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On Conformal Invariants of Higher Order

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Let (M, g) be an n-dimensional Riemannian manifold with fundamental metric tensor g $(n>2)$ and R be the curvature tensor of type $(0, 4)$. Let C and C_0 be the Weyl conformal curvature tensor of type $(0, 4)$ and the so-called Weyl 3-index tensor, respectively. As usual, a Riemannian manifold is said to be *flat* or *of constant curvature* according as the sectional curvature is identically zero or constant, and to be *conformally flat* if it is locally conformally diffeomorphic to a Euclidean space. A well-known theorem due to H. Weyl says that (M, g) is conformally flat if and only if $C=0$ for $n>3$ and $C_0=0$ for $n=3$. The tensors R and C are typical examples of curvature structures of order two.

On the other hand, researches on curvature structures of higher order, e.g. the q-th Gauss-Kronecker curvature tensor R^q, have been developed by many people. Especially, J. A. Thorpe [7] has considered the $2q$-th sectional curvature γ_{2q}, which is defined for each even positive integer $2q \leqq n$, and studied relationships between curvature properties and topological structures of the manifold. The sectional curvature γ_{2q} is a curvature function corresponding to R^q on the Grassmann bundle of $2q$-planes tangent to the manifold, and coincides with the usual sectional curvature if $q=1$. The higher order sectional curvatures are weaker invariants of Riemannian structure than the usual sectional curvature.

Very recently, R. S. Kulkarni [4] has introduced an interesting double form $\operatorname{con}\omega$ for a double form ω, such as $\operatorname{con}R=C$ as a special case $\omega=R$. He also proved that $\operatorname{con}\omega$ has the same algebraic properties as the tensor C. It seems natural to seek for generalizations of classical results (conformal invariants, the theorem of Weyl etc.) on a conformal change of metric to the case of higher order, by making use of the Gauss-Kronecker curvature tensors. This is the purpose of the present work.

Section 1 is devoted to preliminary remarks. We shall recall definitions and fundamental formulas related to curvature structures from a view-point of double forms. In Section 2, we shall define a double form $\operatorname{con}_0\omega$ as a generalization of the Weyl 3-index tensor C_0 and obtain a new differential identity in Proposition 1. In Proposition 2, we shall give the conformal transformation formulas of $\operatorname{con}R^q$ and $\operatorname{con}_0 R^q$.

In this paper, a Riemannian manifold is said to be *q-flat* or *of q-constant curvature* according as the $2q$-th sectional curvature γ_{2q} is identically zero or constant, and to be *q-conformally flat* if $\operatorname{con}R^q=0$ for $n>4q-1$ and $\operatorname{con}_0 R^q=0$

for $n=4q-1$. In Section 4, we shall be concerned with relationships among these notions of higher order. The results in Theorems 1 and 2 are illustrated in the following diagram associated with a sequence of the Gauss-Kronecker curvature tensors $\{R^k\}$ $\left(k=1,\dots,\ q=\left[\frac{n+1}{4}\right]\right)$:

$$\begin{array}{ccccc}
\text{1-flat} & \longrightarrow \cdots \longrightarrow & k\text{-flat} & \longrightarrow \cdots \longrightarrow & q\text{-flat} \\
\downarrow & & \downarrow & & \downarrow \\
\text{1-const. curvature} & \cdots\cdots & k\text{-const. curvature} & \cdots\cdots & q\text{-const. curvature} \\
\downarrow & & \downarrow & & \downarrow \\
\text{1-conformally flat} & \to\cdots\to & k\text{-conformally flat} & \to\cdots\to & q\text{-conformally flat,}
\end{array}$$

where an arrow means implication from one to the next. (As for relations of another type for constancy of higher order sectional curvatures, see [5] and [7].) Furthermore, in Theorem 3 we shall state the conformal dependence of the q-conformally flat metric on the q-flat metric. Theorem 3 is a generalization of the theorem of Weyl. Examples of manifolds with or without some flatness will be presented in Section 5.

We shall assume, throughout this paper, that all manifolds and all objects are of differentiability class C^∞. For terminologies and notations, we generally follow [4] and [7].

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1. Preliminaries

In this section, let us recall main facts on the calculus of double forms due to A. Gray [2], R. S. Kulkarni [4] and O. Kowalski [3], for later use.

Let (M, g) be an n-dimensional smooth Riemannian manifold, $\mathfrak{F}(M)$ the ring of smooth functions on M, and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. For simplicity, we denote the space of sections of a bundle by the same notation as the bundle space. Let Λ^{*p} be the bundle of p-forms and

$$\Lambda^* = \sum_{0\leq p\leq n} \Lambda^{*p}$$

the bundle of differential forms on M. We put

$$\mathfrak{D}^{p,q} = \Lambda^{*p}\otimes\Lambda^{*q}$$

and

$$\mathfrak{D} = \Lambda^*\otimes\Lambda^* = \sum_{0\leq p,q\leq n} \mathfrak{D}^{p,q},$$

where the tensor products are taken over $\mathfrak{F}(M)$. We call an element ω of $\mathfrak{D}^{p,q}$ a *double form of type* (p, q) on M. It is an $\mathfrak{F}(M)$-multilinear map

$$\omega\colon \mathfrak{X}(M)^p \times \mathfrak{X}(M)^q \longrightarrow \mathfrak{F}(M),$$

which is skew-symmetric in the first p variables and also in the last q variables. We shall use the notation

$$\omega(x_1x_2\dots x_p \otimes y_1y_2\dots y_q)$$

to denote the value of ω in the vector fields $x_1,\dots, x_p$ and $y_1,\dots, y_q$. For convenience, we identify Λ^{*p} with $\mathfrak{D}^{p,0}$ unless stated otherwise. Furthermore, we call ω a *curvature structure of order* p if $p=q$ and we have

$$\omega(x_1\dots x_p \otimes y_1\dots y_p) = \omega(y_1\dots y_p \otimes x_1\dots x_p)$$

for all $x_1,\dots, x_p, y_1,\dots, y_p \in \mathfrak{X}(M)$, and denote the set of curvature structures of order p by $\mathfrak{C}^p$. The metric tensor field g is a curvature structure of order one. We put

$$\mathfrak{C} = \sum_{0 \leqq p \leqq n} \mathfrak{C}^p.$$

As de Rham has noted, it is possible to define the exterior product $\omega \wedge \theta$ of two double forms $\omega \in \mathfrak{D}^{p,q}$ and $\theta \in \mathfrak{D}^{r,s}$ by the formula

