

## ***Certain Results on Nonoscillation and Asymptotic Nature of Delay Equations***

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The purpose of this paper is to study the second order delay equation

$$(1) \quad y''(t) + a(t)y_{\tau}(t) = f(t)$$

where the following will be assumed throughout this paper unless otherwise specified:

(i)  $a(t), f(t), \tau(t)$  are assumed to be continuous on the whole real line  $R$ ,

$$(ii) \quad \int_0^{\infty} t|f(t)|dt < \infty,$$

$$(iii) \quad y_{\tau}(t) \equiv y(t - \tau(t)),$$

(iv)  $\tau(t) \geq 0$ ,  $\tau(t)$  is bounded as  $t \rightarrow \infty$ ,  $0 \leq \tau'(t) < 1$ .

Our first result deals with the existence and asymptotic nature of the non-oscillatory solutions of equation (1). We call a function on  $C(T_0, \infty)$ ,  $T_0 > 0$  oscillatory if it has arbitrarily large zeros, otherwise we call it nonoscillatory. The term "solution" will only apply to continuous solutions (of equations under consideration) on some positive half line. The equation

$$(2) \quad y''(t) + a(t)y(t) = f(t)$$

has been very extensively studied. Cohen [2] and Bellman [1] give a very elaborate treatment on this subject. The main problem, however, is created by the presence of the delay term to the point that hitherto known techniques fail to contain equation (1) (see [4]). We shall use an adaptation of a technique given in Singh [5] to arrive at the asymptotic nature of the oscillatory solutions of equation (1). In the third section, it will be shown that this technique applies to higher order equations.

Second section deals with a Lyapunov inequality for a certain class of solutions of the equation

$$(3) \quad y''(t) + a(t)y_{\tau}(t) = 0.$$

The classical Lyapunov inequality states that if  $y(t)$  is nontrivial solution of the second order equation

$$(4) \quad y''(t) + a(t)y(t) = 0,$$

and if  $y(t)$  vanishes at least twice on an interval  $[c, d]$  then

$$(d-c) \int_c^d a^+(t) dt > 4.$$

As shown by Dahiya and Singh [3], this inequality is not true for delay equations. For example the equation

$$y''(t) - y(t - \pi) = 0$$

has

$$y(t) = \sin t$$

as a nontrivial solution in  $(0, \infty)$ . If we consider this solution over the interval  $[0, \pi]$  where it has two zeros then taking  $a(t) \equiv -1$ , we find the conclusion of the inequality is not true.

However this inequality will be shown to be true for a certain class of solutions of equation (3).

The fourth section of this paper is a commentary on the integrability of the nonoscillatory solutions of equation (1) with restrictive conditions on  $a(t)$  and  $f(t)$ .

### 1. Asymptoticity and Existence of Nonoscillatory Solutions

**THEOREM 1.** *Under the given conditions, and  $\int_{t_0}^{\infty} t|a(t)|dt < \infty$ , equation (1) has nonoscillatory solutions asymptotic to  $b_0 + b_1 t$ , where  $b_1 \neq 0$ .*

**PROOF.** From equation (1), we have

$$(5) \quad y''(t) = f(t) - a(t)y_{\tau}(t).$$

Integrating (5) between  $t_0$  and  $t$  where  $t_0 > 1$  and large enough so that

$$\int_{t_0}^{\infty} t|a(t)|dt < 1$$

and

$$\int_{t_0}^{\infty} t|f(t)|dt < 1,$$

we get

$$(6) \quad y'(t) = y'(t_0) + \int_{t_0}^t f(s)ds - \int_{t_0}^t a(s)y(s - \tau(s))ds.$$

Integrating (6) again between  $t_0$  and  $t - \tau(t)$ , ( $t - \tau(t) > t_0$  for large  $t$ ),

$$y(t - \tau(t)) = y(t_0) + y'(t_0)(t - \tau(t) - t_0) + \int_{t_0}^{t - \tau(t)} \int_{t_0}^s f(r) dr ds - \int_{t_0}^{t - \tau(t)} \int_{t_0}^s a(r) y(r - \tau(r)) dr ds$$

from which

$$|y_\tau(t)| \leq |y(t_0)| + t |y'(t_0)| \left( \frac{t - \tau(t) - t_0}{t} \right) + \int_{t_0}^{t - \tau(t)} \int_{t_0}^s |f(r)| dr ds + \int_{t_0}^{t - \tau(t)} \int_{t_0}^s |a(r)| |y_\tau(r)| dr ds.$$

Therefore, there exist positive constants  $c_0$  and  $c_1$  such that

$$(7) \quad |y_\tau(t)| \leq c_0 + c_1 t + \int_{t_0}^t \int_{t_0}^s |f(r)| dr ds + \int_{t_0}^t \int_{t_0}^s |a(r)| |y_\tau(r)| dr ds$$

since

$$0 < \frac{t - \tau(t) - t_0}{t} \leq 1$$

for large  $t$ .

