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Codivisorial and Divisorial Modules over Completely Integrally Closed Domains (I)

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Introduction

Krull domains are, roughly speaking, congruent to Dedekind domains modulo the primes of height $\geqq 2$. This principle has brought on many results on Krull domains, which generalize the corresponding ones on Dedekind domains; for example, the fundamental theorem on ideals in a Dedekind domain can be formulated for a Krull domain as follows: For an ideal $\mathfrak{a}$ of a Krull domain A, there are primes $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ of height 1, which are uniquely determined, so that $A:(A;\mathfrak{a})=A:(A:\mathfrak{p}_1\dots\mathfrak{p}_r)$; here the operation $A:(A:*)$ corresponds to the modulus "the primes of height $\geqq 2$".

It is well known that the notion of divisorial ideals plays an important role in the theory of Krull domains; in fact, the divisorial ideals represent the quotient of the set of ideals modulo the primes of height $\geqq 2$. However it seems to the authors that the importance of the notion of divisorial modules, which generalizes that of divisorial ideals, has not been recognized yet except the case of lattices.

The purpose of this paper is to introduce the notion of divisorial modules over a completely integrally closed domain by means of codivisorial modules and also to develope a theory of them. The key theorem is Theorem 1 (§1) which is valid for a completely integrally closed domain; this is the reason why we are mainly concerned with modules over completely integrally closed domains rather than Krull domains. In §2, we study modules over Krull domains exclusively.

§1. Codivisorial and divisorial modules over a completely integrally closed domain.

1. Let A be a completely integrally closed domain and K be its quotient field. We say that a fractional ideal $\mathfrak{a}$ of A is divisorial if $\mathfrak{a}$ is an intersection of principal ideals. It is well known that, for a fractional ideal $\mathfrak{a}$, $A:(A:\mathfrak{a})$ is the intersection of principal ideals which contain $\mathfrak{a}$ and is the smallest divisorial ideal containing $\mathfrak{a}$; we denote by $\tilde{\mathfrak{a}}$ the ideal $A:(A:\mathfrak{a})$. For fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$, we say that $\mathfrak{a}$ is equivalent to $\mathfrak{b}$ if $\tilde{\mathfrak{a}}=\tilde{\mathfrak{b}}$; this relation is an equivalence relation and is denoted by $\sim$. The set of divisorial ideals can be identified with the quotient set $\boldsymbol{I}(A)/\sim$ of $\boldsymbol{I}(A)$ modulo the relation $\sim$, where $\boldsymbol{I}(A)$ is the set of fractional ideals of A. Since A is completely integrally closed, the set of non-zero divisorial ideals of A is a group under the binary operation $*$ defined as follows: $\mathfrak{a}*\mathfrak{b}=\widetilde{\mathfrak{a}\mathfrak{b}}$.

Let $\mathfrak{a}$ be an integral ideal and a be a non-zero element of A. Since A is completely integrally closed, we have $\mathfrak{a}:\mathfrak{a}=A$, which implies that $\mathfrak{a}:(\mathfrak{a}, a)=(\mathfrak{a}:\mathfrak{a})\cap(\mathfrak{a}:a)=A\cap(\mathfrak{a}:a)$; we denote this integral ideal by $\mathfrak{a}:{}_A a$. Let $\mathrm{L}(\mathfrak{a})$ denote the set $\{a\in A;\ \mathfrak{a}:{}_A a\sim A\}$. Then we have the following

THEOREM 1. $\mathrm{L}(\mathfrak{a})=\tilde{\mathfrak{a}}$.

PROOF. Let a be an element of $\tilde{\mathfrak{a}}$. If y is an element of K such that $yA\supset\mathfrak{a}:{}_A a$, then $yA\supset\mathfrak{a}$ and hence $yA\supset\tilde{\mathfrak{a}}$. So $a=ya_1$ for some element a_1 of A. Put $\mathfrak{a}_1=\mathfrak{a}/y$. Then $\mathfrak{a}_1:{}_A a_1=\mathfrak{a}_1 y:{}_A a_1 y=\mathfrak{a}:{}_A a\subset yA$. Therefore $\mathfrak{a}_1\subset yA$, and hence $\mathfrak{a}\subset y^2A$. Inductively we can easily see $\mathfrak{a}\subset y^nA$ for $n\geqq 3$. Consequently $A[y^{-1}]\subset A:\mathfrak{a}$; so y^{-1} is almost integral over A and hence contained in A. This implies that $A\subset yA$, and so $\widetilde{\mathfrak{a}:{}_A a}=A$. Hence $a\in\mathrm{L}(\mathfrak{a})$.

Conversely we show that $\mathrm{L}(\mathfrak{a})\subset\tilde{\mathfrak{a}}$ in the case $\tilde{\mathfrak{a}}\subsetneqq A$. For any element a of $A-\tilde{\mathfrak{a}}$, it is sufficient to show that $a\notin\mathrm{L}(\mathfrak{a})$, in other words, $\widetilde{\mathfrak{a}:{}_A a}\subsetneqq A$. Since $\tilde{\mathfrak{a}}:{}_A a$ is divisorial and $\tilde{\mathfrak{a}}:{}_A a\supset\mathfrak{a}:{}_A a$, $\widetilde{\mathfrak{a}:{}_A a}\subset\tilde{\mathfrak{a}}:{}_A a\subsetneqq A$ by the choice of a.

COROLLARY 1. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be fractional ideals of* A. *Then*

$$\widetilde{\mathfrak{a}\cap\mathfrak{b}}=\tilde{\mathfrak{a}}\cap\tilde{\mathfrak{b}}.$$

PROOF. We can readily see that the statement can be reduced to the case where $\mathfrak{a}$ and $\mathfrak{b}$ are integral ideals; so we may assume that $\mathfrak{a}$ and $\mathfrak{b}$ are integral ideals and $\mathfrak{b}\not\subset\mathfrak{a}$. First, suppose that both $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent to A. Let a be an element of $\mathfrak{b}-\mathfrak{a}$; then $(\mathfrak{a}\cap\mathfrak{b}):{}_A a=(\mathfrak{a}:{}_A a)\cap(\mathfrak{b}:{}_A a)=\mathfrak{a}:{}_A a$. Since $\mathfrak{a}\subset\mathfrak{a}:{}_A a$ and $\mathfrak{a}\sim A$, $\mathfrak{a}:{}_A a\sim A$; this implies that $a\in\widetilde{\mathfrak{a}\cap\mathfrak{b}}$ by Th. 1. Therefore $\mathfrak{b}\subset\widetilde{\mathfrak{a}\cap\mathfrak{b}}$ and, since $\mathfrak{b}\sim A$, $\widetilde{\mathfrak{a}\cap\mathfrak{b}}=A$. Thus we see that if $\mathfrak{a}\sim A$ and $\mathfrak{b}\sim A$, then so is $\mathfrak{a}\cap\mathfrak{b}$.

Now we proceed to the general case. Since $\widetilde{\mathfrak{a}\cap\mathfrak{b}}$ is contained in both $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{b}}$, we have only to prove that $\tilde{\mathfrak{a}}\cap\tilde{\mathfrak{b}}\subset\widetilde{\mathfrak{a}\cap\mathfrak{b}}$. Let a be an element of $\tilde{\mathfrak{a}}\cap\tilde{\mathfrak{b}}$. Then $(\mathfrak{a}\cap\mathfrak{b}):{}_A a=(\mathfrak{a}:{}_A a)\cap(\mathfrak{b}:{}_A a)$ and both $\mathfrak{a}:{}_A a$ and $\mathfrak{b}:{}_A a$ are equivalent to A; therefore $(\mathfrak{a}\cap\mathfrak{b}):{}_A a\sim A$, which implies $a\in\widetilde{\mathfrak{a}\cap\mathfrak{b}}$ by Th. 1.

COROLLARY 2. *Let* $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{c}$ *be ideals of* A *and* a *be an element of* A. *Then we have*

(i) $\widetilde{\mathfrak{a}:{}_A a}=\tilde{\mathfrak{a}}:{}_A a$.

(ii) *If* $\mathfrak{b}\sim A$ *and* $\mathfrak{c}$ *is divisorial and* $\mathfrak{a}=\mathfrak{b}\cap\mathfrak{c}$, *then* $\mathfrak{c}=\tilde{\mathfrak{a}}$.

The assertions follow immediately from Coroll. 1.

COROLLARY 3. *Let M be an A-module and put $N=\{x \in M; O(x) \sim A\}$, where $O(x)$ is the order ideal of x. Then N is a submodule of M.*

The assertion follows from Coroll. 1 to Th. 1.

2. From now on the submodule N of M given in Coroll. 3 to Th. 1 will be denoted by $\tilde{M}$. We give a definition for A-modules by means of "$\sim$".

DEFINITION 1. *Let M be an A-module. We say that M is codivisorial (resp. pseudo-null) if $\tilde{M}=0$ (resp. $M=\tilde{M}$).*

It is easy to see that the functor "$\sim$" is *left-exact additive functor* from the category of A-modules to the category of pseudo-null A-modules. The relation between *the ideal theoretical* "$\sim$" and *the module theoretical* "$\sim$" is given by the following proposition.

PROPOSITION 1. *Let $\mathfrak{a}$ be an ideal of A. Then $\widetilde{A/\mathfrak{a}}=\tilde{\mathfrak{a}}/\mathfrak{a}$. In particular, $A/\mathfrak{a}$ is codivisorial (resp. pseudo-null) if and only if $\mathfrak{a}$ is divisorial (resp. equivalent to A).*

PROOF. For an element a of A, we let $\bar{a}$ denote the canonical image of a in $A/\mathfrak{a}$. Then $\bar{a}$ belongs to $\widetilde{A/\mathfrak{a}}$ if and only if $O(\bar{a})=\mathfrak{a}:{}_A a \sim A$. Our assertion follows now from Th. 1.

PROPOSITION 2. *Let M be an A-module. Then M is codivisorial if and only if the order ideal $O(x)$ is divisorial for every element x of M.*

PROOF. It is easy to see that M is codivisorial if and only if Ax is codivisorial for every element x of M. Hence the assertion follows from Prop. 1.

The following corollary is a direct consequence of Prop. 2.

