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Dirichlet Integral of Product of Functions on a Self-adjoint Harmonic Space

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Introduction

In the previous paper [2], the author defined a notion of gradient measures for functions on a self-adjoint harmonic space. In case the harmonic space is given by solutions of a second order elliptic partial differential equation of the form

$$\sum_{i,j=1}^{k} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)-qu=0$$

on a Euclidean domain, the mutual gradient measure $\delta_{[f,g]}$ of functions f and g is given by

$$\delta_{[f,g]}=\left(\sum_{i,j=1}^{k} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\right)dx \qquad (dx\text{: the Lebesgue measure}).$$

Thus, in this case, the equality

$$(*) \qquad \delta_{[fg,\phi]}=f\delta_{[g,\phi]}+g\delta_{[f,\phi]}$$

holds. The main purpose of this paper is to show that the equality $(*)$ remains valid for general self-adjoint harmonic spaces. Once this equality is established, we can consider Royden's algebra (cf. [3, Chap. III]) on a self-adjoint harmonic space. We shall also see that if the harmonic structure is considered on a Euclidean domain and satisfies a certain additional condition (see Theorem 5), then the gradient measure is expressed as

$$\delta_{[f,g]}=\sum_{i,j=1}^{k}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\nu_{ij}$$

with a positive-definite system of signed measures (ν_{ij}); and the harmonic functions are "solutions" of the second order elliptic partial differential equation

$$\sum_{i,j=1}^{k}\frac{\partial}{\partial x_i}\left(\nu_{ij}\frac{\partial u}{\partial x_j}\right)-\pi u=0$$

whose coefficients ν_{ij}, π are (signed) measures.

§1. Basic definitions in the previous paper [2].

The base space Ω is a connected, locally compact Hausdorff space with a countable base. We consider a harmonic structure $\mathfrak{H}=\{\mathscr{H}(\omega)\}_{\omega:\text{open}}$ on Ω satisfying Axioms 1, 2 and 3 of M. Brelot. A domain ω in Ω is called a P-domain if it is non-compact and there is a positive potential on ω. We assume

Axiom 4. On any P-domain ω, the condition of proportionality is satisfied.

We furthermore assume that $(\Omega, \mathfrak{H})$ is self-adjoint, i.e., there is a system $\{G_\omega(x, y)\}_{\omega:\text{P-domain}}$ of symmetric Green functions satisfying the consistency condition [2, §1.2, (c)]; this system will be fixed. For a P-domain ω and a signed measure σ on ω, let $U_\omega^\sigma(x)=\int_\omega G_\omega(x, y)d\sigma(y)$ whenever it has a meaning.

A domain ω is called a PC-domain if it is relatively compact and its closure is contained in a P-domain. For an open set ω_0 in Ω, let

$$\mathscr{B}_{\text{loc}}(\omega_0) = \left\{ f;\ \begin{array}{l}\text{for any PC-domain } \omega \text{ such that } \bar{\omega}\subset\omega_0, \text{ there are} \\ \text{two non-negative bounded superharmonic func-} \\ \text{tions } s_1 \text{ and } s_2 \text{ on } \omega \text{ such that } f|\omega=s_1-s_2\end{array} \right\}.$$

To each $f\in\mathscr{B}_{\text{loc}}(\omega_0)$, a signed measure σ_f on ω_0 is associated in such a way that $f|\omega=U_\omega^{\sigma_f}+u$ with $u\in\mathscr{H}(\omega)$ for any PC-domain ω such that $\bar{\omega}\subset\omega_0$. We assume

Axiom 5. The constant function 1 belongs to $\mathscr{B}_{\text{loc}}(\Omega)$ and for any PC-domain ω, $U_\omega^{|\pi|}$ is continuous, where $\pi=\sigma_1$.

For any open set ω_0, $\mathscr{B}_{\text{loc}}(\omega_0)$ is an algebra ([2, Proposition 2.1]). We define

$$\delta_{[f,g]}=\frac{1}{2}(f\sigma_g+g\sigma_f - \sigma_{fg} - fg\pi)$$

for $f, g\in\mathscr{B}_{\text{loc}}(\omega_0)$ as a signed measure on ω_0. We know ([2, Theorem 4.1]) that $\delta_f\equiv\delta_{[f,f]}\geqq 0$ for any $f\in\mathscr{B}_{\text{loc}}(\omega_0)$.

