Goldman proved that a ring R is a Hilbert ring if and only if every maximal ideal in $R[X]$ contracts in R to a maximal ideal. The following proposition shows that an $H(\mathfrak{a})$-ring is characterized similarly.

PROPOSITION 9. *A ring R is an $H(\mathfrak{a})$-ring if and only if every maximal ideal in the $\mathfrak{a}$-polynomial ring over R contracts in R to a maximal ideal.*

PROOF. Suppose first that R is an $H(\mathfrak{a})$-ring. Let $\mathfrak{m}$ be any maximal ideal in the $\mathfrak{a}$-polynomial ring A over R. Since $A/\mathfrak{m}$ is $\mathfrak{a}$-generated over $R/R \cap \mathfrak{m}$, $R/R \cap \mathfrak{m}$ is a $G(\mathfrak{a})$-domain by (b) of Proposition 4; hence $R \cap \mathfrak{m}$ is a maximal ideal in R by assumption. Suppose now that every maximal ideal in A contracts in R to a maximal ideal. Let $\mathfrak{p}$ be a $G(\mathfrak{a})$-ideal in R. There exists a maximal ideal $\mathfrak{m}$ in A such that $\mathfrak{p} = R \cap \mathfrak{m}$ by Corollary to Proposition 6; hence $\mathfrak{p}$ is a maximal ideal in R by assumption.

THEOREM 2. *For a ring R the following statements are equivalent:*

(a) *R is an $H(\mathfrak{a})$-ring and for every maximal ideal m in R we have card $(R/\mathfrak{m}) > \mathfrak{a}$.*

(b) *$R[X]$ is an $H(\mathfrak{a})$-ring.*

(c) *the $\mathfrak{a}$-polynomial ring over R is an $H(\mathfrak{a})$-ring.*

PROOF. (a)⇒(c). Let $A = R[\dots, X_i, \dots]$, $i \in I$, be the $\mathfrak{a}$-polynomial ring over R. It suffices to prove that $A/\mathfrak{P}$ has the property $J^*(\mathfrak{a})$ for every non maximal prime ideal $\mathfrak{P}$ in A. When $\mathfrak{p} = \mathfrak{P} \cap R$ is a maximal ideal in R, we have card$(R/\mathfrak{p}) > \mathfrak{a}$ by assumption. $A/\mathfrak{P} = (R/\mathfrak{p})[\dots, X_i, \dots]/\bar{\mathfrak{P}}$, where $\bar{\mathfrak{P}} = (R/\mathfrak{p}) \otimes_R P$. Since $\bar{\mathfrak{P}}$ is not maximal, $(R/\mathfrak{p})[\dots, X_i, \dots]/\bar{\mathfrak{P}}$ has the property $J^*(\mathfrak{a})$ by Theorem 1. When $\mathfrak{p} = R \cap \mathfrak{P}$ is not maximal in R, $R/\mathfrak{p}$ has the property $J^*(\mathfrak{a})$ by assumption. $A/\mathfrak{P}$ is $\mathfrak{a}$-generated over $R/\mathfrak{p}$, so $A/\mathfrak{P}$ has the property $J^*(\mathfrak{a})$ by (b) of Proposition 4.

(c)⇒(b) follows from (a) of the preceding remark.

(b)⇒(a). Let $\mathfrak{m}$ be any maximal ideal in R. $R/\mathfrak{m}[X]$ is an $H(\mathfrak{a})$-ring; hence card$(R/\mathfrak{m}) > \mathfrak{a}$ by Theorem 1.

PROPOSITION 10. *Let R be an integral domain which satisfies the following conditions:*

(a) *$\dim(R) \geq 1$.*

(b) *Every non zero prime ideal in R contains at least a prime ideal of height one.*

(c) *For any non unit $a \neq 0$ in R, the cardinality of the set of prime ideal of height one containing a is not greater than $\mathfrak{a}$. Then R has the property $J^*(\mathfrak{a})$ if the $\mathfrak{a}$-polynomial ring over R is a Hilbert ring.*

Proof. Suppose R is a $G(\mathfrak{a})$-domain. By the condition (b) and Proposition 1, R has not the property $J(\mathfrak{a})$; therefore for some $J(\mathfrak{a})$-subset D of R we have $Ht_1(R)=H_R(D)$; hence the condition (c) implies $\mathrm{card}(Ht_1(R))\leq\mathfrak{a}$. We put $Ht_1(R)=\{\mathfrak{p}_j; j\in J\}$, and we fix an element j_0 of J. Let a_{j_0} be a non zero element in $\mathfrak{p}_{j_0}$, and for any $j\neq j_0$ we pick a non zero element a_j in $\mathfrak{p}_j$ but not in $\mathfrak{p}_{j_0}$. Then we have $K=Q(R)=R[\dots, 1/a_j,\dots], j\in J$, and $K\supsetneqq R[\dots, 1/a_j,\dots], j\in J-\{j_0\}$. ($Q(*)$ stands for the quotient field of $*$.) Let $A=R[\dots, X_j,\dots], j\in J-\{j_0\}$ and let $\mathfrak{M}$ be the ideal in $A[X_{j_0}]$ generated by $a_jX_j-1, j\in J$. Since $A[X_{j_0}]/\mathfrak{M}=K$, $\mathfrak{M}$ is a maximal ideal in $A[X_{j_0}]$. However $A/A\cap\mathfrak{M}=R[\dots, 1/a_j,\dots], j\in J-\{j_0\}$, $\subsetneqq K$ implies that $A\cap\mathfrak{M}$ is not a maximal ideal in A; hence by Theorem 5 in [3] A is not a Hilbert ring. This leads to a contradiction by our assumption.