$$\begin{aligned}(1.1)\qquad &(\omega \wedge \theta)(x_1\dots x_{p+r} \otimes y_1\dots y_{q+s}) \\ &= \sum_{\rho \in Sh(p,r)} \sum_{\sigma \in Sh(q,s)} \varepsilon_\rho \varepsilon_\sigma \omega(x_{\rho(1)}\dots x_{\rho(p)} \otimes y_{\sigma(1)}\dots y_{\sigma(q)}) \\ &\qquad \times \theta(x_{\rho(p+1)}\dots x_{\rho(p+r)} \otimes y_{\sigma(q+1)}\dots y_{\sigma(q+s)})\end{aligned}$$

for $x_1,\dots, x_{p+r}, y_1,\dots, y_{q+s} \in \mathfrak{X}(M)$. Here, $Sh(p, r)$ denotes the set of all (p, r)-shuffles;

$$Sh(p, r) = \{\rho \in S_{p+r};\ \rho(1) < \cdots < \rho(p) \quad \text{and} \quad \rho(p+1) < \cdots < \rho(p+r)\},$$

where S_{p+r} is the symmetric group of degree $p+r$. It is not difficult to show that the multiplication $\wedge$ is associative and that we have

$$\omega \wedge \theta = (-1)^{pr+qs}\theta \wedge \omega$$

for $\omega \in \mathfrak{D}^{p,q}$ and $\theta \in \mathfrak{D}^{r,s}$. Thus, $\mathfrak{D}$ forms a graded associative ring and in particular $\mathfrak{C}$ is a commutative subring of $\mathfrak{D}$.

Let ω^k denote the k-th exterior power of $\omega \in \mathfrak{C}^p$. Then, by the formula (1.1) we have

$$\begin{aligned}(1.2)\qquad &\omega^k(x_1\dots x_{pk} \otimes y_1\dots y_{pk}) \\ &= \frac{1}{(p!)^{2k}} \sum_{\rho,\sigma \in S_{pk}} \varepsilon_\rho \varepsilon_\sigma \omega(x_{\rho(1)}\dots x_{\rho(p)} \otimes y_{\sigma(1)}\dots y_{\sigma(p)}) \\ &\qquad \dots \omega(x_{\rho\{p(k-1)+1\}}\dots x_{\rho(pk)} \otimes y_{\sigma\{p(k-1)+1\}}\dots y_{\sigma(pk)})\end{aligned}$$

for any vector fields $x_1,\dots, x_{pk}, y_1,\dots, y_{pk}$. The inner product on the bundle Λ^p of p-vectors is defined by the formula

$$<x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p> = \det \| <x_i, y_j> \|$$

for any decomposable p-vectors $x_1 \wedge \cdots \wedge x_p$ and $y_1 \wedge \cdots \wedge y_p$. Putting $\omega = g$ in (1.2), we see that it can be rewritten as follows;

$$(1.3) \qquad <x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p> = \frac{1}{p!} g^p(x_1 \dots x_p \otimes y_1 \dots y_p).$$

Now, we introduce three basic operations on $\mathfrak{D}$.

(I) The *first Bianchi sum* $\mathfrak{S}$ is a map of $\mathfrak{D}^{p,q}$ into $\mathfrak{D}^{p+1,q-1}$ defined as follows. For $\omega \in \mathfrak{D}^{p,q}$, we put $\mathfrak{S}\omega = 0$ if $q = 0$, and put

$$\mathfrak{S}\omega(x_1 \dots x_{p+1} \otimes y_1 \dots y_{q-1}) = \sum_{j=1}^{p+1} (-1)^j \omega(x_1 \dots \hat{x}_j \dots x_{p+1} \otimes x_j y_1 \dots y_{q-1})$$

if $q \geqq 1$, where $x_1,\dots, x_{p+1}, y_1,\dots, y_{q-1} \in \mathfrak{X}(M)$ and the symbol $\wedge$ denotes omission.

(II) The *second Bianchi sum* D is a map of $\mathfrak{D}^{p,q}$ into $\mathfrak{D}^{p+1,q}$ defined as follows. For $\omega \in \mathfrak{D}^{p,q}$, we put

$$D\omega(x_1 \dots x_{p+1} \otimes y_1 \dots y_q) = \sum_{j=1}^{p+1} (-1)^j (\nabla_{x_j}\omega)(x_1 \dots \hat{x}_j \dots x_{p+1} \otimes y_1 \dots y_q),$$

where ∇ denotes the covariant differentiation with respect to the metric g. We remark that D coincides with $-d$ on $\mathfrak{D}^{p,0}$, where d is the exterior differential operator.

(III) The *contraction* c is a map of $\mathfrak{D}^{p,q}$ into $\mathfrak{D}^{p-1,q-1}$ defined as follows. For $\omega \in \mathfrak{D}^{p,q}$ with $p = 0$ or $q = 0$, we put $c\omega = 0$. If both p and $q \geqq 1$, then we put

$$c\omega(x_1 \dots x_{p-1} \otimes y_1 \dots y_{q-1}) = \sum_{k=1}^{n} \omega(e_k x_1 \dots x_{p-1} \otimes e_k y_1 \dots y_{q-1}),$$

where $\{e_1,\dots, e_n\}$ is a locally defined orthonormal frame field with respect to g. We shall say ω to be *effective* if $c\omega = 0$. Let $E^{p,q}$ denote the set of effective elements of $\mathfrak{D}^{p,q}$.

Concerning these operators, the following propositions are well-known (cf. [4, §1, §2]).

PROPOSITION A. *Let $\omega \in \mathfrak{D}^{p,q}$ and $\theta \in \mathfrak{D}^{r,s}$. Then we have*

(a) $\qquad \mathfrak{S} \cdot c = c \cdot \mathfrak{S}$ *on* $\mathfrak{D}$,

(b) $\qquad \mathfrak{S}(\omega \wedge \theta) = \mathfrak{S}\omega \wedge \theta + (-1)^{p+q}\omega \wedge \mathfrak{S}\theta$,

(c) $\qquad D(\omega \wedge \theta) = D\omega \wedge \theta + (-1)^p \omega \wedge D\theta$.

PROPOSITION B.

(a) *For any* $\omega \in \mathfrak{D}^{p,q}$, *we have*

$$c(g \wedge \omega) = g \wedge c\omega + (n-p-q)\omega .$$

(b) *Multiplication with g is injective on* $\sum_{p+q<n} \mathfrak{D}^{p,q}$.