For a PB-domain ω (i.e., a P-domain for which $U_\omega^{|\pi|}$ is bounded), set

$$\mathscr{M}_{BC}(\omega) = \left\{ \sigma;\ \begin{array}{l}\text{signed measure on } \omega \text{ such that } U_\omega^{|\sigma|} \text{ is} \\ \text{bounded continuous and } |\sigma|(\omega)<\infty\end{array} \right\}$$

and

$$\mathscr{P}_{BC}(\omega) = \{U_\omega^\sigma;\ \sigma\in\mathscr{M}_{BC}(\omega)\}.$$

The space $\mathscr{P}_{BC}(\omega)$ is a normed space with respect to

$$\|U_\omega^\sigma\|_{I,\omega}=\left(\int_\omega U_\omega^\sigma\,d\sigma\right)^{1/2}.$$

We define

$$\mathscr{D}_0(\omega)=\left\{f;\ \begin{array}{l}\text{there is a sequence } \{f_n\} \text{ in } \mathscr{P}_{BC}(\omega) \text{ such that}\\ f_n\to f \text{ q.e. on } \omega \text{ and } \|f_n-f_m\|_{I,\omega}\to 0\ (n, m\to\infty)\end{array}\right\},$$

where "q.e." means "except on a polar set". This space is a Hilbert space with respect to the norm

$$\|f\|_{I,\omega}=\lim_{n\to\infty}\|f_n\|_{I,\omega},$$

where $\{f_n\}$ is a sequence for f described in the definition of $\mathscr{D}_0(\omega)$.

For an open set ω_0, we define

$$\mathscr{D}_{\rm loc}(\omega_0)=\left\{f;\ \begin{array}{l}\text{for any PC-domain } \omega \text{ such that } \bar{\omega}\subset\omega_0,\\ f|\omega\in\mathscr{D}_0(\omega)+\mathscr{H}_E(\omega)\end{array}\right\}.$$

Here,

$$\mathscr{H}_E(\omega)=\{u\in\mathscr{H}(\omega);\ \delta_u(\omega)+\int_\omega u^2d|\pi|<\infty\},$$

which is complete with respect to the norm (semi-norm, in case $\pi=0$)

$$\|u\|_{E,\omega}=\{\delta_u(\omega)+\int_\omega u^2d|\pi|\}^{1/2}.$$

To each $f\in\mathscr{D}_{\rm loc}(\omega_0)$, there corresponds a non-negative measure δ_f on ω_0 which is determined as follows: For a PC-domain ω such that $\bar{\omega}\subset\omega_0$, if $f|\omega=g+u$ with $g\in\mathscr{D}_0(\omega)$ and $u\in\mathscr{H}_E(\omega)$ and if $\{g_n\}$ is a sequence in $\mathscr{P}_{BC}(\omega)$ such that $g_n\to g$ q.e. on ω and $\|g_n-g_m\|_{I,\omega}\to 0$ $(n, m\to\infty)$, then

$$\delta_f(A)=\lim_{n\to\infty}\delta_{g_n+u}(A)$$

for every Borel set A in ω (see [2, Theorems 6.2 and 7.1]). This is an extension of the notion δ_f for $f\in\mathscr{B}_{\rm loc}(\omega_0)$. If $f, g\in\mathscr{D}_{\rm loc}(\omega_0)$, then we define

$$\delta_{[f,g]}=\frac{1}{2}(\delta_{f+g}-\delta_f-\delta_g).$$

§2. Gradient measure of product of functions in $\mathscr{B}_{\rm loc}(\omega_0)$.

The purpose of this section is to establish the following results:

THEOREM 1. *Let ω_0 be an open set in Ω. For any f, g, $\phi\in\mathscr{B}_{\rm loc}(\omega_0)$,*

$$\delta_{[fg,\phi]} = f\delta_{[g,\phi]} + g\delta_{[f,\phi]}.$$

COROLLARY. *For any* $f, g \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$,

$$\delta_{fg} = f^2\delta_g + 2fg\delta_{[f,g]} + g^2\delta_f,$$

in particular,

$$\delta_{f^2} = 4f^2\delta_f.$$

The proof of the above theorem will be given by a series of lemmas. First, we consider the perturbed sheaf $\mathfrak{H}^\sim = \{\mathscr{H}^\sim(\omega)\}_{\omega:\,\mathrm{open}}$ which was defined in [2, § 3.2]. We note that if $u \in \mathscr{H}^\sim(\omega_0)$, then $\sigma_u = u\pi$.