COROLLARY. *If M is a codivisorial A-module, then* $\mathrm{Ann}(M)$ *is divisorial.*

PROPOSITION 3. *Let M be an A-module. Then $M/\tilde{M}$ is codivisorial, and has the universal mapping property in the following sense: If N is a codivisorial A-module and f is a homomorphism of M to N, then there is a unique homomorphism g of $M/\tilde{M}$ to N such that $f=gp$, where p is the canonical projection of M to $M/\tilde{M}$.*

PROOF. We may assume that M is not pseudo-null. Let x be an element of $M-\tilde{M}$, and put $\mathfrak{a}=O(x)$. Then we have $\widetilde{Ax}=Ax \cap \tilde{M}=\tilde{\mathfrak{a}}x$ by Prop. 1, and hence we have $Ap(x) \cong Ax/Ax \cap \tilde{M}=Ax/\tilde{\mathfrak{a}}x \cong A/\tilde{\mathfrak{a}}$. Therefore $Ap(x)$ is codivisorial by Prop. 1. This means that $M/\tilde{M}$ is codivisorial. The last assertion follows from the fact that $f(\tilde{M}) \subset \tilde{N}=0$.

PROPOSITION 4. *Let M be an A-module and N be an essential extension of M. If M is codivisorial, then so is N.*

PROOF. Since $\tilde{N} \cap M = \tilde{M} = 0$, $\tilde{N} = 0$ by the essentiality of N over M.

COROLLARY 1. *If M is a codivisorial A-module, then so is $E(M)$, where $E(M)$ is an injective envelope of M.*

This is a direct consequence of Prop. 4.

COROLLARY 2. *$E(K/A)$ is codivisorial.*

The assertion follows immediately from the fact that K/A is codivisorial.

Here we give a criterion for a module to be pseudo-null.

PROPOSITION 5. *Let M be an A-module. Then M is pseudo-null if and only if* $\mathrm{Hom}_A(M, E(K/A)) = 0$.

PROOF. It is sufficient to show the "if" part. Suppose that M is not pseudo-null and put $N = M/\tilde{M}$. Then there is a non-zero element x of N and $\mathrm{O}(x)$ is a proper divisorial ideal by Prop. 2 and Prop. 3 and hence $A : \mathrm{O}(x) \supsetneqq A$. Let a be an element of $(A : \mathrm{O}(x)) - A$. Then $A :_A a \supset \mathrm{O}(x)$. Let f be a homomorphism of A to K/A such that $f(b) = \overline{ab}$ where $\overline{ab}$ is the class of ab in K/A. Since $\mathrm{Ker}(f) = A :_A a \supset \mathrm{O}(x)$, there is a non-zero homomorphism g of $A/\mathrm{O}(x)$ to K/A such that $f = gp$, where p is the canonical projection of A to $A/\mathrm{O}(x)$. Let i be the canonical injection of K/A to $E(K/A)$ and j be the canonical injection of $A/\mathrm{O}(x) \cong Ax$ to N. Then there is a non-zero homomorphism h of N to $E(K/A)$ such that $ig = hj$, and hence hq is a non-zero homomorphism of M to $E(K/A)$, where q is the canonical projection of M to N.

COROLLARY. *Let M and N be A-modules. If M is pseudo-null, then so is* $\mathrm{Tor}_n^A(N, M)$ *for* $n \geqq 0$.

PROOF. First we treat the case $n = 0$. Since M is pseudo-null, we have $\mathrm{Hom}_A(N \otimes_A M, E(K/A)) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, E(K/A))) = 0$; therefore $N \otimes_A M$ is pseudo-null by Prop. 5. Next, when $n \geqq 1$, we consider a projective resolution of N

$$\cdots\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0.$$

Then, since each $P_n \otimes_A N$ is pseudo-null, we can see that $\mathrm{Tor}_n^A(N, M)$ is pseudo-null for every $n \geqq 0$ by noting the fact that submodules and homomorphic images of a pseudo-null module are also pseudo-null.

PROPOSITION 6. (i) *Let $0 \to L \to M \to N$ be an exact sequence of A-modules.*

If L and N are codivisorial, then so is M.

(ii) *Let $L \to M \to N$ be an exact sequence of A-modules. If L and N are pseudo-null, then so is M.*

PROOF. The first assertion follows from the fact that the additive functor "$\sim$" is left-exact. As for the last one, in the exact sequence

$$\mathrm{Hom}_A(N, E(K/A) \longrightarrow \mathrm{Hom}_A(M, E(K/A)) \longrightarrow \mathrm{Hom}_A(L, E(K/A))$$

we have $\mathrm{Hom}_A(L, E(K/A))=0$ and $\mathrm{Hom}_A(N, E(K/A))=0$ by Prop. 5, and hence $\mathrm{Hom}_A(M, E(K/A))=0$. Therefore M is pseudo-null again by Prop. 5.

3. Now we give some definitions for modules and homomorphisms in terms of codivisorial modules and pseudo-null modules.

DEFINITION 2. *Let N be an A-module and M be a submodule of N. We say that M is divisorial in N if N/M is codivisorial. Furthermore M is said to be divisorial if it is divisorial in its injective envelope $E(M)$.*

DEFINITION 3. *Let f be a homomorphism of A-modules. Then f is called pseudo-injective (resp. pseudo-surjective) if* $\mathrm{Ker}(f)$ *(resp.* $\mathrm{Coker}(f)$*) is pseudo-null. Furthermore f is said to be pseudo-isomorphic if it is pseudo-injective and pseudo-surjective.*

We have the following corollaries to Prop. 6.

COROLLARY 1. *Let N be an A-module, M a submodule of N and L a submodule of M. If L is divisorial in M and M is divisorial in N, then L is divisorial in N. In particular, if L is divisorial in M and M is divisorial, then L is divisorial.*

PROOF. Consider the following exact sequence

$$0 \longrightarrow M/L \longrightarrow N/L \longrightarrow N/M \longrightarrow 0.$$

Since M/L and N/M are codivisorial, N/L is codivisorial by Prop. 6. Therefore L is divisorial in N. As for the last one, L is divisorial in $E(M)$ by the above case. Hence L is divisorial in $E(L)$, namely, L is divisorial.

COROLLARY 2. *Let $f: L \to M$ and $g: M \to N$ be homomorphisms of A-modules. If f and g are pseudo-injective (resp. pseudo-surjective or pseudo-isomorphic), then so is gf.*

PROOF. Consider the following exact sequences

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(gf) \xrightarrow{f} \mathrm{Ker}(g)$$

$$\mathrm{Coker}(f) \xrightarrow{g} \mathrm{Coker}(gf) \longrightarrow \mathrm{Coker}(g).$$

Then the assertion follows immediately from Prop. 6.

4. In what follows, we shall denote by $\mathscr{X}(A)$ the A-module $\oplus A/\mathfrak{a}$, where $\mathfrak{a}$ runs over the ideals of A which are equivalent to A. We give some criteria for a module to be codivisorial or divisorial in terms of $\mathscr{X}(A)$.

PROPOSITION 7. *Let M be an A-module. Then M is codivisorial if and only if* $\mathrm{Ext}_A^0(\mathscr{X}(A), M)=\mathrm{Hom}_A(\mathscr{X}(A), M)=0$.

PROOF. It is sufficient to show the "if" part. Let x be an element of $\tilde{M}$ and put $\mathfrak{a}=\mathrm{O}(x)$. Then $\mathfrak{a}\sim A$ by the definition of $\tilde{M}$, and $A/\mathfrak{a}\cong Ax$. Since $\mathrm{Hom}_A(\mathscr{X}(A), M)=0$, $\mathrm{Hom}_A(A/\mathfrak{a}, M)=0$. This means that $x=0$. Hence $\tilde{M}=0$, namely, M is codivisorial.

COROLLARY. *Let M and N be A-modules. If M is codivisorial, then so is* $\mathrm{Hom}_A(N, M)$.

PROOF. By Prop. 7, it suffices to show that

$$\mathrm{Hom}_A(\mathscr{X}(A), \mathrm{Hom}_A(N, M)) \cong \mathrm{Hom}_A(\mathscr{X}(A)\otimes_A N, M) = 0.$$

By Prop. 5, $\mathscr{X}(A)\otimes_A N$ is pseudo-null because $\mathscr{X}(A)$ is pseudo-null. Since M is codivisorial, $\mathrm{Hom}_A(\mathscr{X}(A)\otimes_A N, M)=0$.

PROPOSITION 8. *Let M be an A-module. Then the following conditions are equivalent:*

(i) *M is divisorial.*

(ii) $\mathrm{Ext}_A^1(\mathscr{X}(A), M)=0$.

(iii) *Let N be an A-module and L be a submodule of N. If N/L is pseudo-null, then the following sequence*

$$\mathrm{Hom}_A(N, M) \longrightarrow \mathrm{Hom}_A(L, M) \longrightarrow 0$$

is exact.

PROOF. (i)⇒(ii): Let $\mathfrak{a}$ be an ideal which is equivalent to A, and f be a homomorphism of $\mathfrak{a}$ to M. Then there is a homomorphism g of A to $E(M)$ such that $jf=gi$, where i (resp. j) is the canonical injection of $\mathfrak{a}$ (resp. M) to A (resp. $E(M)$). Therefore there is a homomorphism h of $A/\mathfrak{a}$ to $E(M)/M$ such that $qg=hp$, where p (resp. q) is the canonical projection of A (resp. $E(M)$) to $A/\mathfrak{a}$ (resp. $E(M)/M$). Since $A/\mathfrak{a}$ is pseudo-null and $E(M)/M$ is codivisorial, $h=0$, and hence $g(A)\subset M$. This implies that the sequence $\mathrm{Hom}_A(A, M)\rightarrow\mathrm{Hom}_A(\mathfrak{a}, M)\rightarrow 0$ is exact, i.e., $\mathrm{Ext}_A^1(A/\mathfrak{a}, M)=0$. Hence $\mathrm{Ext}_A^1(\mathscr{X}(A), M)\cong\Pi\,\mathrm{Ext}_A^1(A/\mathfrak{a}, M)=0$, where $\mathfrak{a}$ runs over the ideals of A which are equivalent to A.

(ii)⇒(iii): For an element f of $\mathrm{Hom}_A(L, M)$, by Zorn's lemma, there is a maximal submodule H of N containing L such that there is a homomorphism g of H to M which is an extension of f. Now we see that $H=N$. If otherwise, there is an element x of $N-H$. Put $\mathfrak{a}=H:{}_Ax$. Then $\mathfrak{a}$ is equivalent to A because $\mathfrak{a}\supset L:{}_Ax$ and $L:{}_Ax$ is equivalent to A by the assumption. Let t be a homomorphism of $\mathfrak{a}$ to M such that $t(a)=g(ax)$ for any element a of $\mathfrak{a}$. By the assumption, there is a homomorphism u of A to M which extends t. We define the homomorphism h of $H+Ax$ to M by $h(y+ax)=g(y)+au(1)$ for any element y of H and any element a of A; then h is an extension of g to $H+Ax$. This contradicts the maximality of H.