We notice that as a special case of (c) in Proposition A we have

$$D(f\omega) = -df \wedge \omega + fD\omega \tag{1.4}$$

for any $f \in \mathfrak{F}(M)$ and any $\omega \in \mathfrak{D}^{p,q}$, and also that the property (b) in Proposition B means a *cancellation law with respect to g* in $\mathfrak{D}$, that is,

$$g \wedge \omega = 0 \quad \textit{implies} \quad \omega = 0$$

if $\omega \in \mathfrak{D}^{p,q}$, $p+q<n$. We define

$$\mathfrak{C}_1 = \mathfrak{C} \cap \operatorname{kernel} \mathfrak{S}, \quad \mathfrak{C}_2 = \mathfrak{C} \cap \operatorname{kernel} D \quad \text{and} \quad \mathfrak{C}_0 = \mathfrak{C}_1 \cap \mathfrak{C}_2 .$$

Then, from the identities (b) and (c) in Proposition A it follows that both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are subrings of $\mathfrak{C}$. We shall call an element of $\mathfrak{C}_1$ or $\mathfrak{C}_2$ the *curvature structure satisfying the first* or *the second Bianchi identity*, respectively.

Let us put

$$\tilde{\delta} = c \cdot D + D \cdot c . \tag{1.5}$$

Then the explicit expression of the map $\tilde{\delta}\colon \mathfrak{D}^{p,q} \to \mathfrak{D}^{p,q-1}$ is given by

$$\tilde{\delta}\omega(x_1 \dots x_p \otimes y_1 \dots y_{q-1}) = -\sum_{k=1}^{n} (\nabla_{e_k}\omega)(x_1 \dots x_p \otimes e_k y_1 \dots y_{q-1}) \tag{1.6}$$

for any $\omega \in \mathfrak{D}^{p,q}$ $(q \geqq 1)$. This formula implies

$$\tilde{\delta} \cdot c + c \cdot \tilde{\delta} = 0 . \tag{1.7}$$

Now, let us define the *inner product* $\omega \llcorner v \in \mathfrak{D}^{p,q-1}$ of a double form $\omega \in \mathfrak{D}^{p,q}$ $(q \geqq 1)$ with a vector field v by the equation

$$(\omega \llcorner v)(x_1 \dots x_p \otimes y_1 \dots y_{q-1}) = \omega(x_1 \dots x_p \otimes y_1 \dots y_{q-1} v)$$

for any vector fields $x_1, \dots, x_p$, $y_1, \dots, y_{q-1}$. Then, from the formula (1.1) we obtain the following identity due to O. Kowalski (cf. [3, Prop. 2]):

$$(\omega \wedge \theta) \llcorner v = \omega \wedge (\theta \llcorner v) + (-1)^s (\omega \llcorner v) \wedge \theta \tag{1.8}$$

for $\omega \in \mathfrak{D}^{p,q}$ and $\theta \in \mathfrak{D}^{r,s}$. From (1.8) we have inductively

$$\omega^k \llcorner v = k\omega^{k-1} \wedge (\omega \llcorner v) \tag{1.9}$$

for any curvature structure ω of order $p \geqq 1$ and any positive integer k.

Let G_p denote the Grassmann bundle of p-planes tangent to the manifold

M. For a curvature structure $\omega \in \mathfrak{C}^p$, we define the *curvature function* K_ω: $G_p \to \mathbf{R}$ associated with ω by

$$K_\omega(\sigma) = \frac{\omega(x_1 \dots x_p \otimes x_1 \dots x_p)}{\|x_1 \wedge \cdots \wedge x_p\|^2} \tag{1.10}$$

for any p-plane σ at each point $m \in M$, where $\{x_1, \dots, x_p\}$ is a base of σ. The value $K_\omega(\sigma)$ is independent of the choice of $\{x_1, \dots, x_p\}$. This curvature function K_ω generically determines ω in the sense that, for two ω, $\theta \in \mathfrak{C}_1^p$, the equality $K_\omega = K_\theta$ implies $\omega = \theta$ (cf. [4, Prop. 2.1]). In particular, by (1.3) and (1.10) we have for any $\omega \in \mathfrak{C}_1^p$

$$K_\omega = \text{const. } \kappa \quad \textit{if and only if} \quad \omega = \frac{\kappa}{p!} g^p . \tag{1.11}$$

2. Generalizations of Weyl's tensors

First of all, let us recall classical facts about the Weyl conformal curvature tensor C and the Weyl 3-index tensor C_0, which are basic for this paper (for the details, see [1, § 28]).

Let R_{xy} be the curvature operator given by the formula

$$R_{xy} = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$$

for any two vector fields x and y. The curvature tensor R of type (0, 4) is defined by the formula

$$R(xy \otimes uv) = \langle R_{xy}u, v \rangle$$

for any vector fields x, y, u, v, and it is an element of $\mathfrak{C}_0^2$. The Weyl conformal curvature tensor C is a double form of type (2, 2) given by

$$C = R - \frac{1}{n-2} g \wedge cR + \frac{c^2 R}{2(n-2)(n-1)} g^2 . \tag{2.1}$$

It is an effective element of $\mathfrak{C}_1^2$ and vanishes identically if $n=3$. The tensor C_0 is defined by the formula

$$C_0 = (n-2) D\theta ,$$

where we have put

$$\theta = \frac{1}{n-2} cR - \frac{c^2 R}{2(n-2)(n-1)} g . \tag{2.2}$$

By the well-known identity (cf. [1, Eq. (28.16)])

$$\delta C = \frac{n-3}{n-2} C_0 , \tag{2.3}$$

the tensor C_0 vanishes identically if $C=0$ and $n>3$, but it does not vanish, in general, if $n=3$.