(7) can now be written as

$$(8) \quad |y_\tau(t)| \leq c_0 + c_1 t + \int_{t_0}^t (t - r) |f(r)| dr + \int_{t_0}^t (t - r) |a(r)| |y_\tau(r)| dr$$

and since  $\frac{1}{t} < 1$  for  $t > 1$ , we have from above

$$|y_\tau(t)| \leq (c_0 + c_1)t + t \int_{t_0}^t |f(r)| dr + t \int_{t_0}^t |a(r)| |y_\tau(r)| dr$$

from where we get

$$(9) \quad \frac{|y_\tau(t)|}{t} \leq (c_0 + c_1 + L) + \int_{t_0}^t r |a(r)| \frac{|y_\tau(r)|}{r} dr$$

where by condition (iii)

$$\int_{t_0}^t |f(r)| dr \leq L$$

for some positive constant  $L$ . Let  $k \equiv c_0 + c_1 + L$ . By Gronwall's inequality [1, p. 107], we get from (9)

$$\frac{|y_{\tau}(t)|}{t} \leq k \cdot \exp\left(\int_{t_0}^t r|a(r)|dr\right)$$

or

$$(10) \quad \frac{|y_{\tau}(t)|}{t} \leq k_0$$

where  $k_0$  is some positive constant.

Due to (10) and conditions (ii) and (iii), it follows that the integrals on the right hand side of (6) converge and hence  $y'(t)$  tends to a limit as  $t \rightarrow \infty$ .  $t_0$  can be chosen so that  $y'(t_0) + \int_{t_0}^t f(s)ds - \int_{t_0}^t a(s)y_{\tau}(s)ds \rightarrow$  a non zero limit as  $t \rightarrow \infty$ . Hence  $y'(t) \rightarrow b_1 \neq 0$  as  $t \rightarrow \infty$  and the conclusion follows.

## 2. Lyapunov Inequality

**THEOREM 2.** *Let  $y(t)$  be an oscillatory solution of equation (3), namely*

$$y''(t) + a(t)y_{\tau}(t) = 0$$

*such that if  $t_1 < t_2$  are two maximas of  $|y(t)|$  then  $|y(t_2)| \geq |y(t_1)|$ . Let  $\alpha < \beta$  be two consecutive zeros of  $y(t)$  in an interval  $[c, d]$  then*

$$(\beta - \alpha) \int_{\alpha}^{\beta} |a(t)| dt \geq 4.$$

**PROOF.** Without any loss we can assume that  $y(t) > 0$  in  $[\alpha, \beta]$ . Let

$$M = \max y(t) \quad \text{for } t \in [\alpha, \beta]$$

Due to continuity of  $y(t)$ , this maximum is achieved at some point  $\delta \in (\alpha, \beta)$ .

Now since  $y(\alpha) = 0$ , we have

$$M = \int_{\alpha}^{\delta} y'(t) dt$$

or

$$(11) \quad M \leq \int_{\alpha}^{\delta} |y'(t)| dt.$$

Again since  $y(\beta) = 0$

$$M = - \int_{\delta}^{\beta} y'(t) dt$$

or

$$(12) \quad M \leq \int_{\delta}^{\beta} |y'(t)| dt.$$

From (11) and (12)

$$(13) \quad 2M \leq \int_{\alpha}^{\beta} |y'(t)| dt.$$

Squaring (13) we get

$$4M^2 \leq \left( \int_{\alpha}^{\beta} |y'(t)| dt \right)^2 \leq (\beta - \alpha) \int_{\alpha}^{\beta} (y'(t))^2 dt$$

by Schwartz's inequality, or

$$4M^2 \leq (\beta - \alpha) \int_{\alpha}^{\beta} y'(t) \cdot y'(t) dt.$$

Integrating by parts we get

$$(14) \quad 4M^2 \leq (\beta - \alpha) \left[ \left( y'(t)y(t) \right)_{\alpha}^{\beta} - \int_{\alpha}^{\beta} y''(t)y(t) dt \right].$$