(iii)⇒(i): Put $N=p^{-1}(\widetilde{E(M)/M})$, where p is the canonical projection of $E(M)$ to $E(M)/M$. Then N is an essential extension of M and N/M is pseudo-null. By the assumption, there is a homomorphism f of N to M such that $fi=id._M$, where i is the canonical injection of M to N. Then f is surjective. Since i is an essential extension, f is injective and hence it is isomorphic; thus we have $M=N$. Consequently $E(M)/M\cong E(M)/M/\widetilde{E(M)/M}$, and hence $E(M)/M$ is codivisorial by Prop. 3. Therefore M is divisorial.

COROLLARY 1. *Let M be a divisorial A-module. Then M is injective if and only if* $\mathrm{Ext}_A^1(A/\mathfrak{a}, M)=0$ *for any divisorial ideal $\mathfrak{a}$ of A.*

PROOF. It suffices to show the "if" part. Let $\mathfrak{a}$ be an ideal of A. Since $\tilde{\mathfrak{a}}/\mathfrak{a}$ is pseudo-null by Prop. 1, the sequence $\mathrm{Hom}_A(\tilde{\mathfrak{a}}, M)\to\mathrm{Hom}_A(\mathfrak{a}, M)\to 0$ is exact by Prop. 8 (iii). By the assumption, $\mathrm{Hom}_A(A, M)\to\mathrm{Hom}_A(\tilde{\mathfrak{a}}, M)\to 0$ is exact. Hence the sequence $\mathrm{Hom}_A(A, M)\to\mathrm{Hom}_A(\mathfrak{a}, M)\to 0$ is exact. Therefore $\mathrm{Ext}_A^1(A/\mathfrak{a}, M)=0$ for any ideal $\mathfrak{a}$ of A. This implies that M is injective.

COROLLARY 2. *Let M be a divisorial A-module and N be an A-module. If* $\mathrm{Hom}_A(\mathrm{Tor}_1^A(\mathscr{Q}(A), N), M)=0$, *then* $\mathrm{Hom}_A(N, M)$ *is divisorial.*

PROOF. By Prop. 8 (ii), it is sufficient to show that the sequence: $\mathrm{Hom}_A(A, \mathrm{Hom}_A(N, M))\to\mathrm{Hom}_A(\mathfrak{a}, \mathrm{Hom}_A(N, M))\to 0$ is exact for any ideal $\mathfrak{a}$ of A which is equivalent to A. Since the sequence

$$0\longrightarrow \mathrm{Tor}_1^A(A/\mathfrak{a}, N)\longrightarrow \mathfrak{a}\otimes_A N\longrightarrow \mathfrak{a}N\longrightarrow 0$$

is exact, so is the sequence

$$0\longrightarrow \mathrm{Hom}_A(\mathfrak{a}N, M)\longrightarrow \mathrm{Hom}_A(\mathfrak{a}\otimes_A N, M)\longrightarrow \mathrm{Hom}_A(\mathrm{Tor}_1^A(A/\mathfrak{a}, N), M).$$

By the assumption, $\mathrm{Hom}_A(\mathrm{Tor}_1^A(A/\mathfrak{a}, N), M)=0$ and hence $\mathrm{Hom}_A(\mathfrak{a}N, M)=\mathrm{Hom}_A(\mathfrak{a}\otimes_A N, M)$. Since $\mathrm{Hom}_A(A, \mathrm{Hom}_A(N, M))\cong\mathrm{Hom}_A(N, M)$ and $\mathrm{Hom}_A(\mathfrak{a}, \mathrm{Hom}_A(N, M))\cong\mathrm{Hom}_A(\mathfrak{a}\otimes_A N, M)$, it is sufficient to show that $\mathrm{Hom}_A(N, M)\to\mathrm{Hom}_A(\mathfrak{a}N, M)\to 0$ is exact. This follows from Prop. 8 (iii) because $N/\mathfrak{a}N$ is pseudo-null.

5. Now we give a definition for A-modules in terms of the torsion functor and $\mathscr{X}(A)$.

DEFINITION 4. *Let M be an A-module. We say that M is weakly flat if* $\mathrm{Tor}_1^A(\mathscr{X}(A), M)=0$.

We have the following corollary to Prop. 8.

COROLLARY 3. *Let M be a divisorial A-module and N be an A-module. If M is codivisorial or N is weakly flat, then* $\mathrm{Hom}_A(N, M)$ *is divisorial.*

PROOF. By Coroll. 2 to Prop. 8, it is sufficient to show that $\mathrm{Hom}_A(\mathrm{Tor}_1^A\cdot(\mathscr{X}(A), N), M)=0$. This follows from the facts that $\mathrm{Tor}_1^A(\mathscr{X}(A), N)$ is pseudo-null by Coroll. to Prop. 5, and M is codivisorial or $\mathrm{Tor}_1^A(\mathscr{X}(A), N)=0$ by Def. 4.

Now we study a relation between weakly flat modules and divisorial modules. We need the following well-known lemma. Let M be an A-module and put $M^*=\mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Q}/\boldsymbol{Z})$. Then M^* is called *the character module* of M. We have

LEMMA 1. (i) *The sequence of A-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow N^* \longrightarrow M^* \longrightarrow L^* \longrightarrow 0$$

is exact.

(ii) *Let M be an A-module. Then M is flat if and only if M^* is injective.*

See J. J. ROTMAN [4], Lemma 3.34 and Th. 3.35.

PROPOSITION 9. *Let M be an A-module. Then M is weakly flat if and only if M^* is divisorial.*

PROOF. Let $\mathfrak{a}$ be an ideal of A which is equivalent to A. Consider the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(A/\mathfrak{a}, M) \longrightarrow \mathfrak{a} \otimes_A M \longrightarrow \mathfrak{a}M \longrightarrow 0 .$$

We have the exact sequences by Lemma 1 (i)

$$0 \longrightarrow (\mathfrak{a}M)^* \longrightarrow (\mathfrak{a} \otimes_A M)^* \longrightarrow \mathrm{Tor}_1^A(A/\mathfrak{a}, M)^* \longrightarrow 0$$

$$0 \longrightarrow (M/\mathfrak{a}M)^* \longrightarrow M^* \longrightarrow (\mathfrak{a}M)^* \longrightarrow 0 .$$

Since $(\mathfrak{a}\otimes_A M)^* \cong \mathrm{Hom}_A(\mathfrak{a}, M^*)$ and $M^* \to (\mathfrak{a}M)^* \to 0$ is exact, $\mathrm{Hom}_A(A, M^*) \to \mathrm{Hom}_A(\mathfrak{a}, M^*) \to 0$ is exact if and only if $(\mathfrak{a}M)^* \to (\mathfrak{a}\otimes_A M)^* \to 0$ is exact, in other words, $\mathrm{Tor}_1^A(A/\mathfrak{a}, M)^* = 0$, i.e., $\mathrm{Tor}_1^A(A/\mathfrak{a}, M) = 0$ by Lemma 1 (i). Hence M^* is divisorial if and only if $\mathrm{Tor}_1^A(\mathscr{Z}(A), M) = 0$ by Prop. 8 (ii).

COROLLARY 1. *Let M be a weakly flat A-module. Then M is flat if and only if* $\mathrm{Tor}_A(A/\mathfrak{a}, M) = 0$ *for any divisorial ideal* $\mathfrak{a}$ *of* A.

PROOF. Since M^* is divisorial by Prop. 9, M^* is injective if and only if $\mathrm{Ext}_1^A(A/\mathfrak{a}, M) = 0$ for any divisorial ideal $\mathfrak{a}$ of A by Coroll. 1 to Prop. 8. By the argument of the proof of Prop. 9, $\mathrm{Ext}_1^A(A/\mathfrak{a}, M^*) = 0$ if and only if $\mathrm{Tor}_1^A(A/\mathfrak{a}, M) = 0$. Therefore $\mathrm{Tor}_1^A(A/\mathfrak{a}, M) = 0$ for any divisorial ideal $\mathfrak{a}$ of A if and only if M^* is injective, namely, M is flat by Lemma 1 (ii).

COROLLARY 2. *Let M be a weakly flat A-module and N be a pseudo-null A-module. Then* $\mathrm{Tor}_1^A(N, M) = 0$.

PROOF. Let $0 \to H \to L \to N \to 0$ be an exact sequence of A-modules where L is projective. Since M^* is divisorial by Prop. 9, the sequence $\mathrm{Hom}_A(L, M^*) \to \mathrm{Hom}_A(H, M^*) \to 0$ is exact by Prop. 8 (iii). Since $\mathrm{Hom}_A(L, M^*) \cong (L\otimes_A M)^*$ and $\mathrm{Hom}_A(H, M^*) \cong (H\otimes_A M)^*$, the sequence $(L\otimes_A M)^* \to (H\otimes_A M)^* \to 0$ is exact and hence $0 \to H\otimes_A M \to L\otimes_A M$ is exact by Lemma 1 (i). Therefore $\mathrm{Tor}_1^A(N, M) = 0$ because L is projective.

REMARK 1. It is easy to see that for an A-module M, M^* is codivisorial if and only if $\mathrm{Tor}_0^A(\mathscr{Z}(A), M) = \mathscr{Z}(A)\otimes_A M = 0$.

6. Here we study some homological properties of the functor "$\sim$". Put $\mathscr{N}(M) = \tilde{M}$ for any A-module M and consider the right derived functors $\{\mathrm{R}^n\mathscr{N}\}$ $n \geqq 0$ of $\mathscr{N}$. Then $\mathrm{R}^0\mathscr{N} = \mathscr{N}$ because the additive functor $\mathscr{N}$ is left exact. We have the well-known homological result.

PROPOSITION 10. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of A-modules. Then the sequence*

$$0 \to \tilde{L} \to \tilde{M} \to \tilde{N} \to \mathrm{R}^1\mathscr{N}(L) \to \mathrm{R}^1\mathscr{N}(M) \to \mathrm{R}^1\mathscr{N}(N) \to \mathrm{R}^2\mathscr{N}(L) \to \cdots\cdots$$

$$\cdots\cdots \to \mathrm{R}^n\mathscr{N}(L) \to \mathrm{R}^n\mathscr{N}(M) \to \mathrm{R}^n\mathscr{N}(N) \to \mathrm{R}^{n+1}\mathscr{N}(L) \to \cdots\cdots$$

is exact.