LEMMA 1. *If* $u, v \in \mathscr{H}^\sim(\omega_0)$, *then*

$$\delta_{[u^2,v]} = 2u\delta_{[u,v]}.$$

PROOF. Let $\tilde{\sigma}_f \equiv \sigma_f - f\pi$ for $f \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$. If $u \in \mathscr{H}^\sim(\omega_0)$, then $\tilde{\sigma}_u = 0$; a continuous function w on ω_0 is $\mathfrak{H}^\sim$-superharmonic if and only if $\tilde{\sigma}_w \geqq 0$ (see [2, Proposition 3.7]). First we shall show that

$$\tilde{\sigma}_{u^3} = 3u\tilde{\sigma}_{u^2} \tag{1}$$

for $u \in \mathscr{H}^\sim(\omega_0)$. Since u is continuous, given $\varepsilon > 0$, each $x_0 \in \omega_0$ has an open neighborhood V $(\subset \omega_0)$ such that

$$u(x_0) - \varepsilon \leqq u \leqq u(x_0) + \varepsilon$$

on V. Consider the function $w = u(x_0) + \varepsilon - u$ on V. Since $w \in \mathscr{H}^\sim(V)$ and $w \geqq 0$ on V, $-w^3$ is $\mathfrak{H}^\sim$-superharmonic on V, so that $\tilde{\sigma}_{w^3} \leqq 0$ on V. It follows that $\tilde{\sigma}_{u^3} \geqq 3(u(x_0) + \varepsilon)\tilde{\sigma}_{u^2}$ on V. Noting that $\tilde{\sigma}_{u^2} \leqq 0$, we have

$$\tilde{\sigma}_{u^3} \geqq 3(u + 2\varepsilon)\tilde{\sigma}_{u^2} \tag{2}$$

on V. Since such V's cover ω_0, (2) holds on ω_0. Since ε is arbitrary, we obtain the inequality $\tilde{\sigma}_{u^3} \geqq 3u\tilde{\sigma}_{u^2}$. Similarly, by considering $w = u - u(x_0) + \varepsilon$ on V, we obtain the converse inequality $\tilde{\sigma}_{u^3} \leqq 3u\tilde{\sigma}_{u^2}$. Hence we have (1).

Next, let $u, v \in \mathscr{H}^\sim(\omega_0)$. For any real t, $\tilde{\sigma}_{(u+tv)^3} = 3(u+tv)\tilde{\sigma}_{(u+tv)^2}$ by (1). Using (1) for u and v and taking the definition of $\tilde{\sigma}_f$ into account, we deduce

$$3t(\sigma_{u^2v} - 2u\sigma_{uv} - v\sigma_{u^2} + 2u^2v\pi)$$
$$= -3t^2(\sigma_{uv^2} - u\sigma_{v^2} - 2v\sigma_{uv} + 2uv^2\pi).$$

From the arbitrariness of t, it follows that

$$\sigma_{u^2v} = 2u\sigma_{uv} + v\sigma_{u^2} - 2u^2v\pi. \tag{3}$$

On the other hand,

$$\delta_{[u^2,v]}=\frac{1}{2}\left(u^2\sigma_v+v\sigma_{u^2}-\sigma_{u^2v}-u^2v\pi\right)=\frac{1}{2}\left(v\sigma_{u^2}-\sigma_{u^2v}\right).$$

Hence, by (3),

$$\delta_{[u^2,v]}=\frac{2u}{2}(uv\pi-\sigma_{uv})=2u\delta_{[u,v]}.$$

Now, let ω be a PC-domain and consider the spaces

$$\mathscr{P}_B(\omega)=\{U_\omega^\sigma;\sigma\in\mathscr{M}_B(\omega)\}\quad\text{and}\quad\mathscr{B}_E(\omega)=\mathscr{P}_B(\omega)+\mathscr{H}_{BE}(\omega)$$

(see [2, § 1.3 and § 2.5] for $\mathscr{M}_B(\omega)$ and $\mathscr{H}_{BE}(\omega)$).

We remark that if $f\in\mathscr{P}_B(\omega)$ and $g\in\mathscr{B}_E(\omega)$ then $fg\in\mathscr{P}_B(\omega)$ by virtue of [2, Corollary to Proposition 2.2] and [2, Lemma 2.9]. Also, if u, $v\in\mathscr{H}_{BE}(\omega)$, then $U_\omega^{\sigma_{uv}}\in\mathscr{P}_B(\omega)$ by [2, Lemma 2.7]. Therefore, for f, $g\in\mathscr{B}_E(\omega)$, $U_\omega^{\sigma_{fg}}\in\mathscr{P}_B(\omega)$. These facts will be frequently used in what follows.

LEMMA 2. *Let ω be a PC-domain, u, $v\in\mathscr{H}_{BE}(\omega)$ and $g\in\mathscr{P}_B(\omega)$. If $uv\in\mathscr{B}_E(\omega)$, then*

$$\int_\omega u\,d\sigma_{vg}=\int_\omega uv\,d\sigma_g-\int_\omega g\,d\sigma_{uv}.$$