Proposition 11. *Let R be a noetherian ring. If the $\mathfrak{a}$-polynomial ring $A=R[\dots, X_i,\dots], i\in I$, over R is a Hilbert ring, then A is an $H(\mathfrak{a})$-ring.*

Proof. We show that R satisfies the condition (a) of Theorem 2. Let $\mathfrak{m}$ be any maximal ideal in R. Since $(R/\mathfrak{m})[\dots, X_i,\dots], i\in I$, is a Hilbert ring and $\mathrm{card}(I)=\mathfrak{a}$, the cardinality of $R/\mathfrak{m}$ is greater than $\mathfrak{a}$ by Proposition 2 in [1]. Let $\mathfrak{p}$ be a non maximal prime ideal in R. Proposition 10 implies that $R/\mathfrak{p}$ has the property $J^*(\mathfrak{a})$; hence R is an $H(\mathfrak{a})$-ring.

Remark. Let R be a $G(\aleph_0)$-domain and K be the quotient field of R. The set $W=\{\{u_1, u_2,\dots\}\subset K; K=R[u_1, u_2,\dots]\}$ is not empty, because R is a $G(\aleph_0)$-domain. We say that R is a $G'(\aleph_0)$-domain if $\{u_n, u_{n+1},\dots\}$ is an element of W for any $\{u_1, u_2,\dots\}\in W$ and for any positive integer n. The following proposition is an immediate consequence of Corollary 2 to Proposition 1 in [1] and Theorem 5 in [3].

Proposition. *Let R be a one dimensional $G(\aleph_0)$-domain. If K is an algebraically closed field, and if $\mathrm{card}(R/\mathfrak{m})>\aleph_0$ for any maximal ideal in R, then $R[X_1, X_2,\dots]$ is a Hilbert ring if and only if R is a $G'(\aleph_0)$-domain.*

3. Valuation rings with the property $J^*(\aleph_0)$

Proposition 12. *Let R be a valuation ring. Then the following statements are equivalent:*

(a) *R has the property $J^*(\aleph_0)$.*

(b) *If $D=\{a_i; i=1,2,\dots\}$ is a $J(\aleph_0)$-subset of R, then $\bigcap_{i=1}^{\infty} Ra_i\supsetneqq(0)$.*

(c) *$K((X))=Q(R[[X]])$, where $K=Q(R)$. ($Q(*)$ stands for the quotient field of $*$.)*

Proof. (a)$\Rightarrow$(b). We can take a non zero prime ideal $\mathfrak{p}$ in R such that $\mathfrak{p}$ is not an element of $H_R^*(D)$. Therefore, for any i, we have $\mathfrak{p}\not\supset Ra_i$; hence $Ra_i\supset\mathfrak{p}$;

thus $\bigcap_{i=1}^{\infty} Ra_i \supset \mathfrak{p}$.

(b)⇒(a). Let $D=\{a_i; i=1, 2,\dots\}$ be a $J(\aleph_0)$-subset of R such that $P(R)=H_R^*(D)$. Then we have $\mathfrak{p}\supset \bigcap_{i=1}^{\infty} Ra_i$ for any non zero prime ideal $\mathfrak{p}$ in R; thus $\mathfrak{p}_1=\bigcap_{\mathfrak{p}\in P(R)} \mathfrak{p} \supsetneqq (0)$. Clearly $ht(\mathfrak{p}_1)=1$; hence $R_{\mathfrak{p}_1}$ is a valuation ring of rank one. Take a non zero element a in $\mathfrak{p}_1$, then $\bigcap_{i=1}^{\infty} a^i R_{\mathfrak{p}_1}=(0)$; hence $\bigcap_{i=1}^{\infty} a^i R=(0)$, because $\bigcap_{i=1}^{\infty} a^i R_{\mathfrak{p}_1} \supset \bigcap_{i=1}^{\infty} a^i R$. This contradicts to the assertion (b).

As for the equivalence of (b) and (c), see Theorem 1 in [2].

COROLLARY. *Let R be a valuation ring. If R has the property $J^*(\aleph_0)$, then $R_{\mathfrak{p}}$ has the property $J^*(\aleph_0)$ for any non zero prime ideal $\mathfrak{p}$ in R.*

This follows immediately from the equivalence of (a) and (c) in Proposition 12.

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