PROPOSITION 11. *Let M be an A-module.*
(i) *If M is divisorial, then* $\mathrm{R}^1\mathscr{N}(M) = 0$.
(ii) *If M is codivisorial and* $\mathrm{R}^1\mathscr{N}(M) = 0$, *then M is divisorial.*

PROOF. Consider a minimal injective resolution of M

$$0 \longrightarrow M \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} \cdots\cdots \longrightarrow E_n \xrightarrow{d_n} E_{n+1} \longrightarrow \cdots\cdots .$$

(i): Since M is divisorial, $\widetilde{E_0/M}=0$ and hence $\widetilde{E_1}=\widetilde{E(E_0/M)}=0$ by Coroll. 1 to Prop. 4. In particular, $\mathrm{R}^1\mathscr{N}(M)\cong(\mathrm{Ker}(d_1)\cap\tilde{E}_1)/d_0(\tilde{E}_0)=0$.

(ii): Since $\mathrm{R}^1\mathscr{N}(M)=0$, $\mathrm{Ker}(d_1)\cap\tilde{E}_1=d_0(\tilde{E}_0)$. By the assumption that M is codivisorial, $\tilde{E}_0=0$ by Coroll. 1 to Prop. 4. Therefore $\mathrm{Ker}(d_1)\cap\tilde{E}_1=0$ and hence $\tilde{E}_1=0$ because $E_1=E(\mathrm{Ker}(d_1))$. This implies that M is divisorial.

COROLLARY 1. *Let $0\to L\to M\to N\to 0$ be an exact sequence of A-modules.*

(i) *If L is divisorial, then the sequence $0\to\tilde{L}\to\tilde{M}\to\tilde{N}\to 0$ is exact. In particular, if L is divisorial and M is codivisorial, then L is divisorial in M.*

(ii) *If M is codivisorial and both L and N are divisorial, then M is divisorial.*

The assertion follows immediately from Prop. 10 and 11.

COROLLARY 2. *Let $\mathfrak{a}$ be an ideal of A. Then $\mathfrak{a}$ is a divisorial ideal if and only if it is a divisorial module.*

PROOF. If $\mathfrak{a}$ is a divisorial ideal, then $\mathfrak{a}$ is a divisorial module by Prop. 1 and Coroll. 1 to Prop. 6, because A is a divisorial module. Conversely if $\mathfrak{a}$ is a divisorial module, then it is divisorial in A by Coroll. 1 to Prop. 11, i.e., it is a divisorial ideal by Prop. 1.

7. Here we give some definitions for A-modules. Let N be an A-module and M be a submodule of N. Put $D_A(M;N)=p^{-1}(\widetilde{N/M})$, where p is the canonical projection of N to N/M. Then we say that $D_A(M;N)$ is *the divisorial envelope* of M in N. Furthermore we denote $D_A(M;E(M))$ by $D_A(M)$ and it is called *a divisorial envelope* of M. Simply we denote $D_A(M;N)$ (resp. $D_A(M)$) by $D(M;N)$ (resp. $D(M)$) unless there is fear of confusion. The following proposition is a characterization of $D(M;N)$.

PROPOSITION 12. *Let N be an A-module and M be a submodule of N. Then $D(M;N)$ is divisorial in N and if L is a submodule of N which contains M and is divisorial in N, then $D(M;N)\subset L$. This means that $D(M;N)$ is the smallest submodule of N containing M which is divisorial in N.*

PROOF. Since $N/D(M;N)\cong N/M/\widetilde{N/M}$, $D(M;N)$ is divisorial in N by Prop. 3. Consider the exact sequence $0\to L/M\to N/M\to N/L\to 0$. By the assumption, N/L is codivisorial, so $\widetilde{N/M}\subset L/M$ by Prop. 3. Hence $D(M;N)\subset L$.

COROLLARY 1. *Let M be an A-module and L be an essential extension of M. If L is divisorial, then $D(M)\subset L$. Therefore M is divisorial if and only* *if* $M=D(M)$.

The assertion follows immediately from Prop. 12.

COROLLARY 2. *Let M be an A-module and N be an essential extension of M. If N is divisorial, then $D(M;N)=D(M)$. In particular, if $\mathfrak{a}$ is an ideal of A, then $D(\mathfrak{a})=D(\mathfrak{a};A)=\tilde{\mathfrak{a}}$.*

PROOF. It is easy to see that $D(M)\subset D(M;N)$ by Coroll. 1 to Prop. 6 and Coroll. 1 to Prop. 12. Conversely, since $D(M)$ is divisorial by the definition of a divisorial envelope, $D(M;N)\subset D(M)$ by Prop. 12. The last assertion follows from Prop. 1.

COROLLARY 3. Let Λ be a set.

(i) *Let N be an A-module and $\{M_\lambda; \lambda\in\Lambda\}$ be a family of submodules of N. Put $M=\cap M_\lambda$ where λ runs over the elements of Λ. Then $D(M;N)\subset \cap D(M_\lambda;N)$.*

(ii) *Let N_λ be an A-module and M_λ be a submodule of N_λ for any element λ of Λ. Put $M=\oplus M_\lambda$ and $N=\oplus N_\lambda$, where λ runs over the elements of Λ. Then $D(M;N)=\oplus D(M_\lambda: N_\lambda)$.*

PROOF. It can be easily seen by Coroll. 1 to Th. 1 that the functor $\mathscr{N}$ commutes with any direct sum (not necessarily finite) of A-modules. Hence the second assertion follows from the definition of $D(M;N)$. As for the first one, since $N/\cap D(M_\lambda;N)$ can be considered as a submodule of $\Pi N/D(M_\lambda;N)$, it suffices to show that $\Pi N/D(M_\lambda;N)$ is codivisorial. By virtue of Prop. 2, this follows from the facts that each $N/D(M_\lambda;N)$ is codivisorial and any intersection of divisorial ideals is also divisorial.

8. Now we give some definitions for A-modules and A-homomorphisms in terms of pseudo-isomorphisms and essential extensions.

DEFINITION 5. *Let f be an A-homomorphism. Then f is said to be essentially isomorphic if it is an essential extension and is pseudo-isomorphic.*

Let M be an A-module and put $\mathscr{F}_M=\{(N,f)$; N is an A-module and f is a homomorphism of M to N and is essentially isomorphic.$\}$. Let (L,g) and (N,f) be elements of $\mathscr{F}_M$. Then we say that (L,g) *is equivalent to* (N,f), and denote it by $(L,g)=(N,f)$, if there is an isomorphism h of L to N such that $f=gh$; we say that (N,f) *is larger than* (L,g), and denote it by $(N,f)>(L,g)$ or $(L,g)<(N,f)$, if there is a homomorphism j of L to N such that $f=gj$. It is easy to see that j is necessarily an essential isomorphism. Furthermore we say that (L,g) *is pseudo-equivalent to* (N,g), and denote it by $(L,g)\sim(N,f)$ if $(L,g)>(N,f)$ and $(L,g)<(N,f)$. An element of $\mathscr{F}_M$ is said to be *an essentially isomorphic exten-*

sion of M. Let (N,f) be an element of $\mathscr{F}_M$. (N,f) is called *a maximal essentially isomorphic extension* (resp. *a pseudo-maximal essentially isomorphic extension*) *of* M, simply *maximal* (resp. *pseudo-maximal*), *in* $\mathscr{F}_M$, if there is no element (L,g) of $\mathscr{F}_M$ such that $(L,g)>(N,f)$ and (L,g) is not equivalent (resp. pseudo-equivalent) to (N,f). Now we give alternative criterion of a divisorial envelope in terms of maximal essentially isomorphic extensions and pseudo-maximal essentially isomorphic extensions.

PROPOSITION 13. *Let M be an A-module and (N,f) be an element of $\mathscr{F}_M$. Then the following conditions are equivalent*:

(i) *(N,f) is pseudo-maximal in $\mathscr{F}_M$.*

(ii) *(N,f) is pseudo-equivalent to $(D(M),i)$ where i is the canonical injection of M to $D(M)$.*

(iii) *(N,f) is maximal in $\mathscr{F}_M$.*

(iv) *(N,f) is equivalent to $(D(M),i)$.*

PROOF. First we show that $(D(M),i)$ is maximal in $\mathscr{F}_M$. Let (L,g) be an element of $\mathscr{F}_M$ which is larger than $(D(M),i)$. Then we may consider $D(M)$ as a submodule of L and $L/D(M)$ is pseudo-null. Since $E(M)=E(L)$ and $D(M)$ is divisorial in $E(M)$, $L/D(M)$ is codivisorial and hence $L/D(M)=0$. This implies that $(D(M),i)$ is maximal in $\mathscr{F}_M$. Now we show that $(D(M),i)$ is larger than every element of $\mathscr{F}_M$. Let (N,f) be an element of $\mathscr{F}_M$. Then we may consider N as a submodule of $E(M)$. Since f is pseudo-isomorphic, N/M is pseudo-null and hence $N/M\subset\widetilde{E(M)}/M$. This implies that $(N,f)<(D(M),i)$. Therefore our assertions follows immediately from the above facts.

COROLLARY. *Let N be a codivisorial A-module and M be a submodule. Then $(D(M;N),j)$ is an essentially isomorphic extension of M, where j is the canonical injection of M to $D(M;N)$. In particular, if N is codivisorial and divisorial, then $D(M;N)=D(M)$.*

PROOF. It can be easily seen that N is an essential extension of M if N is codivisorial and N/M is pseudo-null. Therefore if N is codivisorial, then $D(M;N)$ is an essentially isomorphic extension of M. The last assertion follows from Coroll. 1 to Prop. 6 and Prop. 13.

9. We have already known in Prop. 4 that an essential extension preserves the codivisoriality. However this is not true in general for the case of the pseudo-nullity (cf. the remark after the next theorem). Here we shall study some properties of a ring A, which preserves the pseudo-nullity.

THEOREM 2. *The following conditions for the ring A are equivalent*:

(i) *If E is injective, then so is $\tilde{E}$.*

(ii) $E(\tilde{M})=\widetilde{E(M)}$ *for any module* M.

(iii) *Let* N *be an essential extension of* M. *If* M *is pseudo-null, then so is* N.

(iv) *Let* $\mathfrak{a}$ *be an ideal of* A *which is not equivalent to* A. *Then* $\mathfrak{a}:{}_A a$ *is divisorial for some element* a *of* $A-\tilde{\mathfrak{a}}$.