Let

$$\bar{g} = e^{2\phi} g \qquad (\phi \in \mathfrak{F}(M))$$

be an another metric conformally equivalent to g. As usual, we indicate by a bar overhead the corresponding geometric objects with respect to the metric $\bar{g}$. Then, we know the transformation formulas

$$\bar{C} = e^{2\phi} C \tag{2.4}$$

and

$$\bar{C}_0 = C_0 + (n-2) C \llcorner \operatorname{grad} \phi . \tag{2.5}$$

Now, the process (2.1) deriving C from R has been generalized to a map on $\mathfrak{D}$ by Kulkarni (cf. [4, § 2]), as follows. Let $p+q+1 \leqq n$ and $h = \min(p, q)$. Then, the *conformal map* con is by definition a map of $\mathfrak{D}^{p+1,q+1}$ into itself such that

$$\operatorname{con} \omega = \omega + \sum_{r=1}^{h+1} \frac{(-1)^r g^r \wedge c^r \omega}{r! \prod_{j=0}^{r-1} (n-p-q+j)} \tag{2.6}$$

for any double form $\omega \in \mathfrak{D}^{p+1,q+1}$. We remark that $\operatorname{con}\omega$ depends only on the conformal class of the metric g. The following proposition due to Kulkarni (cf. [4, § 2, § 3]) shows that the formal algebraic identities for the tensor C actually hold good for the double form $\operatorname{con}\omega$, and it plays an important role in this paper.

PROPOSITION C. *Let* $p+q+1 \leqq n$ *and* $\omega \in \mathfrak{D}^{p+1,q+1}$.

(a) *The map* con *is a projection of* $\mathfrak{D}^{p+1,q+1}$ *onto* $E^{p+1,q+1}$.

(b) *There are unique elements* $\alpha \in \mathfrak{D}^{p,q}$ *and* $\beta \in E^{p+1,q+1}$ *such that*

$$\omega = \beta + g \wedge \alpha . \tag{2.7}$$

Moreover, $\beta = \operatorname{con}\omega$.

(c) *If* $n = p+q+1$, *then* $\operatorname{con}\omega = 0$.

(d) *If* $\mathfrak{S}\omega = 0$, *then* $\mathfrak{S}\cdot\operatorname{con}\omega = 0$.

Following O. Kowalski, we call the correspondence $\omega \mapsto \alpha$ given by the formula (2.7) the *deviation map*, and we denote the element α by $\operatorname{dev}\omega$. From (2.6) and (2.7), it follows that the explicit expression of the map dev: $\mathfrak{D}^{p+1,q+1} \to \mathfrak{D}^{p,q}$ is given by

$$\operatorname{dev}\omega=\sum_{r=1}^{h+1}\frac{(-1)^{r-1}g^{r-1}\wedge c^{r}\omega}{r!\prod_{j=0}^{r-1}(n-p-q+j)}. \tag{2.8}$$

From the property (b) in Proposition C, we see that

$$\operatorname{con}\omega=0 \quad \textit{if and only if} \quad \omega=g\wedge\operatorname{dev}\omega. \tag{2.9}$$

Also, we notice that

$$\theta=\operatorname{dev}R.$$

As a generalization of the Weyl 3-index tensor C_0, we define a map con_0: $\mathfrak{D}^{p+1,q+1}\to\mathfrak{D}^{p+1,q}$ for $p+q+1\leqq n$ by

$$\operatorname{con}_0=D\cdot\operatorname{dev}.$$

The main purpose of this section is to prove

PROPOSITION 1. *Let* $p+q+1<n$ *and* $\omega\in\mathfrak{D}^{p+1,q+1}$. *Then we have*

$$\tilde{\delta}\cdot\operatorname{con}\omega=(n-p-q-1)(\operatorname{con}_0\omega+\operatorname{dev}\cdot D\omega).$$

COROLLARY 1. *Let* $p+q+1<n$ *and* $\omega\in\mathfrak{D}^{p+1,q+1}$. *If* ω *satisfies the second Bianchi identity, then we have*

$$\tilde{\delta}\cdot\operatorname{con}\omega=(n-p-q-1)\operatorname{con}_0\omega. \tag{2.10}$$

Moreover, suppose that $\operatorname{con}\omega=0$. *Then we have* $\operatorname{con}_0\omega=0$.

The formula (2.10) is a generalization of the identity (2.3). In fact, by putting $\omega=R$ we get (2.3), because we have

$$C_0=(n-2)\operatorname{con}_0R.$$

To prove Proposition 1, we shall need three lemmas. First, we have

LEMMA 1. *For any* $\omega\in\mathfrak{D}^{p,q}$, *we have*

$$\tilde{\delta}(g^r\wedge\omega)=(-1)^r(g^r\wedge\tilde{\delta}\omega-rg^{r-1}\wedge D\omega) \qquad (r\geqq 1).$$

PROOF. By the formula (a) in Proposition B, we get inductively

$$c(g^r\wedge\omega)=g^r\wedge c\omega+r(n-p-q-r+1)g^{r-1}\wedge\omega$$

for any $\omega\in\mathfrak{D}^{p,q}$ and any positive integer r. Since $Dg^r=0$, by making use of (c) in Proposition A, (a) in Proposition B and the above identity, we have

$$\begin{aligned}c\cdot D(g^r\wedge\omega)&=(-1)^r c(g^r\wedge D\omega)\\&=(-1)^r\{g^r\wedge c\cdot D\omega+r(n-p-q-r)g^{r-1}\wedge D\omega\}\end{aligned}$$

and

$$D\cdot c(g^r \wedge \omega) = D\{g^r \wedge c\omega + r(n-p-q-r+1)g^{r-1} \wedge \omega\}$$
$$= (-1)^r\{g^r \wedge D\cdot c\omega - r(n-p-q-r+1)g^{r-1} \wedge D\omega\}.$$

Therefore, we have Lemma 1 from the definition of $\tilde{\delta}$. q.e.d.

Next, we have

LEMMA 2. $\quad \tilde{\delta}\cdot c^r = \dfrac{1}{r+1}\{D\cdot c^{r+1} + (-1)^r c^{r+1}\cdot D\} \qquad (r \geqq 0)$.

PROOF. Since we have

$$\tilde{\delta}\cdot c^r = (c\cdot D + D\cdot c)\cdot c^r = c\cdot D\cdot c^r + D\cdot c^{r+1},$$

it suffices to verify the following relation:

$$c\cdot D\cdot c^r = -\frac{1}{r+1}\{rD\cdot c^{r+1} + (-1)^{r+1}c^{r+1}\cdot D\}. \tag{2.11}$$

