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A Note on Finite Groups which Act Freely on Closed Surfaces

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§1. Introduction

The purpose of this note is to study what kind of finite groups can act freely on closed surfaces.

Let X be a given closed surface. Suppose that a finite group G acts freely on X. Then, it is well known that the orbit space $Y=X/G$ is also a closed surface and there is a normal covering

$$p: X \longrightarrow Y = X/G, \tag{1.1}$$

that is, the image $p_*\pi_1(X)$ of the induced monomorphism $p_*: \pi_1(X)\to\pi_1(Y)$ of the fundamental groups is a normal subgroup of $\pi_1(Y)$ and $\pi_1(Y)/p_*\pi_1(X)\cong G$. Therefore,

$$\chi(X) = \chi(Y)g \qquad (g \geqq 1), \tag{1.2}$$

where χ means the Euler characteristic and $g=\# G$ is the order of G. Also, we see easily the following.

(1.3) In the case that X is orientable, Y is orientable if and only if the action of G preserves the orientation of X.

Conversely, suppose that

(1.4.1) Y is a closed surface satisfying (1.2) for some integer $g\geqq 1$, and N is a normal subgroup of $\pi_1(Y)$ of index g, and

(1.4.2) N is isomorphic to $\pi_1(X)$.

Then, we have a normal covering $p': X'\to Y$ with the covering group

$$G = \pi_1(Y)/N, \tag{1.5}$$

and the closed surface X' satisfies $\chi(X')=\chi(X)$, $\pi_1(X')\cong\pi_1(X)$ by (1.4.1–2). Therefore, we see that X' is homeomorphic to X by the classification theorem of closed surfaces, and so G acts freely on X.

Thus, we have the following

THEOREM 1.6. *Let X be a closed surface. Then, a finite group G acts freely on X if and only if G is given by* (1.5) *under the assumptions* (1.4.1–2).

Furthermore, in the case that X is orientable, G acts on X preserving or *reversing the orientation according as* Y *in* (1.4.1–2) *is orientable or non-orientable.*

Here, we say that G acts on X reversing the orientation, if some element of G reverses the orientation of X.

Since two orientable closed surfaces are homeomorphic if their Euler characteristics coincide, we have the following

COROLLARY 1.7. *Let* X *be a closed orientable surface. Then a finite group* G *acts freely on* X *preserving the orientation if and only if* G *is given by* (1.5), *under the assumption* (1.4.1) *with the additional assumption that* Y *is orientable.*

Using these results and the elementary group theory, we obtain the following results, some of which may be known.

THEOREM 1.8. *The finite group which acts freely on the Klein bottle* U_2 *is the cyclic group*

$$Z_n \text{ of order } n = 2(2s+1) \quad \text{or} \quad 2s+1 \qquad (s \geqq 0),$$

and then the orbit surface is always homeomorphic to U_2.

THEOREM 1.9. *The finite group which acts freely on the torus* T_1 *reversing the orientation is one of the following groups*:

$$\{x, y; xyx = y, x^s = y^{2t}\}, \ \{x, y; xyx = y, x^s = y^{2t} = 1\}, \quad (s, t \geqq 1),$$

and then the orbit surface is always U_2.

Here, the notation

$$\{x_1,\dots, x_n; R_1,\dots, R_k\}$$

means the group with generators $x_1,\dots, x_n$ and defining relations $R_1,\dots, R_k$.

The groups in this theorem for $t=1$, $s>1$ are the generalized quaternion groups and the dihedral groups.

THEOREM 1.10. *The finite abelian group, which acts freely on the orientable closed surface* T_m $(m\geqq 0)$ *of genus* m *preserving the orientation, is the direct sum*

$$Z_{s_1}\oplus\cdots\oplus Z_{s_{2n}}, \quad n \geqq 0, \qquad m-1 = (n-1)s_1\dots s_{2n},$$

of the cyclic groups Z_{s_i} *of order* $s_i\geqq 1$, *and then the orbit surface is* T_n. *Also, any finite group which acts freely on* T_m $(0\leqq m\leqq 6)$ *preserving the orientation is an abelian group.*

Concerning this theorem, P. A. Smith [2, Ch. 15] calculated the number of certain classes of free abelian actions on 2-manifolds. Also, we notice in § 3 the results for $m \geqq 7$.