(v) *If* M *is not pseudo-null, then it has a non-zero codivisorial submodule.*

(vi) *Let* $\mathfrak{a}$ *be an ideal of* A. *Then there is an ideal* $\mathfrak{b}$ *of* A *which is equivalent to* A *such that* $\mathfrak{a}=\tilde{\mathfrak{a}}\cap\mathfrak{b}$.

PROOF. (v)$\Rightarrow$(iii): Suppose that N is not pseudo-null. Then there is a non-zero codivisorial submodule L of N. Since N is an essential extension of M, $L\cap M$ is a non-zero codivisorial submodule of M. Hence M is not pseudo-null.

(iii)$\Rightarrow$(ii): Since $E(\tilde{M})$ is an essential extension of $\tilde{M}$, $E(\tilde{M})$ is pseudo-null and hence $E(\tilde{M})\subset\widetilde{E(M)}$ by the maximality of $\widetilde{E(M)}$. $\widetilde{E(M)}$ is an essential extension $\tilde{M}$ because $\widetilde{E(M)}\cap M=\tilde{M}$. Therefore $E(\tilde{M})=\widetilde{E(M)}$.

(ii)$\Rightarrow$(i): Let E be an injective A-module. Then $E(\tilde{E})=\widetilde{E(E)}=\tilde{E}$. Hence $\tilde{E}$ is injective.

(i)$\Rightarrow$(vi): Put $E=E(A/\mathfrak{a})$. Then $E=E_d\oplus\tilde{E}$ for some codivisorial submodule E_d of E. Let x be the class of 1 in A/a and put $x=x_1+x_2$ where $x_1\in E_d$ and $x_2\in\tilde{E}$. Then $\mathfrak{a}=\mathrm{O}(x)=\mathrm{O}(x_1)\cap\mathrm{O}(x_2)$. $\mathrm{O}(x_1)$ is divisorial by Prop. 2, and $\mathrm{O}(x_2)$ is equivalent to A because $\tilde{E}$ is pseudo-null. The assertion follows from Coroll. 2 to Th. 1.

(vi)$\Rightarrow$(iv): Let $\mathfrak{a}$ be an ideal of A which is not equivalent to A. Then there is an ideal $\mathfrak{b}$ of A, which is equivalent to A, such that $\mathfrak{a}=\tilde{\mathfrak{a}}\cap\mathfrak{b}$. Since $\tilde{\mathfrak{a}}\subsetneqq A$, $\tilde{\mathfrak{a}}\not\supset\mathfrak{b}$. Take an element a of $\mathfrak{b}-\tilde{\mathfrak{a}}$. Then $\tilde{\mathfrak{a}}:{}_A a=\mathfrak{a}:{}_A a$ and hence $\mathfrak{a}:{}_A a$ is divisorial.

(iv)$\Rightarrow$(v): Suppose that $M\neq\tilde{M}$ and take an element x of $M-\tilde{M}$. Then $\mathrm{O}(x)$ is not equivalent to A. Therefore $\mathrm{O}(x):{}_A a$ is divisorial for some element a of $A-\widetilde{\mathrm{O}(x)}$. In other words, $\mathrm{O}(ax)$ is a proper divisorial ideal. Hence Aax is a non-zero codivisorial submodule of M by Prop. 1.

REMARK 2. A completely integrally closed domain does not necessarily satisfy the conditions of Th. 2.

EXAMPLE 1. Let V be a valuation ring whose value group is $\boldsymbol{R}$, and v be its valuation. Then (iv) of Th. 2 is false for an ideal $\{a\in V; v(a)>2\}$ because any divisorial ideal of V is principal. Furthermore an V-module M is injective if and only if it is divisorial and divisible by Coroll. 1 to Prop. 8.

Here we give a definition for a ring.

DEFINITION 6. *A domain* A *is said to be strongly integrally closed if* A *is* *completely integrally closed and satisfies the conditions of* Th. 2.

Now we shall study modules over a strongly integrally closed domain. We understand that, in the rest of this section, A is always a strongly integrally closed domain, unless otherwise specified. We have

PROPOSITION 14. *Let M be an A-module. Then $E(M/\tilde{M})\cong E(M)/E(\tilde{M})$.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{M} & \longrightarrow & M & \longrightarrow & M/\tilde{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \widetilde{E(M)} & \longrightarrow & E(M) & \longrightarrow & E(M)/\widetilde{E(M)} & \longrightarrow & 0. \end{array}$$

Then there is a homomorphism f of $M/\tilde{M}$ to $E(M)/\widetilde{E(M)}$ such that $fp=qi$, where p (resp. q) is the canonical projection of M (resp. $E(M)$) to $M/\tilde{M}$ (resp. $E(M)/\widetilde{E(M)}$) and i is the canonical injection of M to $E(M)$. It is sufficient, by (ii) of Th. 2, to show that f is an essential extension. Since $\widetilde{E(M)}\cap M=\tilde{M}$, f is injective; thus we may consider $M/\tilde{M}$ as a submodule of $E(M)/\widetilde{E(M)}$. We denote $q(x)$ by $\bar{x}$ for any element x of $E(M)$. Suppose that $A\bar{x}\cap(M/\tilde{M})=0$. Then for any element y of $Ax\cap M$, we have $\bar{y}=0$ and hence $Ax\cap M\subset\tilde{M}$. This implies that $Ax\cap M$ is pseudo-null, and so Ax is pseudo-null, by (iii) of Th. 2, because Ax is an essential extension of $Ax\cap M$. Therefore $Ax\subset\widetilde{E(M)}$; thus $A\bar{x}=0$. This completes the proof.

COROLLARY. *Let M be an A-module. Then we have $E(M)\cong E(\tilde{M})\oplus E(M/\tilde{M})$.*

The assertion follows immediately from Prop. 14.

PROPOSITION 15. *Let M be a pseudo-null A-module. Then $\mathrm{R}^n\mathcal{N}(M)=0$ for $n\geqq 1$.*

PROOF. We consider a minimal injective resolution of M

$$0\longrightarrow M\longrightarrow E_0\longrightarrow E_1\longrightarrow E_2\cdots\cdots\longrightarrow E_n\longrightarrow E_{n+1}\longrightarrow\cdots\cdots.$$

Then each E_n is pseudo-null by Th. 2. Therefore $E_n=\tilde{E}_n$ and hence $\mathrm{R}^n\mathcal{N}(M)=0$ for $n\geqq 1$.

COROLLARY 1. *Let M be an A-module. Then $\mathrm{R}^n\mathcal{N}(M)\cong\mathrm{R}^n\mathcal{N}(M/\tilde{M})$ for $n\geqq 1$. In particular, $\mathrm{R}^1\mathcal{N}(M)=0$ if and only if $M/\tilde{M}$ is divisorial.*

PROOF. The first assertion follows from Prop. 10 and 15. As for the last one, the proof follows easily from Prop. 11.

COROLLARY 2. *Let M be an A-module. Then $D(M)\cong D(\tilde{M})\oplus D(M/\tilde{M})$ and $D(\tilde{M})=E(\tilde{M})$.*

PROOF. Consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{M} \arrow[r] \arrow[d] & E(\tilde{M}) \arrow[r] \arrow[d] & E(\tilde{M})/\tilde{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & E(M) \arrow[r] \arrow[d] & E(M)/M \arrow[r] \arrow[d] & 0 \\
0 & M/\tilde{M} \arrow[l] \arrow[r] \arrow[d] & E(M/\tilde{M}) \arrow[r] \arrow[d] & E(M/\tilde{M})/M/\tilde{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Each column is exact by Prop. 14. Since $E(\tilde{M})/\tilde{M}$ is pseudo-null, the sequence

$$0\longrightarrow E(\tilde{M})/\tilde{M}\longrightarrow \widetilde{E(M)/M}\longrightarrow \widetilde{E(M/\tilde{M})/M/\tilde{M}}\longrightarrow 0$$

is exact by Coroll. 1 to Prop. 15. Hence it is easy to see the sequence:

$$0\longrightarrow D(\tilde{M})\longrightarrow D(M)\longrightarrow D(M/\tilde{M})\longrightarrow 0$$

is exact. Since $D(\tilde{M})=p^{-1}(\widetilde{E(\tilde{M})/\tilde{M}})=p^{-1}(E(\tilde{M})/\tilde{M})=E(\tilde{M})$, where p is the canonical projection of $E(\tilde{M})$ to $E(\tilde{M})/\tilde{M}$, $D(\tilde{M})$ is injective. Therefore $D(M)\cong D(\tilde{M})\oplus D(M/\tilde{M})$.

REMARK 3. Let A be a completely integrally closed domain. Then the following conditions are equivalent:

(i) A is a Dedekind domain.
(ii) Every ideal of A is divisorial.
(iii) Every A-module is divisorial.
(iv) Every A-module is codivisorial.
(v) A has no proper ideal which is equivalent to A.
(vi) $\mathscr{X}(A)=0$.
(vii) There is no non-zero pseudo-null A-module.
(viii) Pseudo-isomorphisms are isomorphisms.
(ix) An essential isomorphism is an isomorphism.
(x) $D(M)=M$ for any A-module M.
(xi) The functor "$\sim$" is an exact functor.

REMARK 4. Let A be a completely integrally closed domain. Then any inductive limit of a direct system of pseudo-null A-modules is also pseudo-null.