We prove this by induction with respect to r. If $r=0$, then (2.11) is trivial. From the relation (1.7) it follows

$$\tilde{\delta}\cdot c + c\cdot\tilde{\delta} = D\cdot c^2 + 2c\cdot D\cdot c + c^2\cdot D = 0,$$

from which we obtain

$$c\cdot D\cdot c = -\frac{1}{2}(D\cdot c^2 + c^2\cdot D). \tag{2.12}$$

Accordingly, (2.11) is true when $r=1$. Suppose that we get (2.11) for $r=0, 1, \ldots, t$, where $t \geqq 1$. Then we have

$$c\cdot D\cdot c^{t+1} = -\frac{1}{t+1}\{tD\cdot c^{t+2} + (-1)^{t+1}c^t\cdot(c\cdot D\cdot c)\},$$

which implies by (2.12)

$$c\cdot D\cdot c^{t+1} = -\frac{1}{t+1}\left\{tD\cdot c^{t+2} + \frac{(-1)^t}{2}(c^t\cdot D\cdot c^2 + c^{t+2}\cdot D)\right\}. \tag{2.13}$$

On the other hand, by (2.11) for $r=t-1$ we get

$$c\cdot D\cdot c^{t+1} = (c\cdot D\cdot c^{t-1})\cdot c^2$$
$$= -\frac{1}{t}\{(t-1)D\cdot c^{t+2} + (-1)^t c^t\cdot D\cdot c^2\},$$

from which we obtain

$$c^t \cdot D \cdot c^2 = (-1)^{t+1}\{tc \cdot D \cdot c^{t+1} + (t-1)D \cdot c^{t+2}\}.$$

Substituting this into (2.13), we find that (2.11) is true when $r=t+1$. q.e.d.

Combining Lemma 1 with Lemma 2, we have

LEMMA 3. *For any $\omega \in \mathfrak{D}^{p,q}$ and any positive integer r, we have*

$$\tilde{\delta}(g^r \wedge c^r \omega) = \frac{1}{r+1} D(g^r \wedge c^{r+1}\omega) + rD(g^{r-1} \wedge c^r \omega) + \frac{1}{r+1} g^r \wedge c^{r+1} \cdot D\omega .$$

PROOF OF PROPOSITION 1. Apply $\tilde{\delta}$ on both the sides of (2.6), and use Lemma 3. Then, since $c^{h+2}\omega = 0$, we obtain

$$\begin{aligned}\tilde{\delta} \cdot \mathrm{con}\,\omega = &\left(1 - \frac{1}{n-p-q}\right) D \cdot c\omega - \frac{1}{n-p-q}\left(\frac{1}{2} - \frac{1}{n-p-q+1}\right) D(g \wedge c^2 \omega) + \cdots \\ &+ \frac{(-1)^h}{h! \prod_{j=0}^{h-1}(n-p-q+j)} \left(\frac{1}{h+1} - \frac{1}{n-p-q+h}\right) D(g^h \wedge c^{h+1}\omega) \\ &+ c \cdot D\omega - \frac{1}{2(n-p-q)} g \wedge c^2 \cdot D\omega + \frac{1}{3!(n-p-q)(n-p-q+1)} g^2 \\ &\wedge c^3 \cdot D\omega - \cdots + \frac{(-1)^{h+1}}{(h+2)! \prod_{j=0}^{h}(n-p-q+j)} g^{h+1} \wedge c^{h+2} \cdot D\omega ,\end{aligned}$$

where the last term vanishes if $p \geqq q$, but it remains if $p < q$. In both the cases, we can verify that the sum of the first two lines in the right-hand side of the above equation is equal to

$$(n-p-q-1) D \cdot \mathrm{dev}\,\omega ,$$

and the sum of the last two lines is equal to

$$(n-p-q-1)\, \mathrm{dev} \cdot D\omega ,$$

respectively. Thus, we have Proposition 1. q.e.d.

3. Conformal change of a metric

In the following, we shall apply the maps con and con_0 on the q-th Gauss-Kronecker curvature tensor R^q and generalize classical results on the conformal change of the metric:

$$\bar{g} = e^{2\phi} g \qquad (\phi \in \mathfrak{F}(M)). \tag{3.1}$$

In this section, we consider the transformation formulas of $\mathrm{con}\,R^q$ and $\mathrm{con}_0\,R^q$

under (3.1).

We need some initial preparations due to Kulkarni (for the details, see [4, § 6]). For a vector field x, we put

$$S_x = \bar{\nabla}_x - \nabla_x .$$

Then, considered as a derivation on the tensor algebra over (M, g), S_x is determined as follows:

$$\text{(3.2)}\qquad \begin{aligned} &\text{(a)}\quad S_x f = 0 \qquad \text{for any}\quad f \in \mathfrak{F}(M),\\ &\text{(b)}\quad S_x y = (x\phi)y + (y\phi)x - <x, y> \operatorname{grad}\phi \qquad \text{for any}\quad y \in \mathfrak{X}(M),\\ &\text{(c)}\quad (S_x\theta)y = -\theta(S_x y) \qquad \text{for any}\quad \theta \in \Lambda^{*1}. \end{aligned}$$

It follows from (3.2) that if $\omega \in \mathfrak{D}^{p,q}$, $u \in \Lambda^p$ and $v \in \Lambda^q$ then

$$\text{(3.3)}\qquad (S_x\omega)(u \otimes v) = -\omega(S_x u \otimes v) - \omega(u \otimes S_x v).$$

Furthermore, it is known (cf. [4, § 6, Lemma 2]) that

$$\text{(3.4)}\qquad \sum_{j=1}^{p+1} (-1)^j S_{x_j}(x_1 \ldots \hat{x}_j \ldots x_{p+1}) = 0$$

for all $x_1, \ldots, x_{p+1} \in \mathfrak{X}(M)$, where $x_1 \ldots \hat{x}_j \ldots x_{p+1}$ denotes of course the p-vector $x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{p+1}$.

Owing these considerations, we have

LEMMA 4. *Suppose that $\omega \in \mathfrak{D}^{p,q}$ $(q \geqq 1)$ satisfies the first Bianchi identity. Then we have*

$$\bar{D}\omega = D\omega + (q-1)d\phi \wedge \omega + (-1)^q (g \wedge \omega) \llcorner \operatorname{grad}\phi .$$