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§ 2. Proofs of Theorems 1.8 and 1.9.

In this section, we consider the Klein bottle U_2 or the torus T_1.

LEMMA 2.1. *If a finite group G acts freely on U_2 or on T_1 reversing the orientation, then the orbit surface is homeomorphic to U_2.*

PROOF. Since $\chi(U_2)=\chi(T_1)=0$, the result follows immediately from (1.2) and (1.3). *q.e.d.*

As is well known, the fundamental group of U_2 is given by

(2.2) $$\pi_1(U_2) = H = \{x, y; xyx = y\}.$$

We see easily the following

LEMMA 2.3. *In the group H, the relation $x^n y^m = y^m x^{(-1)^m n}$ holds for any integers m and n. Moreover, any element of H can be represented uniquely by the form $x^n y^m$ for some integers m and n.*

LEMMA 2.4. *Let a map f: $H \to H$ be given by*

$$f(x) = x^i y^j, \qquad f(y) = x^k y^l.$$

Then, f is a monomorphism such that Imf *is a normal subgroup of H of finite index if and only if $i = \pm 1$ or ± 2, $j=0$ and l is an odd integer.*

PROOF. If f is a homomorphism, the relation $xyx=y$ implies $2j+l=l$ and $i+(-1)^j k+(-1)^{j+l} i=k$ by the above lemma, which show that $j=0$ and l is odd. Then, we have

$$f(x^a y^b) = x^{ai+k} y^{bl} \ (b\colon \text{odd}), \qquad = x^{ai} y^{bl} \ (b\colon \text{even}).$$

Therefore, we have $i \neq 0$, if f is monomorphic. Furthermore, if Imf is a normal subgroup of H, then $xf(y)x^{-1}$, $yf(y)y^{-1} \in \mathrm{Im} f$ and so $ai=2$ and $a'i=-2k$ for some a, a'. These show that $i = \pm 1$ or ± 2, and the necessity is proved.

The sufficiency is proved easily. *q.e.d.*

PROOF OF THEOREM 1.8. By Theorem 1.6 and Lemma 2.1, the finite group G which acts freely on U_2 is given as the quotient group $H/\mathrm{Im} f$, where f: $H \to H$

is a monomorphism of the above lemma. Therefore,

$$G = H/\mathrm{Im}f = \{x, y; xyx = y, x^i = x^k y^{2s+1} = 1\},$$

where $i=1$ or 2 and $s \geqq 0$. Then, we can easily verify that this group G is given by

$$G = \begin{cases} Z_{2s+1}, \text{ generated by } y, & \text{if } i = 1, \\ Z_{2(2s+1)}, \text{ generated by } y, & \text{if } i = 2 \text{ and } k \text{ is odd}, \\ Z_{2(2s+1)}, \text{ generated by } xy, & \text{if } i = 2 \text{ and } k \text{ is even}. \end{cases}$$

Thus, the proof of Theorem 1.8 is completed. *q.e.d.*

Now, we can verify the following lemma by the routine calculations by using Lemma 2.3.

LEMMA 2.5. *Let a map $f: \pi_1(T_1) = Z \oplus Z \to \pi_1(U_2) = H$ be given by*

$$f(a) = x^i y^j, \quad f(b) = x^k y^l, \textit{ for the generators } a, b \textit{ of } Z \oplus Z.$$

Then, f is a monomorphism such that Imf *is a normal subgroup of H with finite index if and only if j and l are even integers, $d = il - jk \neq 0$ and d is a divisor of $il + jk$, $2ij$ and $2kl$.*

PROOF. If f is a homomorphism, the equality $f(a)f(b) = f(b)f(a)$ implies

$$i(1-(-1)^l) = k(1-(-1)^j).$$

If j is odd and l is even, then this equality implies $k=0$. Then, we see that $f(b) = y^l$ and $f(2a) = y^{2j}$. By the same way $f(a) = y^j$ and $f(2b) = y^{2l}$, if j is even and l is odd. Also $f(2a) = y^{2j}$ and $f(2b) = y^{2l}$, if j and l are odd. Therefore, f is not monomorphic for these cases.