Since  $y(\beta) = y(\alpha) = 0$ , we obtain from (14) after using equation (3),

$$4M^2 \leq (\beta - \alpha) \int_{\alpha}^{\beta} y(t)y_{\tau}(t)a(t) dt$$

or

$$(15) \quad 4M^2 \leq (\beta - \alpha) \int_{\alpha}^{\beta} y(t) |y_{\tau}(t)| |a(t)| dt.$$

Since

$$\max |y_{\tau}(t)| \leq \max y(t) \quad \text{for } t \in [\alpha, \beta],$$

it follows from (15)

$$4M^2 \leq (\beta - \alpha) \int_{\alpha}^{\beta} M \cdot M \cdot |a(t)| dt$$

or

$$4 \leq (\beta - \alpha) \int_{\alpha}^{\beta} |a(t)| dt.$$

This proves the theorem.

*Example 1.* Consider the equation

$$(16) \quad y''(t) + 2e^{\frac{\pi}{2}} y\left(t - \frac{\pi}{2}\right) = 0.$$

This has

$$y(t) = e^t \sin t$$

as an oscillatory solution satisfying the conditions of the theorem.

$$\int_0^\pi |a(t)| dt = \int_0^\pi 2e^{\frac{\pi}{2}} dt = 2\pi e^{\frac{\pi}{2}}$$

$$\text{Thus } (\pi - 0) \int_0^\pi |a(t)| dt = 2\pi^2 e^{\frac{\pi}{2}} > 4.$$

### 3. More on Asymptoticity

The method of section 2 can be extended to  $n$ th order equations of the form

$$(17) \quad y^{(n)}(t) + a(t)y_\tau(t) = f(t).$$

In fact we shall prove the following theorem:

**THEOREM 3.** *Let*

$$(18) \quad \int_0^\infty t^{n-1} |f(t)| dt < \infty$$

$$(19) \quad \int_0^\infty t^{n-1} |a(t)| dt < \infty.$$

*Then equation (17) has nonoscillatory solutions asymptotic to  $b_0 + b_1 t + b_2 t^2 + \dots + b_{n-1} t^{n-1}$ ,  $b_{n-1} \neq 0$ .*

**PROOF.** Let  $t$  be large so that  $t - \tau(t) > t_0 > 0$ .

Integrating (17)  $(n-1)$  times between  $t_0$  and  $t$  we have

$$(20) \quad y'(t) = b'_0 + b'_1 t + b'_2 t^2 + \dots + b'_{n-2} t^{n-2} \\ + \int_{t_0}^t \frac{(t-s)^{n-2}}{(n-2)!} f(s) ds - \int_{t_0}^t \frac{(t-s)^{n-2}}{(n-2)!} a(s) y_\tau(s) ds,$$

where  $b'_0, b'_1, \dots, b'_{n-2}$  are appropriate constants. Integrating (20) between  $t_0$  and  $t - \tau(t)$  we have

$$y_\tau(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \dots + \beta_{n-1} t^{n-1} + \int_{t_0}^{t-\tau(t)} \int_{t_0}^s \frac{(t-r)^{n-2}}{(n-2)!} f(r) dr ds \\ - \int_{t_0}^{t-\tau(t)} \int_{t_0}^s \frac{(t-r)^{n-2}}{(n-2)!} a(r) y_\tau(r) dr ds,$$

where  $\beta_0, \beta_1, \dots, \beta_{n-1}$  is another set of appropriate constants. Now in a manner of Theorem 2, there exist positive constants  $b_0, b_1, \dots, b_{n-1}$  such that

$$\frac{|y_\tau(t)|}{t^{n-1}} \leq b_0 + b_1 + b_2 + \dots + b_{n-1} + L_0 + \frac{1}{(n-1)!} \int_{t_0}^t (t-s)^{n-1} |a(s)| \frac{|y_\tau(s)|}{t^{n-1}} ds$$

where  $\frac{1}{(n-1)!} \int_{t_0}^t (t-s)^{n-1} |f(s)| ds < L_0$ , for some positive constant  $L_0$ .

The rest of the proof is more or less the same as of Theorem 2. The proof of Theorem 3 is now complete.