PROOF. From the definitions of D and S_x, we get

$$\begin{aligned} \{(\bar{D} - D)\omega\}(x_1 \ldots x_{p+1} \otimes y_1 \ldots y_q) &= \sum_{j=1}^{p+1} (-1)^j (S_{x_j}\omega)(x_1 \ldots \hat{x}_j \ldots x_{p+1} \otimes y_1 \ldots y_q)\\ &= -\sum_{j=1}^{p+1} (-1)^j \omega(x_1 \ldots \hat{x}_j \ldots x_{p+1} \otimes S_{x_j}(y_1 \ldots y_q)), \end{aligned}$$

by making use of the equations (3.3) and (3.4). Since we have by (b) in (3.2)

$$\begin{aligned} S_{x_j}(y_1 \ldots y_q) = q(x_j\phi)y_1 \ldots y_q &+ \sum_{k=1}^{q} (-1)^{k-1} (y_k\phi) x_j y_1 \ldots \hat{y}_k \ldots y_q\\ &- \sum_{k=1}^{q} (-1)^{k-1} <x_j, y_k> G y_1 \ldots \hat{y}_k \ldots y_q , \end{aligned}$$

being $G = \operatorname{grad}\phi$, it follows

$$\{(\bar{D}-D)\omega\}(x_1 \ldots x_{p+1} \otimes y_1 \ldots y_q)$$
$$= -q \sum_{j=1}^{p+1} (-1)^j (x_j\phi)\omega(x_1 \ldots \hat{x}_j \ldots x_{p+1} \otimes y_1 \ldots y_q)$$
$$- \sum_{k=1}^{q} (-1)^{k-1} (y_k\phi)\{\sum_{j=1}^{p+1} (-1)^j \omega(x_1 \ldots \hat{x}_j \ldots x_{p+1} \otimes x_j y_1 \ldots \hat{y}_k \ldots y_q)\}$$
$$+ \sum_{j=1}^{p+1} \sum_{k=1}^{q} (-1)^{j+k-1} < x_j, y_k > \omega(x_1 \ldots \hat{x}_j \ldots x_{p+1} \otimes G y_1 \ldots \hat{y}_k \ldots y_q).$$

On the right-hand side of the above equation, we have

$$\text{the first sum} = q(d\phi \wedge \omega)(x_1 \ldots x_{p+1} \otimes y_1 \ldots y_q) \qquad \text{(by (1.1))},$$
$$\{\quad\} \text{ in the second sum} = \mathfrak{S}\omega(x_1 \ldots x_{p+1} \otimes y_1 \ldots \hat{y}_k \ldots y_q),$$
$$\text{the third sum} = (-1)^q \{g \wedge (\omega \llcorner G)\}(x_1 \ldots x_{p+1} \otimes y_1 \ldots y_q),$$

respectively. Since the second sum vanishes by the assumption $\mathfrak{S}\omega = 0$, we find

$$\bar{D}\omega - D\omega = q\, d\phi \wedge \omega + (-1)^q g \wedge (\omega \llcorner G)$$
$$= (q-1) d\phi \wedge \omega + (-1)^q (g \wedge \omega) \llcorner G \qquad \text{(by (1.8))},$$

because of the identity $g \llcorner G = d\phi$. q.e.d.

It is well-known (cf. [1, Eq. (28.5)]) that the curvature tensors $\bar{R}$ and R are related by the formula

$$\bar{R} = e^{2\phi}\{R + g \wedge \kappa(\phi)\}, \tag{3.5}$$

where $\kappa(\phi)$ is an element of $\mathfrak{C}_1^1$ defined for any $\phi \in \mathfrak{F}(M)$ by

$$\kappa(\phi)(x \otimes y) = < \nabla_x G, y > - < G, x > < G, y > + \frac{1}{2} < G, G > < x, y >$$

for any vector fields x and y. By straightforward calculations, we can obtain the identity

$$D\kappa(\phi) = \{R + g \wedge \kappa(\phi)\} \llcorner \operatorname{grad} \phi. \tag{3.6}$$

It follows from (3.5) that the q-th Gauss-Kronecker curvature tensors $\bar{R}^q$ and R^q are related by

$$\bar{R}^q = e^{2q\phi}\{R^q + g \wedge \eta(\phi)\}, \tag{3.7}$$

where $\eta(\phi)$ is an element of $\mathfrak{C}_1^{2q-1}$ defined for any $\phi \in \mathfrak{F}(M)$ by

$$\eta(\phi) = \sum_{r=1}^{q} \binom{q}{r} R^{q-r} \wedge g^{r-1} \wedge \kappa(\phi)^r. \tag{3.8}$$

By making use of the equations (1.9), (3.6) and (3.8), we can obtain the following identity due to Kowalski (cf. [3, p. 342]):

$$D\eta(\phi) = \{R+g \wedge \kappa(\phi)\}^q \llcorner \operatorname{grad}\phi . \tag{3.9}$$

Now, let us give the transformation formulas of $\operatorname{con} R^q$ and $\operatorname{con}_0 R^q$ under the conformal change (3.1) of the metric g. We notice that both $\operatorname{con} R^q$ and $\operatorname{con}_0 R^q$ are defined for the Riemannian manifold (M, g) of dimension $n \geqq 4q-1$.

PROPOSITION 2. *Under the conformal change* (3.1) *of metric, we have*

$$\overline{\operatorname{con}}\, \bar{R}^q = e^{2q\phi} \operatorname{con} R^q \tag{3.10}$$

and

$$\overline{\operatorname{con}_0}\, \bar{R}_q = e^{2(q-1)\phi}\{\operatorname{con}_0 R^q + (\operatorname{con} R^q) \llcorner \operatorname{grad}\phi\} . \tag{3.11}$$

The formula (3.10) was first due to Kulkarni (cf. [4, Prop. 8.1]). The formulas (3.10) and (3.11) are generalizations of the formulas (2.4) and (2.5) respectively.

PROOF OF PROPOSITION 2. By the remark following the definition (2.6) of $\operatorname{con}\omega$ we have

$$\overline{\operatorname{con}}\, \bar{R}^q = \operatorname{con} \bar{R}^q .$$

Hence we get by (3.7)

$$\overline{\operatorname{con}}\, \bar{R}^q = e^{2q\phi}\{\operatorname{con} R^q + \operatorname{con}(g \wedge \eta(\phi))\} ,$$

and moreover $\operatorname{con}(g \wedge \eta(\phi))=0$ by the property (b) in Proposition C. Thus, we have (3.10). Next, from the definition of the map dev, it follows that

$$\begin{aligned} \bar{g} \wedge \overline{\operatorname{dev}}\, \bar{R}^q &= \bar{R}^q - \overline{\operatorname{con}}\, \bar{R}^q \\ &= e^{2q\phi} g \wedge \{\operatorname{dev} R^q + \eta(\phi)\} \qquad \text{(by (3.7), (3.10))}. \end{aligned}$$

Substitute (3.1) into the left-hand side of the above equation, and then apply the cancellation law with respect to g. Then we have

$$\overline{\operatorname{dev}}\, \bar{R}^q = e^{2(q-1)\phi}\{\operatorname{dev} R^q + \eta(\phi)\} . \tag{3.12}$$