For even j and l, we have $f(a^m b^n) = x^{im+kn} y^{jm+ln}$. Therefore

$$d = il - jk \neq 0,$$

if f is monomorphic. Furthermore, if Imf is a normal subgroup, then $y(\mathrm{Im}f)y^{-1} \subset \mathrm{Im}f$ and so $x^{-i}y^j, x^{-k}y^l \in \mathrm{Im}f$. These show that d is a divisor of $il+jk$, $2ij$ and $2kl$ as desired, and the necessity is proved.

The sufficiency is proved easily. *q.e.d.*

PROOF OF THEOREM 1.9. By Theorem 1.6, Lemmas 2.1 and 2.5, a finite group G which acts freely on T_1 reversing the orientation is given by

$$(*) \qquad G = \{x, y; xyx = y, y^{2j} = x^i, y^{2l} = x^k\},$$

where $d = il - jk \neq 0$ and d is a divisor of $il+jk$, $2ij$ and $2kl$.

Now, we prove that G of $(*)$ is one of the groups of Theorem 1.9. We notice the following fact, which is easily seen.

(2.6) The relations $xyx=y$ and $y^{2j}=x^i$ imply $y^{4j}=1=x^{2i}$.

(I) The case $k=0$. (The proof for $i=0$ is similar.) Then, $il \neq 0$, and assume $j \neq 0$, since the desired result is trivial if $j=0$. The defining relations of $(*)$ are

$$xyx = y, \quad y^{2j} = x^i, \quad y^{2l} = 1,$$

where $ijl \neq 0$ and l is a divisor of $2j$. Therefore, these relations are reduced to

$$\begin{cases} xyx = y,\ y^{2l} = x^i = 1, & \text{if } 2j/l \text{ is even}, \\ xyx = y,\ y^l = x^i, & \text{if } 2j/l \text{ is odd}, \end{cases}$$

and l is even for the latter case, as desired.

(II) The case $ik \neq 0$. Assume the defining relations of $(*)$ are

$$xyx = y, \quad y^{2j} = x^i, \quad y^{2l} = x^k \qquad (i > 0, k > 0).$$

Then, we have $x^s = y^{2a}$ for some integer a, where $s=$g.c.m. $\{i, k\}$, and so

$$y^{2j} = \begin{cases} 1, & \text{if } i/s \text{ is even}, \\ x^s, & \text{if } i/s \text{ is odd}, \end{cases} \qquad y^{2l} = \begin{cases} 1, & \text{if } k/s \text{ is even}, \\ x^s, & \text{if } k/s \text{ is odd}. \end{cases}$$

Hence, in the case where i/s and k/s are odd, the defining relations of $(*)$ are reduced to

$$xyx = y, \quad y^{2j} = x^s, \quad y^{2(j-l)} = 1.$$

If $j=0$ or $j=l$, the desired result is trivial. If $i \neq 0$, l, then we have $y^{2t}=1$ for $t=$g.c.m. $\{2|j|, |j-l|\}$, and these relations are reduced to

$$\begin{cases} xyx = y, \quad y^{2t} = x^s = 1, & \text{if } 2|j|/t \text{ is even}, \\ xyx = y, \quad y^{2t} = x^s, & \text{if } 2|j|/t \text{ is odd}, \end{cases}$$

and t is even for the latter case. Therefore, we have the desired result. We can prove by the same way the result for i/s or k/s even.

Thus, we have proved that G of $(*)$ is one of the groups of Theorem 1.9. The converse is seen in the above proof (I). *q.e.d*

§3. Proof of Theorem 1.10

In this section, we use the following notations.

F_{2n} = the free group generated by $x_1,\dots, x_n, y_1,\dots, y_n$,

$r_n = [x_1, y_1]\dots[x_n, y_n] \in F_{2n}$,

$\{w_1,\dots, w_k\}$ = the minimal normal subgroup of F_{2n} containing the elements $w_1,\dots, w_k$ of F_{2n},

where $[x_i, y_i]$ is the commutator of x_i and y_i.