§2. Codivisorial and divisorial modules over a Krull domain

1. In this section, we shall study modules over a Krull domain. We recall that a completely integrally closed domain is a Krull domain if and only if the set of divisorial ideals satisfies the maximum condition. We understand that, throughout this section, A is always a Krull domain. Let M be an A-lattice. Then we say that M is a divisorial lattice if it is reflexive, i.e., $M=A:(A:M)=\mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$. $A:(A:M)$ is the smallest divisorial lattice in $M\otimes_A K=E(M)$, where K is the quotient field of A, which contains M. By R. M. FOSSUM [3], Chap. I, Coroll. 5.5. (e), M is a divisorial lattice if and only if it is a divisorial module. Hence $D(M)=A:(A:M)$. Let K be the quotient field of A and $\mathrm{Ht}_1(A)$ be the set of non-zero divisorial prime ideals, i.e., prime ideals of height 1 in A. Then we have

PROPOSITION 16. *Let M be a codivisorial A-module. Then*
(i) $\mathrm{Ass}(M)\subset\mathrm{Ht}_1(A)\cup\{0\}$.
(ii) *$M=0$ if and only if* $\mathrm{Ass}(M)=\phi$.
(iii) *If M is finitely generated, then* $\mathrm{Ass}(M)$ *is a finite set.*

PROOF. The first two assertions follow from Prop. 2. As for the last one, let $\{x_1,\dots, x_n\}$ be a system of generators of M. We use the induction on n. In the case that $n=1$, we put $\mathfrak{a}=\mathrm{O}(x_1)$. Then $A/\mathfrak{a}\cong M$ and $\mathfrak{a}$ is a divisorial ideal of A by Prop. 1. We denote by $\mathfrak{p}^{(n)}$ the n-th symbolic power of $\mathfrak{p}$, where $\mathfrak{p}$ is an element of $\mathrm{Ht}_1(A)$. Since it is easy to see the assertion if $\mathfrak{a}=0$, we may assume that $\mathfrak{a}\neq 0$. By R. M. FOSSUM [3], Chap. I, Coroll. 5.7., there are finite elements, say $\mathfrak{p}_1,\dots, \mathfrak{p}_r$, of $\mathrm{Ht}_1(A)$ such that $\mathfrak{a}=\mathfrak{p}^{(n_1)}\cap\cdots\cap\mathfrak{p}^{(n_r)}$ for some positive integers $n_1,\dots, n_r$. Then it can be easily seen that $\mathrm{Ass}(M)=\mathrm{Ass}(A/\mathfrak{a})=\{\mathfrak{p}_1,\dots, \mathfrak{p}_r\}$. In general case, put $N=D(Ax_1; M)$. Then N is divisorial in M, namely, M/N is codivisorial. It is easy to see that N is an essential extension of Ax_1, and hence $\mathrm{Ass}(Ax_1)=\mathrm{Ass}(N)$, and so $\mathrm{Ass}(N)$ is a finite set by the above case. Consider the exact sequence

$$0\longrightarrow N\longrightarrow M\longrightarrow M/N\longrightarrow 0.$$

Then we have $\mathrm{Ass}(M)\subset\mathrm{Ass}(N)\cup\mathrm{Ass}(M/N)$. Since M/N is generated by $(n-1)$ elements, $\mathrm{Ass}(M/N)$ is a finite set by inductive hypothesis. Therefore $\mathrm{Ass}(M)$ is a finite set.

LEMMA 2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero fractional ideals of A. Then we have* $\widetilde{\mathfrak{a}:\mathfrak{b}}=\tilde{\mathfrak{a}}:\tilde{\mathfrak{b}}$.

PROOF. $\widetilde{\mathfrak{a}:\mathfrak{b}} = A:[A:(\mathfrak{a}:\mathfrak{b})] \subset A:[(A:\mathfrak{a})\mathfrak{b}] = (A:\mathfrak{b}):(A:\mathfrak{a}) = \{A:[A:(A:\mathfrak{b})]\}:(A:\mathfrak{a}) = A:\{(A:\mathfrak{a})[A:(A:\mathfrak{b})]\} = [A:(A:\mathfrak{a})]:[A:(A:\mathfrak{b})] = \tilde{\mathfrak{a}}:\tilde{\mathfrak{b}}$. Hence $\widetilde{\mathfrak{a}:\mathfrak{b}} \subset \tilde{\mathfrak{a}}:\tilde{\mathfrak{b}}$.

Conversely we shall show that $\tilde{\mathfrak{a}}:\tilde{\mathfrak{b}} \subset \widetilde{\mathfrak{a}:\mathfrak{b}}$. We recall that for any non-zero fractional ideals $\mathfrak{c}$, $\mathfrak{d}$ of A, $\tilde{\mathfrak{c}} = \cap\, \mathfrak{c}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the elements of $\mathrm{Ht}_1(A)$, and $(\mathfrak{d}:\mathfrak{c})_{\mathfrak{p}} = \mathfrak{d}_{\mathfrak{p}}:\mathfrak{c}_{\mathfrak{p}}$ if $\mathfrak{d}$ is divisorial, where $\mathfrak{p}$ is an element of $\mathrm{Ht}_1(A)$.[1] Therefore $\widetilde{\tilde{\mathfrak{a}}(A:\mathfrak{b})}_{\mathfrak{p}} = \tilde{\mathfrak{a}}_{\mathfrak{p}}(A:\mathfrak{b})_{\mathfrak{p}} = \tilde{\mathfrak{a}}_{\mathfrak{p}}(A_{\mathfrak{p}}:\tilde{\mathfrak{b}}_{\mathfrak{p}}) = \tilde{\mathfrak{a}}_{\mathfrak{p}}:\tilde{\mathfrak{b}}_{\mathfrak{p}} = (\tilde{\mathfrak{a}}:\tilde{\mathfrak{b}})_{\mathfrak{p}}$, for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(A)$, because $A_{\mathfrak{p}}$ is a principal valuation ring. Hence $\tilde{\mathfrak{a}}:\tilde{\mathfrak{b}} = \widetilde{\tilde{\mathfrak{a}}(A:\mathfrak{b})}$. It is easy to see that $\widetilde{\tilde{\mathfrak{a}}(A:\mathfrak{b})} = \widetilde{\mathfrak{a}(A:\mathfrak{b})} \subset \widetilde{\mathfrak{a}:\mathfrak{b}}$, and so $\tilde{\mathfrak{a}}:\tilde{\mathfrak{b}} \subset \widetilde{\mathfrak{a}:\mathfrak{b}}$.

Now we have a corollary to Prop. 16.

COROLLARY. *Let $\mathfrak{a}$ be an ideal of A. Then* $\mathrm{Ass}(A/\tilde{\mathfrak{a}}) \subset \mathrm{Ass}(A/\mathfrak{a})$.

PROOF. We may assume that $\mathrm{Ass}(A/\tilde{\mathfrak{a}}) \neq \phi$, i.e., $\tilde{\mathfrak{a}} \subsetneqq A$ by Prop. 1 and 16. Let $\mathfrak{p}$ be an element of $\mathrm{Ass}(A/\tilde{\mathfrak{a}})$. Then $\mathfrak{p} = \tilde{\mathfrak{a}}:_A a$ for some element a of $A - \tilde{\mathfrak{a}}$. Put $\mathfrak{b} = \mathfrak{a}:_A a$. Then $\tilde{\mathfrak{b}} = \mathfrak{p}$ by Coroll. 2 to Th. 1. Since $\mathfrak{b}:_A \mathfrak{p} = \mathfrak{b}:\mathfrak{p}$, $\widetilde{\mathfrak{b}:_A \mathfrak{p}} = \mathfrak{p}:\mathfrak{p}$ by Lemma 2, because $\mathfrak{p}$ is divisorial. Therefore $\mathfrak{b}:_A \mathfrak{p}$ is equivalent to A because A is completely integrally closed. Since $\mathfrak{p}$ is divisorial, $\mathfrak{b}:_A \mathfrak{p} \not\subset \mathfrak{p}$. Take an element b of $(\mathfrak{b}:_A \mathfrak{p}) - \mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{p}:_A b = \mathfrak{b}:_A b = \mathfrak{a}:_A ab$. This implies that $\mathrm{Ass}(A/\tilde{\mathfrak{a}}) \subset \mathrm{Ass}(A/\mathfrak{a})$.

PROPOSITION 17. *A Krull domain is strongly integrally closed.*

PROOF. (iv) of Theorem 2 follows from Coroll. 2 to Th. 1 and Coroll. to Prop. 16.

PROPOSITION 18.[2] *Let M be an A-module. Then M is pseudo-null if and only if $M_{\mathfrak{p}} = 0$ for any element $\mathfrak{p}$ of* $\mathrm{Ht}_1(A)$.

The assertion follows immediately from Prop. 3 and 16.

COROLLARY. *Let f be an A-homomorphism. Then f is pseudo-injective (resp. pseudo-surjective or pseudo-isomorphic) if and only if $f_{\mathfrak{p}}$ is injective (resp. surjective or isomorphic) for any element $\mathfrak{p}$ of* $\mathrm{Ht}_1(A)$.

The assertion follows immediately from Prop. 18.

2. Now we shall study relations between the A-modules and the $S^{-1}A$-modules, where S is a multiplicatively closed subset of A. We understand that,

1) See R. M. FOSSUM [3], Lemma 5.3., Coroll. 5.4., and Coroll. 5.5. (b), (c).

2) As for the case that A is a noetherian Krull domain, i.e., A is a noetherian normal domain, and M is a finitely generated, this proposition was given by N. BOURBAKI. See N. BOURBAKI, [2], § 4, n°4, Prop. 9 and Def. 2.

in the rest of this section, S is always a multiplicatively closed subset of A. We recall that $S^{-1}A$ is a Krull domain and $S^{-1}A=\cap A_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over the elements of the set $\{\mathfrak{p}\in \mathrm{Ht}_1(A); \mathfrak{p}\cap S=\phi\}$, and that $S^{-1}\mathfrak{a}$ is a divisorial $S^{-1}A$-ideal (resp. is equivalent to $S^{-1}A$) for any divisorial ideal $\mathfrak{a}$ of A (resp. for any ideal $\mathfrak{a}$ of A which is equivalent to A). (See R. M. FOSSUM, [3] Prop. 1.8. and Coroll. 5.5. (b), (c))

PROPOSITION 19. *If M is a codivisorial A-module, then $S^{-1}M$ is also a codivisorial $S^{-1}A$-module.*

The assertion follows immediately from Prop. 2.

COROLLARY 1. *Let M be an A-module. Then $S^{-1}\tilde{M}=\widetilde{S^{-1}M}$. In particular, $S^{-1}(M/\tilde{M})\cong S^{-1}M/\widetilde{S^{-1}M}$.*

PROOF. Consider the exact sequence

$$0\longrightarrow S^{-1}\tilde{M}\longrightarrow S^{-1}M\longrightarrow S^{-1}(M/\tilde{M})\longrightarrow 0.$$

Then we have $S^{-1}\tilde{M}\supset\widetilde{S^{-1}M}$ by Prop. 3 and 19. Conversely it can be easily seen that $S^{-1}\tilde{M}\subset\widetilde{S^{-1}M}$ by the fact that $S^{-1}\mathfrak{a}\sim S^{-1}A$ for any ideal $\mathfrak{a}$ of A which is equivalent to A.

COROLLARY 2. *Let N be an A-module and M be a submodule of N. Then $S^{-1}D_A(M;N)=D_{S^{-1}A}(S^{-1}M;S^{-1}N)$.*

The assertion follows immediately from Coroll. 1 to Prop. 19.