Further, apply the second Bianchi map $\bar{D}$ on both the sides of (3.12). Then, from the equation (1.4) we obtain

$$\begin{aligned} \overline{\operatorname{con}_0}\, \bar{R}^q = & -d(e^{2(q-1)\phi}) \wedge \{\operatorname{dev} R^q + \eta(\phi)\} \\ & + e^{2(q-1)\phi}\, \bar{D}\{\operatorname{dev} R^q + \eta(\phi)\} . \end{aligned}$$

Since both $\operatorname{dev} R^q$ and $\eta(\phi)$ satisfy the first Bianchi identity, we can apply Lemma 4 on the second term in the right-hand side of the above equation, and we obtain

$$\begin{aligned}\overline{\operatorname{con}}_0 \bar{R}^q &= e^{2(q-1)\phi}\Big[D\{\operatorname{dev} R^q+\eta(\phi)\}-[g \wedge \{\operatorname{dev} R^q+\eta(\phi)\}] \llcorner G\Big] \\ &= e^{2(q-1)\phi}\{\operatorname{con}_0 R^q+(R^q-g \wedge \operatorname{dev} R^q) \llcorner G\} \qquad \text{(by (3.9))} \\ &= e^{2(q-1)\phi}\{\operatorname{con}_0 R^q+(\operatorname{con} R^q) \llcorner G\}.\end{aligned}$$

Thus, we have (3.11). q.e.d.

Since we get

$$\operatorname{con} R^q = 0 \qquad \text{for} \quad n = 4q-1 \tag{3.13}$$

by the property (c) in Proposition C, we have

COROLLARY 2. *Let $n=4q-1$. Then, under the conformal change* (3.1) *of metric, we have*

$$\overline{\operatorname{con}}_0 \bar{R}^q = e^{2(q-1)\phi} \operatorname{con}_0 R^q. \tag{3.14}$$

The formula (3.10) says that $\operatorname{con} R^q$ is a conformal invariant for Riemannian manifolds of dimension $n \geqq 4q-1$, but it is a trivial one by (3.13) when $n=4q-1$. The formula (3.14) shows that $\operatorname{con}_0 R^q$ is a conformal invariant for $(4q-1)$-dimensional Riemannian manifolds.

All the results obtained in Sections 2 and 3 are generalizations of the corresponding classical ones, except for the theorem of Weyl which will be considered in the next section.

4. q-conformal flatness

The object of this section is to define the concept of q-conformal flatness for Riemannian manifold (M, g) of dimension $n \geqq 4q-1$, and to obtain several basic theorems.

The $2q$-th sectional curvature γ_{2q} of Thorpe [7] is given by

$$\begin{aligned}\gamma_{2q}(\sigma) = \frac{(-1)^q}{2^q\{(2q)!\}} \sum_{\tau,\mu \in S_{2q}} \varepsilon_\tau \varepsilon_\mu R(e_{\tau(1)} e_{\tau(2)} \otimes e_{\mu(1)} e_{\mu(2)}) \\ \cdots R(e_{\tau(2q-1)} e_{\tau(2q)} \otimes e_{\mu(2q-1)} e_{\mu(2q)})\end{aligned}$$

for any $2q$-plane $\sigma \in G_{2q}$, where $\{e_1, \ldots, e_{2q}\}$ is an orthonormal base of σ. In the case $q=1$, γ_2 is the usual sectional curvature. By putting $\omega=R$ in (1.2), we find that the above formula can be rewritten as

$$\gamma_{2q}(\sigma) = \frac{(-2)^q}{(2q)!} R^q(e_1 \ldots e_{2q} \otimes e_1 \ldots e_{2q}).$$

Hence, γ_{2q} is equal to the curvature function $K_\omega\colon G_{2q}\to\mathbf{R}$ associated with the curvature structure

$$\omega=\frac{(-2)^q}{(2q)!}R^q. \tag{4.1}$$

DEFINITION 1. A Riemannian manifold (M, g) of dimension $n\geqq 2q$ is said to be *q-flat* or *of q-constant curvature*, according as the $2q$-th sectional curvature γ_{2q} is identically zero or constant. The metric g is called *q-flat* if (M, g) is q-flat.

LEMMA 5. *A Riemannian manifold (M, g) is of q-constant curvature κ_{2q} if and only if*

$$R^q=\frac{(-1)^q}{2^q}\kappa_{2q}g^{2q}. \tag{4.2}$$

In particular, (M, g) is q-flat if and only if $R^q=0$.

PROOF. Since $R^q\in\mathfrak{C}_0^{2q}$, we have Lemma 5 by the equations (1.11) and (4.1). q.e.d.

DEFINITION 2. Let $n\geqq 4q-1$. An n-dimensional Riemannian manifold is said to be *q-conformally flat* if

$$\operatorname{con} R^q=0 \qquad \text{for} \quad n>4q-1$$

and

$$\operatorname{con}_0 R^q=0 \qquad \text{for} \quad n=4q-1.$$

Owing to the formulas (3.10) and (3.14), it is of conformal nature whether a given manifold is q-conformally flat or not. Also, we remark that Corollary 1 implies $\operatorname{con}_0 R^q=0$ for q-conformally flat Riemannian manifolds of dimension $n>4q-1$.

LEMMA 6. *Let $n\geqq 2p+1$ $(p\geqq 1)$. For a curvature structure ω on (M, g) such that*

$$\omega=g\wedge\theta, \quad \textit{where} \quad \theta\in\mathfrak{C}_2^p,$$

we have $\operatorname{con}\omega=\operatorname{con}_0\omega=0$.

PROOF. By the property (b) in Proposition C, we have

$$\operatorname{con}\omega=0 \qquad \text{and} \quad \operatorname{dev}\omega=\theta.$$

The assumption $D\theta=0$ implies

$$\operatorname{con}_0\omega=D\cdot\operatorname{dev}\omega=0.$$

Thus, we have Lemma 6. q.e.d.