As is well known, the fundamental group and the Euler characteristic of the orientable closed surface T_m of genus m are given by

$$\pi_1(T_m) = F_{2m}/\{r_m\}, \qquad \chi(T_m) = 2-2m. \tag{3.1}$$

By Corollary 1.7 and (3.1), we see immediately

PROPOSITION 3.2. *A finite group G acts freely on T_m $(m \geqq 0)$ preserving the orientation if and only if*

$$G = F_{2n}/N', \quad N' \ni r_n, \quad m-1 = (n-1)\# G.$$

PROOF OF THEOREM 1.10. The first half of Theorem 1.10 is an immediate consequence of the above proposition. The last half is also so, since any group of order $\leqq 5$ is abelian. *q.e.d.*

It is difficult to determine the groups which act freely on T_m $(m \geqq 7)$ preserving the orientation. We notice finally the results for $m \leqq 31$, which is obtained by using the following proposition and the known classification theorem of non-abelian groups of lower order.

PROPOSITION 3.3. *Let G be a finite group and assume that the number of generators of G is less than $n+1$. Then, G acts freely on T_m preserving the orientation, where $m = 1+(n-1)\# G$.*

PROOF. By the assumption, we see that G is isomorphic to a quotient group F_{2n}/K, where K contains $x_1y_1^{-1},\dots, x_ny_n^{-1}$. Then, K contains $r_n = [x_1, y_1]\dots[x_n, y_n]$, and so the desired result follows immediately from Proposition 3.2. *q.e.d.*

THEOREM 3.4. *A finite non-abelian group G acts freely on T_m $(m \leqq 31)$ preserving the orientation if and only if $\# G$ is a divisor of $m-1$.*

PROOF. The necessity is an immediate consequence of (1.2), (1.3) and (3.1). Conversely, assume $\# G$ is a divisor of $m-1$. If G is generated by 2 elements, then G acts freely on T_m preserving the orientation by the above proposition. Since $\# G \leqq 30$, G is generated by 2 or 3 elements and only the following groups are generated by 3 elements by [1, Table 1]:

$$A=\{x, y, z; x^2=y^2=z^2=(zx)^2=(xy)^2=(yz)^4\},\ \# A=16,$$

$$B=\{x, y, z; x^2=y^2=(xy)^2, z^2=x^{-1}zxz=y^{-1}zyz=1\},\ \# B=16,$$

$$C=\{x, y, z; x^2=y^2=z^2=1, xyz=yzx=zxy\},\ \# C=16,$$

$$D=\{x, y, z; x^2=y^2=z^2=(xyz)^2=(xy)^3=(xz)^3=1\},\ \# D=18,$$

$$E=\{x, y, z; x^2=y^2=z^2=(yz)^6=(zx)^2=(xy)^2=1\},\ \# E=24.$$

Therefore, it remains to show that these groups G act on T_m for $m=1+\# G$.

Take the normal subgroup K of F_4 as follows:

$$K=\{x_1^2, y_1^2, x_2^2, x_2y_2^{-1}, (x_2x_1)^2, (x_1y_1)^2, (y_1x_2)^4\}, \quad \text{for} \quad G=A,$$

$$K=\{x_2y_2^{-1}, x_2^2x_1^{-2}, x_1^2(x_2x_1)^{-2}, y_1^2, x_2^{-1}y_1x_2y_1, x_1^{-1}y_1x_1y_1\}, \quad \text{for} \quad G=B,$$

$$K=\{x_1^2, x_2^2, x_2y_2^{-1}, (x_2^{-1}y_1)^2, (x_1y_1)(y_1x_1)^{-1}, (y_1x_1)(x_2^{-1}y_1x_1x_2)^{-1}\}, \quad \text{for} \quad G=C,$$

$$K=\{x_1^2, x_2^2, y_2^2, y_1(x_2y_2)^{-1}, (x_1y_1)^2, (x_1x_2)^3, (x_1y_2)^3\}, \quad \text{for} \quad G=D,$$

$$K=\{x_1^2, y_1^2, x_2^2, x_2y_2^{-1}, (x_2x_1)^2, (x_1y_1)^2, (y_1x_2)^6\}, \quad \text{for} \quad G=E.$$

Then, it is easy to see that $K \ni r_2$ and $F_4/K \cong G$. Therefore, we have the desired result by Proposition 3.2. *q.e.d.*

References

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