PROPOSITION 20. *Let M be a codivisorial A-module and N be an essential extension of M. Then $S^{-1}N$ is an essential extension of $S^{-1}M$.*

PROOF. Since N is codivisorial by Coroll. 1 to Prop. 4, $S^{-1}N$ is a codivisorial $S^{-1}A$-module by Prop. 19. We may assume that $S^{-1}N\neq 0$. Let x be a non-zero element of $S^{-1}N$ and put $x=y/s$, where $y\in N$ and $s\in S$. Set $\mathscr{F}=\{\mathfrak{p}\in \mathrm{Ass}_A(Ay); \mathfrak{p}\cap S=\phi\}$. Then $\mathrm{Ass}_{S^{-1}A}(S^{-1}Ax)=\{S^{-1}\mathfrak{p}; \mathfrak{p}\in\mathscr{F}\}$. By Prop. 16, $\mathrm{Ass}_{S^{-1}A}(S^{-1}Ax)\neq\phi$. And hence $\mathscr{F}\neq\phi$. Let $\mathfrak{p}$ be an element of $\mathscr{F}$ and take an element a of A such that $\mathfrak{p}=\mathrm{O}(ay)$. By the assumption, $M\cap Aay\neq 0$. Take a non-zero element z of $M\cap Aay$ and put $z=bay$ for some element b of A. Then $z/1\neq 0$ in $S^{-1}M$. If otherwise, there is an element t of S such that $tz=0$ and hence $tbay=0$, i.e., $tb\in\mathrm{O}(ay)=\mathfrak{p}$. Since $\mathfrak{p}\cap S=\phi$, $b\in\mathfrak{p}$. In other words, $z=bay=0$. This contradicts the choice of z. Hence $S^{-1}Ax\cap S^{-1}M\neq 0$. This implies that $S^{-1}N$ is an essential extension of $S^{-1}M$.

COROLLARY. *Let N be a codivisorial A-module and M be a submodule of N. Then N is an essential extension of M if and only if $N_{\mathfrak{p}}$ is an essential* *extension of* $M_{\mathfrak{p}}$ *for any element* $\mathfrak{p}$ *of* $\mathrm{Ht}_1(A)$.

PROOF. By Prop. 20, it is sufficient to show the "if" part. Let L be a submodule of N such that $L \cap M=0$. Then $L_{\mathfrak{p}} \cap M_{\mathfrak{p}}=(L \cap M)_{\mathfrak{p}}=0$ and hence $L_{\mathfrak{p}}=0$ for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(A)$ because $N_{\mathfrak{p}}$ is an essential extension of $M_{\mathfrak{p}}$. Therefore L is pseudo-null by Prop. 18. Hence $L=0$ because L is codivisorial. This implies that N is an essential extension of M.

3. Now we shall study codivisorial injective A-modules.
The following results are due to I. BECK.

PROPOSITION 21. *Let E be a non-zero codivisorial and injective A-module. Then E is indecomposable if and only if it is isomorphic to K or $E(A/\mathfrak{p}) \cong K/A_{\mathfrak{p}}$ for some element $\mathfrak{p}$ of* $\mathrm{Ht}_1(A)$.

See I. BECK [1], Prop. 2.3, 2.5 and 2.6.

PROPOSITION 22. *Let E be a non-zero codivisorial A-module. Then E is injective if and only if it is isomorphic to a direct sum of indecomposable codivisorial injective modules.*

See I. BECK [1], Prop. 2.4, 2.5 and 2.6.

THEOREM 3. *A direct sum of codivisorial injective A-modules is injective.*

See I. BECK [1], Prop. 2.7.

COROLLARY 1. *Let M be a codivisorial A-module. Then we have $S^{-1}E_A(M) \cong E_{S^{-1}A}(S^{-1}M)$.*

PROOF. $S^{-1}E_A(M)$ is an essential extension of $S^{-1}M$ by Prop. 20. Hence it is sufficient to show that $S^{-1}E_A(M)$ is injective. This follows immediately from Prop. 21 and 22 and Th. 3.

Let M be an A-lattice. We recall that $S^{-1}(A:(A:M))=S^{-1}A:(S^{-1}A:S^{-1}M)$, namely, $S^{-1}D_A(M)=D_{S^{-1}A}(S^{-1}M)$. The following corollary to Th. 3 is the generalization of the case of lattices.

COROLLARY 2. *Let M be a codivisorial A-module. Then we have $S^{-1}D_A(M) \cong D_{S^{-1}A}(S^{-1}M)$.*

The assertion follows immediately from Coroll. 2 to Prop. 19 and Coroll. 1 to Th. 3.

COROLLARY 3. *Let M be a codivisorial and divisorial A-module. Then $S^{-1}M$ is a codivisorial and divisorial $S^{-1}A$-module.*

The assertion follows from Coroll. 1 to Prop. 12 and Coroll. 2 to Th. 3.

COROLLARY 4. *Let Λ be a set and M_λ be a codivisorial A-module for any $\lambda \in \Lambda$. Then we have $E(\oplus M_\lambda) \cong \oplus E(M_\lambda)$ and $D(\oplus M_\lambda) \cong \oplus D(M_\lambda)$, where λ runs over the elements of Λ.*

PROOF. It can be easily seen that $\oplus E(M_\lambda)$ is an essential extension of $\oplus M_\lambda$. Since each $E(M_\lambda)$ is codivisorial and injective by Coroll. 1 to Prop. 4, $\oplus E(M_\lambda)$ is injective by Th. 3, and hence $E(\oplus M_\lambda) \cong \oplus E(M_\lambda)$. The last assertion follows immediately from Coroll. 3 to Prop. 12.

PROPOSITION 23. *Let M be a divisorial $S^{-1}A$-module. Then M is a divisorial A-module. In particular, $S^{-1}A$ is a divisorial A-module.*

PROOF. Let $\mathfrak{a}$ be an ideal of A which is equivalent to A and f be a homomorphism of $\mathfrak{a}$ to M. Then $S^{-1}\mathfrak{a}$ is equivalent to $S^{-1}A$ and hence, by the assumption, there is a homomorphism g of $S^{-1}A$ to M such that $S^{-1}f = g\,S^{-1}i$ by Prop. 8, where i is the canonical injection of $\mathfrak{a}$ to A. Let j be the canonical injection of A to $S^{-1}A$. Then $f = gji$. This implies that $\mathrm{Ext}^1_A(A/\mathfrak{a}, M) = 0$ for any ideal $\mathfrak{a}$ of A which is equivalent to A, i.e., $\mathrm{Ext}^1_A(\mathscr{Z}(A), M) = 0$. Therefore M is divisorial by Prop. 8.

COROLLARY. *Let M be an $S^{-1}A$-module.*

(i) *M is a codivisorial $S^{-1}A$-module if and only if it is a codivisorial A-module.*

(ii) *M is a codivisorial and divisorial $S^{-1}A$-module if and only if it is a codivisorial and divisorial A-module.*

In particular, any $A_\mathfrak{p}$-module is a codivisorial and divisorial A-module, where $\mathfrak{p}$ is an element of $\mathrm{Ht}_1(A)$.

PROOF. The first assertion follows from Prop. 19 and 2 and the fact that $\mathfrak{a} \cap A$ is a divisorial ideal of A for any divisorial ideal $\mathfrak{a}$ of $S^{-1}A$. As for the last one, the assertion follows immediately from Coroll. 3 to Prop. 22 and Prop. 23.

THEOREM 4. *Let M be a codivisorial A-module.*

(i) *If M is torsion free, then $D(M) \cong \cap M_\mathfrak{p}$, where $\mathfrak{p}$ runs over the elements of* $\mathrm{Ht}_1(A)$.

(ii)[3] *If M is a torsion module, then $D(M) \cong \oplus M_\mathfrak{p}$, where $\mathfrak{p}$ runs over the elements of* $\mathrm{Ass}(M)$.

In particular, if M is a coirreducible torsion module, then $D(M) \cong M_\mathfrak{p}$, where $\mathrm{Ass}(M) = \{\mathfrak{p}\}$.

3) This assertion was essentially obtained by I. BECK. See I. BECK [1], Prop. 1.9.

PROOF. (i): Since M is torsion free, $E(M)\cong E(M_{\mathfrak{p}})\cong M\otimes_A K$. Put $N=\cap M_{\mathfrak{p}}$. Then N is divisorial by Coroll. 3 to Prop. 12, because for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(A)$, $M_{\mathfrak{p}}$ is a divisorial A-module by Coroll. to Prop. 23. Therefore it suffices to show that (N, i) is an essentially isomorphic extension of M by Prop. 13, where i is the canonical injection of M to N. It is easy to see that i is an essential extension. Let $\mathfrak{q}$ be an element of $\mathrm{Ht}_1(A)$. Then we have $N_{\mathfrak{q}}\subset M_{\mathfrak{q}}\cap(\cap M_{\mathfrak{p}}\otimes_A A_{\mathfrak{q}})$, where $\mathfrak{p}$ runs over the elements of $\mathrm{Ht}_1(A)-\{\mathfrak{q}\}$. Since $A_{\mathfrak{p}}\otimes_A A_{\mathfrak{q}}=K$, $N_{\mathfrak{q}}\subset M_{\mathfrak{q}}$ and hence $N_{\mathfrak{q}}=M_{\mathfrak{q}}$. This implies that i is pseudo-isomorphic by Coroll. to Prop. 18.

(ii): Let $i_{\mathfrak{p}}$ be the canonical homomorphism of M to $M_{\mathfrak{p}}$ and put $N=\oplus M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the elements of $\mathrm{Ass}(M)$. Let $i=\Pi i_{\mathfrak{p}}$ be the canonical homomorphism of M to $\Pi M_{\mathfrak{p}}$. Then it can be easily seen that $i(M)\subset N$ by Prop. 16 (iii). Since N is divisorial by Coroll. to Prop. 23 and Coroll. 4 to Th. 3, it is sufficient to show that i is essentially isomorphic by Prop. 13. This follows from Coroll. to Prop. 18 and Coroll. to Prop. 20.

COROLLARY 1.[4)] *Let B be a subring of K which contains A. Then B is a divisorial A-module if and only if it is a subintersection of A, i.e., $B=\cap A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the elements of a subset of* $\mathrm{Ht}_1(A)$.

The assertion follows immediately from Th. 4 (i) and the fact that $A_{\mathfrak{p}}$ is a principal valuation ring for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(A)$.

The following corollary to Th. 4 is originally due to I. BECK. See I. BECK [1], Prop. 2.9.