THEOREM 1. (i) *If* $n \geqq 2(q+1)$, *an* n*-dimensional* q*-flat Riemannian manifold is* $(q+1)$*-flat;*

(ii) *If* $n \geqq 4q+3$, *an* n*-dimensional* q*-conformally flat Riemannian manifold is* $(q+1)$*-conformally flat.*

PROOF. (i) is an immediate consequence of Lemma 5. Since $n > 4q-1$, the q-conformal flatness in (ii) means $\mathrm{con}\, R^q = 0$ by Definition 2. Hence, by (2.9) we have

$$R^q = g \wedge \mathrm{dev}\, R^q, \tag{4.3}$$

and also by Corollary 1 we get

$$\mathrm{con}_0\, R^q = D\cdot \mathrm{dev}\, R^q = 0. \tag{4.4}$$

Multiplying R to both the sides of the equation (4.3), we obtain

$$R^{q+1} = g \wedge (R \wedge \mathrm{dev}\, R^q),$$

where $R \wedge \mathrm{dev}\, R^q$ is an element of $\mathfrak{C}_2^{2q+1}$ because of the equation (4.4). Hence, we have $\mathrm{con}\, R^{q+1} = \mathrm{con}_0\, R^{q+1} = 0$ by Lemma 6. Thus, the manifold is $(q+1)$-conformally flat by Definition 2. q.e.d.

It follows from Definition 1 that a q-flat Riemannian manifold is of q-constant curvature. Moreover, we have

THEOREM 2. *Let* $n \geqq 4q-1$ *and* $q \geqq p \geqq 1$. *Then, an* n*-dimensional Riemannian manifold of* p*-constant curvature is* q*-conformally flat.*

PROOF. Let us assume $\gamma_{2p} = \text{const.}\, \kappa_{2p}$. Then, from (4.2) we obtain

$$R^q = g \wedge \theta,$$

where we have put

$$\theta = \frac{(-1)^p}{2^p} \kappa_{2p} g^{2p-1} \wedge R^{q-p}.$$

Since $\theta \in \mathfrak{C}_2^{2q-1}$, the manifold is q-conformally flat by Lemma 6 and Definition 2. q.e.d.

The diagram in the introduction is obtained by Theorems 1 and 2.

Corresponding to the theorem of Weyl, we have

THEOREM 3. *Let* $n \geqq 4q-1$. *Then, an* n*-dimensional Riemannian manifold* (M, g) *is* q*-conformally flat if the metric* g *is conformally related to some* q*-flat metric on* M. *Conversely, if* (M, g) *is* q*-conformally flat and, in an open subset* U *of* M, *there exists a solution* ϕ *of the differential equation*

$$\eta(\phi)+\operatorname{dev} R^q = 0, \tag{4.5}$$

where $\eta(\phi)$ *is the element of* $\mathfrak{C}_1^{2q-1}$ *given by* (3.8), *then the metric* $\bar{g}=e^{2\phi}g$ *is* q*-flat in* U.

PROOF. Suppose that the metric g is conformally related to some q-flat metric $\bar{g}$ by (3.1). Then we have

$$\overline{\operatorname{con}}\,\bar{R}^q = \overline{\operatorname{con}_0}\,\bar{R}^q = 0,$$

because the tensor $\bar{R}^q$ vanishes identically on M by Lemma 5. On account of the transformation formulas (3.10) and (3.14), we have from the above equations

$$\operatorname{con} R^q = 0 \qquad \text{for} \quad n > 4q-1$$

and

$$\operatorname{con}_0 R^q = 0 \qquad \text{for} \quad n = 4q-1.$$

Thus, (M, g) is q-conformally flat by Definition 2.

Conversely, if the equation (4.5) admits a solution ϕ in U, then we put $\bar{g}=e^{2\phi}g$. By (3.12) and (4.5) we find $\overline{\operatorname{dev}}\,\bar{R}^q=0$ in U, from which we have

$$\bar{R}^q = \overline{\operatorname{con}}\,\bar{R}^q \tag{4.6}$$

in U by (2.7). Since we have assumed that (M, g) is q-conformally flat, by the equation (3.13) and Definition 2 we get

$$\operatorname{con} R^q = 0 \qquad \text{for} \quad n \geqq 4q-1,$$

which implies $\bar{R}^q=0$ by (3.10) and (4.6). Thus, the metric $\bar{g}$ is q-flat by Lemma 5.

q.e.d.

REMARK. In the classical case $q=1$, J.A. Schouten has proved by making use of the identity (2.3) that the differential equation (4.5) admits a solution ϕ on a neighborhood at each point of M, if the manifold (M, g) is 1-conformally flat (cf. [1, p. 92]). In the case $q\geqq 2$, (4.5) is a system of non-linear partial differential equations of second order with coefficients in $\mathfrak{F}(M)$. Though we have obtained (2.10) as a generalization of the identity (2.3), the present author does not yet know whether (4.5) still admits a solution ϕ or not.

5. Examples

We assume, throughout this section, that (M, g) is a product Riemannian manifold of two Riemannian manifolds (M_a, g_a) $(a=1, 2)$.

In the paper [6, Th. 2], we have obtained

PROPOSITION 3. *Let (M, g) be a product Riemannian manifold of (M_1, g_1) and (M_2, g_2) with constant sectional curvatures κ_2' and κ_2'', respectively. Suppose that both M_1 and M_2 are of dimension $\geqq 2q\,(q \geqq 1)$. Then, (M, g) is q-conformally flat if and only if*

$$\kappa_2' + \kappa_2'' = 0\,. \tag{5.1}$$

As a corollary to Proposition 3, we have the following example.

EXAMPLE 1. Under the assumptions of Proposition 3, (M, g) is q-conformally flat if and only if (M, g) is conformally flat. In fact, (5.1) is a sufficient condition for (M, g) to be 1-conformally flat.

EXAMPLE 2 (cf. [7, Example b]). Let (M_1, g_1) be an arbitrary manifold of dimension $n_1 < 2q$, and (M_2, g_2) be a flat manifold of dimension $n_2 \geqq 4q - n_1 - 1$. Then the q-th Gauss-Kronecker curvature tensor R^q of (M, g) vanishes identically. Hence, (M, g) is q-flat by Lemma 5. Accordingly, (M, g) is q-conformally flat by Theorem 2. Thus, the restriction for dimension is essential in Proposition 3.

EXAMPLE 3. Let (M_1, g_1) be the Euclidean unit $(2q+1)$-sphere and (M_2, g_2) be a Euclidean space of dimension $n_2 \geqq 2(q+1)$. Then, the product Riemannian manifold (M, g) of these two manifolds is not q-conformally flat by Proposition 3. However, as seen in Example 2, (M, g) is $(q+1)$-conformally flat. Thus, the notion "$(q+1)$-conformal flatness" is really weaker than the notion "q-conformal flatness" for Riemannian manifolds of sufficiently high dimension.

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