#### 4. Integrability of Nonoscillatory Solutions

In this section we consider equation (1)

$$(1) \quad y''(t) + a(t)y_\tau(t) = f(t)$$

when  $a(t) > 0$  and  $f(t) > 0$ .

**THEOREM 4.** *Suppose  $a(t) > 0$ ,  $a'(t) \geq 0$ ,  $a(t)a''(t) \leq 2(a'(t))^2$ . Further suppose that*

$$(21) \quad \int_0^\infty |f(t)| dt < \infty.$$

*Then all nonoscillatory solutions of equation (1) are integrable.*

**PROOF.** Let  $y(t)$  be a nonoscillatory solution of equation (1). Dividing equation (1) by  $a(t)$  and integrating between  $t_0$  and  $t$  we get

$$\frac{y'(t)}{a(t)} - \frac{y'(t_0)}{a(t_0)} + \int_{t_0}^t \frac{y'(s)a'(s)}{a^2(s)} ds + \int_{t_0}^t y_\tau(s) ds \leq \int_{t_0}^t \frac{|f(s)|}{a(s)} ds$$

or

$$(22) \quad \frac{y'(t)}{a(t)} - \frac{y'(t_0)}{a(t_0)} + \frac{y(t)a'(t)}{a^2(t)} - \frac{y(t_0)a'(t_0)}{a^2(t_0)} - \int_{t_0}^t y(s) \cdot \left(\frac{a'(s)}{a^2(s)}\right)' ds + \int_{t_0}^t y_\tau(s) ds < \infty$$

by condition (21). Since  $y(t)$  is a nonoscillatory solution, we can assume without any loss that  $y(t)$  is nonnegative eventually. The case when  $y(t) < 0$  eventually can be handled in a similar manner. Suppose to the contrary

$$(23) \quad \int_{t_0}^\infty y_\tau(t) dt = \infty.$$

Now

$$a'(t) \geq 0, a(t)a''(t) \leq 2(a'(t))^2 \Rightarrow \left(\frac{a'(t)}{a^2(t)}\right)' \leq 0$$

Thus on the left hand side of (22)

$$\frac{y'(t)}{a(t)} \rightarrow -\infty$$

and since  $a'(t) \geq 0$ , it follows that

(24)  $\lim_{t \rightarrow \infty} y'(t) = -\infty$  — a contradiction since this means  $y(t)$  is eventually negative; now boundedness of  $\tau(t)$  and  $0 \leq \tau'(t) < 1$  imply from  $\int^{\infty} y_{\tau}(t) dt < \infty$  that  $y(t)$  is integrable. The proof is complete.

*Remark.* We like to point out that if  $a(t) > 0$  and monotonic and  $a(t)a''(t) \leq 2(a'(t))^2$ , then the theorem will not be true if  $a'(t) < 0$  as the following example shows.

*Example 2.* The equation

$$y''(t) + \frac{1}{t}y'(t) = \frac{2}{t^3} + \frac{1}{t^2}$$

$$a(t) = \frac{1}{t}, \quad f(t) = \frac{2}{t^3} + \frac{1}{t^2}$$

$$a'(t) < 0, \quad a(t) > 0$$

$$a''(t)a(t) = \frac{2}{t^4}$$

$$2(a'(t))^2 = \frac{2}{t^4}$$

so that  $a''(t)a(t) \leq 2(a'(t))^2$ . All conditions of the theorem are satisfied except on  $a'(t)$ . This equation has

$$y(t) = \frac{1}{t}$$

as a nonoscillatory solution which is not integrable on  $[t_0, \infty]$ .

This example completes this section.

### References

- [ 1 ] R. Bellman, *Stability Theory of Differential Equations*, McGraw Hill, New York, 1953.
- [ 2 ] Donald Cohen, The asymptotic behavior of a class of nonlinear differential equations, *Proc. Amer. Math. Soc.* **18** (1967), 607–609.
- [ 3 ] R. S. Dahiya and Bhagat Singh, A Lyapunov inequality and nonoscillation theorem



for a second order nonlinear differential-difference equation, *J. Mathematical and Physical Sci.* **7** (1973), 163–170.

- [ 4 ] R. S. Dahiya and Bhagat Singh, On oscillatory behavior of even order delay equations, *J. Math. Anal. Appl.* **42** (1973), 183–190.
- [ 5 ] Y. P. Singh, The asymptotic behavior of solutions of a nonlinear nth order differential equation, *Math. J. Okayama Univ.* **15** (1971), 71–73.

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