COROLLARY 2. $E(K/A)\cong D(K/A)\cong\oplus E(A/\mathfrak{p})$, *where $\mathfrak{p}$ runs over the elements of* $\mathrm{Ht}_1(A)$.

PROOF. By Th. 4 (ii), $D(K/A)\cong\oplus(K/A)_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the elements of $\mathrm{Ass}(K/A)=\mathrm{Ht}_1(A)$. Since $E(A/\mathfrak{p})\cong K/A_{\mathfrak{p}}\cong(K/A)_{\mathfrak{p}}$, $D(K/A)$ is injective by Th. 3. Hence the assertion follows immediately from the above fact.

4. Now we shall give criteria of injective modules and flat modules over a Krull domain, and study some properties of the derived functors of $\mathcal{N}$ and the extension functors.

PROPOSITION 24. *Let M be an A-module. Then M is injective if and only if it is divisorial and divisible.*

PROOF. It is sufficient to show the "if" part. By Coroll. 2 to Prop. 15, we may assume that M is codivisorial, divisorial and divisible. Since $E_A(M)/M$

4) This result was obtained by J. AHMED in a quite different way. See J. AHMED, Modules sur les anneaux de Krull. C.R. Acad. Sei. Paris Ser. A–B, 276 (1973).

is codivisorial, it suffices to show that $E_A(M)/M$ is pseudo-null, i.e., $(E_A(M)/M)_{\mathfrak{p}} = 0$ for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(A)$ by Prop. 18. By Coroll. 1 to Th. 3, $E_A(M)_{\mathfrak{p}} \cong E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ because M is codivisorial. Since M is divisible, $M_{\mathfrak{p}}$ is divisible $A_{\mathfrak{p}}$-module, i.e., injective $A_{\mathfrak{p}}$-module because $A_{\mathfrak{p}}$ is a principal valuation ring. Hence $M_{\mathfrak{p}} = E_A(M)_{\mathfrak{p}}$, and so $(E_A(M)/M)_{\mathfrak{p}} = 0$.

COROLLARY 1. *Let E be an injective A-module and M be a torsion free A-module. If* $\mathrm{Hom}_A(\mathrm{Tor}_A^1(\mathscr{Z}(A), M), E) = 0$, *then* $\mathrm{Hom}_A(M, E)$ *is injective.*

PROOF. Since E is injective and M is torsion free, $\mathrm{Hom}_A(M, E)$ is divisible. Hence the assertion follows from Coroll. 2 to Prop. 8 and Prop. 24.

COROLLARY 2. *Let E be an injective A-module and M be a torsion free A-module. If E is codivisorial or M is flat, then* $\mathrm{Hom}_A(M, E)$ *is injective.*

The assertion follows from Coroll. 3 to Prop. 8 and Coroll. 1 to Prop. 24.

PROPOSITION 25. *Let M be an A-module. Then M is flat if and only if it is weakly flat and torsion free.*

PROOF. It is sufficient to show the "if" part. By Prop. 9, M^* is divisorial. Since M is torsion free, M^* is divisible by Lemma 1 (i). Hence M^* is injective by Prop. 24. Therefore M is flat by Lemma 1 (ii).

The following proposition and its corollary 1 are originally due to I. BECK. See I. BECK [1], Prop. 3.7 and 3.8.

PROPOSITION 26. *Let M be an A-module (which is not necessarily codivisorial). Then $E_A(M)_{\mathfrak{p}} \cong E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for any element $\mathfrak{p}$ of* $\mathrm{Ht}_1(A)$.

PROOF. By Coroll. to Prop. 14 and Prop. 17, $E_A(M) \cong E_A(\tilde{M}) \oplus E_A(M/\tilde{M})$. Let $\mathfrak{p}$ be an element of $\mathrm{Ht}_1(A)$. Then $E_A(M)_{\mathfrak{p}} \cong E_A(\tilde{M})_{\mathfrak{p}} \oplus E_A(M/\tilde{M})_{\mathfrak{p}}$. Since $E_A(\tilde{M})$ is pseudo-null by Th. 2 and Prop. 17, $E_A(\tilde{M})_{\mathfrak{p}} = 0$ by Prop. 18. Since $M/\tilde{M}$ is codivisorial by Prop. 3, $E_A(M/\tilde{M})_{\mathfrak{p}} = E_{A_{\mathfrak{p}}}((M/\tilde{M})_{\mathfrak{p}})$ by Coroll. 1 to Th. 3. By Prop. 18, $\tilde{M}_{\mathfrak{p}} = 0$ and hence $M_{\mathfrak{p}} \cong (M/\tilde{M})_{\mathfrak{p}}$. Therefore $E_A(M)_{\mathfrak{p}} \cong E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.

COROLLARY 1. *Let M be an A-module and let*

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots\cdots \longrightarrow E_n \longrightarrow E_{n+1} \longrightarrow \cdots\cdots$$

be a minimal injective resolution of M. Then E_n is pseudo-null for $n \geqq 2$.

The assertion follows from Prop. 26 and the fact that $A_{\mathfrak{p}}$ is a principal valuation ring and so $\mathrm{gldim}(A) = 1$.

COROLLARY 2. *Let M be an A-module. Then $\mathrm{R}^n\mathscr{N}(M) = 0$ for $n \geqq 3$.*

The assertion follows immediately from Coroll. 1 to Prop. 26 and the definition of $\mathrm{R}^n\mathcal{N}$.

PROPOSITION 27. *Let M be an A-module. Then M is a codivisorial and divisorial module of injective dimension at most one if and only if* $\mathrm{R}^n\mathcal{N}(M)=0$ *for* $n=0, 1, 2$.

PROOF. It is sufficient to show the "if" part. Since $\tilde{M}=\mathrm{R}^0\mathcal{N}(M)=0$, M is codivisorial. By Prop. 11 (ii), M is divisorial because $\mathrm{R}^1\mathcal{N}(M)=0$ and M is codivisorial. Since $\mathrm{R}^1\mathcal{N}(E(M)/M)\cong\mathrm{R}^2\mathcal{N}(M)=0$, $E(M)/M$ is divisorial by Prop. 11 (ii) because M is divisorial, i.e., $E(M)/M$ is codivisorial. Therefore $E(M)/M$ is injective by Prop. 24 because $E(M)/M$ is divisible. This implies that $\operatorname{inj\,dim}_A(M)\leqq 1$.

PROPOSITION 28. *Let M be an A-module and $n\geqq 2$ be an integer. Then* $\operatorname{inj\,dim}_A(M)\leqq n$ *if and only if* $\mathrm{Ext}_A^n(\mathscr{Z}(A), M)=0$.

PROOF. It is sufficient to show the "if" part. Let

$$0\longrightarrow M\longrightarrow E_0\xrightarrow{d_0}E_1\xrightarrow{d_1}\cdots\cdots E_n\xrightarrow{d_n}E_{n+1}\longrightarrow\cdots\cdots$$

be a minimal injective resolution of M and put $I_n=\mathrm{Ker}(d_n)$. Then $\mathrm{Ext}_A^1(\mathscr{Z}(A), I_{n-1})\cong\mathrm{Ext}_A^n(\mathscr{Z}(A), M)=0$. Therefore I_{n-1} is divisorial by Prop. 8, i.e., I_n is codivisorial. Since I_n is pseudo-null by Coroll. 1 to Prop. 26, $I_n=0$. Hence $E_n=E(I_n)=0$, i.e., $\operatorname{inj\,dim}_A(M)\leqq n$.

PROPOSITION 29. *Let $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda\in\Lambda}$ be a direct system of codivisorial A-modules and $\{M, f_\lambda\}_{\lambda\in\Lambda}$ its inductive limit. Then M is codivisorial.*

PROOF. Let x be an element of M. Then there is an element λ_0 of Λ and an element x_{λ_0} of M_{λ_0} such that $f_{\lambda_0}(x_{\lambda_0})=x$. Put $\Lambda'=\{\lambda\in\Lambda; \lambda\geqq\lambda_0\}$, and put $x_\lambda=f_{\lambda_0,\lambda}(x_{\lambda_0})$ for any $\lambda\in\Lambda'$. Then Λ' is cofinal to Λ. It is easy to see that for elements λ and λ' of Λ', $\mathrm{O}(x_\lambda)\supset\mathrm{O}(x_{\lambda'})$ if $\lambda\geqq\lambda'$, and $\varinjlim_{\lambda\in\Lambda'}\mathrm{O}(x_\lambda)=\cup_{\lambda\in\Lambda'}\mathrm{O}(x_\lambda)=\mathrm{O}(x)$. Since each $\mathrm{O}(x_\lambda)$ is a divisorial ideal of A by Prop. 2, $\mathrm{O}(x)=\mathrm{O}(x_\lambda)$ for some element λ of Λ' because any ascending chain of divisorial ideals of A breaks off at finite step. Hence $\mathrm{O}(x)$ is divisorial. Therefore M is codivisorial by Prop. 2.

REMARK 5. Let M be a codivisorial A-module. Then $\mathrm{Hom}_A(N, M)$ is codivisorial for any A-module N, by Coroll. to Prop. 7. But $\mathrm{Ext}_A^n(N, M)$ is not necessarily codivisorial for $n\geqq 1$.

EXAMPLE 2. Let A be a regular local ring of Krull $\dim(A)=n\geqq 2$. Then $\operatorname{inj\,dim}(A)=n$ and hence $\mathrm{Ext}_A^m(N, A)\neq 0$ for any m with $2\leqq m\leqq n$, and for some finitely generated A-module N. Since $\mathrm{Ext}_A^m(N, A)_\mathfrak{p}\cong\mathrm{Ext}_{A_\mathfrak{p}}^m(N_\mathfrak{p}, A_\mathfrak{p})$ for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(A)$, $\mathrm{Ext}^m_A(N, A)=0$ because $A_\mathfrak{p}$ is a principal valuation ring. Hence $\mathrm{Ext}^m_A(N, M)$ is pseudo-null by Prop. 18. Therefore $\mathrm{Ext}^m_A(N, A)$ is not codivisorial.

REMARK 6. Example 1 shows that Prop. 24 can not characterize a Krull domain.

REMARK 7. Let A be a completely integrally closed domain. Then the following conditions are equivalent:

(i) A is a Krull domain.

(ii) Any direct sum of codivisorial and injective A-modules is injective.

(iii) Any direct sum of codivisorial and divisorial A-modules is divisorial, and any codivisorial, divisorial and divisible A-module is injective.

To see this, we make use of a method similar to one due to H. BASS who gave the well-known criterion for a ring to be noetherian in terms of injective modules by it.

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