PROOF. Put $p=U_\omega^{\sigma_{uv}}$ and $h=uv-p$. Then, $p\in\mathscr{P}_B(\omega)$ and $h\in\mathscr{H}_{BE}(\omega)$. Hence, [2, Corollary to Proposition 2.2] and [2, Proposition 2.4] imply

$$\sigma_{pg}(\omega)=\int_\omega pg\,d\pi$$

and

$$\sigma_{hg}(\omega)=\int_\omega h\,d\sigma_g+\int_\omega hg\,d\pi.$$

Hence

$$\sigma_{uvg}(\omega)=\sigma_{pg}(\omega)+\sigma_{hg}(\omega)=\int_\omega uvg\,d\pi+\int_\omega h\,d\sigma_g.$$

On the other hand, since $vg\in\mathscr{P}_B(\omega)$, [2, Proposition 2.4] implies

$$\sigma_{uvg}(\omega)=\int_\omega u\,d\sigma_{vg}+\int_\omega uvg\,d\pi.$$

Hence

$$\int_\omega u\,d\sigma_{vg}=\int_\omega h\,d\sigma_g=\int_\omega uv\,d\sigma_g-\int_\omega p\,d\sigma_g$$
$$=\int_\omega uv\,d\sigma_g-\int_\omega g\,d\sigma_p=\int_\omega uv\,d\sigma_g-\int_\omega g\,d\sigma_{uv}.$$

LEMMA 3. *Let ω be a PC-domain, $f\in\mathscr{P}_B(\omega)$ and $u\in\mathscr{H}_{BE}(\omega)$. Then*

$$\int_\omega u\,d\sigma_{f^2}=0.$$

PROOF. Applying [2, Corollary to Proposition 2.2] to f and fu, we have

$$\sigma_{f^2u}(\omega)=\int_\omega f^2u\,d\pi.$$

On the other hand, [2, Proposition 2.4] implies

$$\sigma_{f^2u}(\omega)=\int_\omega u\,d\sigma_{f^2}+\int_\omega f^2u\,d\pi.$$

Hence we have the required equality.

LEMMA 4. *Let ω be a PC-domain, $f\in\mathscr{P}_B(\omega)$ and $g\in\mathscr{B}_E(\omega)$. Then*

$$\delta_{[f^2,g]}(\omega)=2\int_\omega f\,d\delta_{[f,g]}.$$

PROOF. Let $\nu=2\delta_{[f^2,g]}-4f\delta_{[f,g]}$. We are to show that $\nu(\omega)=0$. By the definition of $\delta_{[\cdot,\cdot]}$, we have

$$\nu=-f^2\sigma_g+g\sigma_{f^2}-\sigma_{f^2g}-2fg\sigma_f+2f\sigma_{fg}+f^2g\pi.$$

Since $fg, f\in\mathscr{P}_B(\omega)$, $\int_\omega fg\,d\sigma_f=\int_\omega f\,d\sigma_{fg}$. By [2, Corollary to Proposition 2.2], $\sigma_{f^2g}(\omega)=\int_\omega f^2g\,d\pi$. Hence

$$\nu(\omega)=-\int_\omega f^2d\sigma_g+\int_\omega g\,d\sigma_{f^2}.$$

Let $g=u+g_0$ with $u\in\mathscr{H}_{BE}(\omega)$ and $g_0\in\mathscr{P}_B(\omega)$. Then,

$$\int_\omega g_0d\sigma_{f^2}=\int_\omega f^2d\sigma_{g_0}=\int_\omega f^2d\sigma_g.$$

By the above lemma, $\int_\omega u\,d\sigma_{f^2}=0$. Hence $\nu(\omega)=0$.

LEMMA 5. *Let ω be a PC-domain, $f\in\mathscr{P}_B(\omega)$, $u\in\mathscr{H}_{BE}(\omega)$ and $g\in\mathscr{B}_E(\omega)$. If $ug\in\mathscr{B}_E(\omega)$, then*

$$\delta_{[fu,g]}(\omega) = \int_\omega f\, d\delta_{[u,g]} + \int_\omega u\, d\delta_{[f,g]}.$$

PROOF. Put $v=2(\delta_{[fu,g]}-f\delta_{[u,g]}-u\delta_{[f,g]})$. Since $\sigma_u=0$,

$$v = (g\sigma_{fu}-fu\sigma_g)+(f\sigma_{ug}-ug\sigma_f)-\sigma_{fug}+u\sigma_{fg}+fug\pi.$$

By [2, Proposition 2.4],

$$\sigma_{fug}(\omega) = \int_\omega u\, d\sigma_{fg}+\int_\omega fug\, d\pi.$$

Let $g=v+g_0$ with $v\in\mathscr{H}_{BE}(\omega)$ and $g_0\in\mathscr{P}_B(\omega)$. Then

$$\int_\omega g_0 d\sigma_{fu} = \int_\omega fu\, d\sigma_{g_0} = \int_\omega fu\, d\sigma_g \quad\text{and}\quad \int_\omega f\, d\sigma_{ug_0} = \int_\omega ug_0 d\sigma_f.$$

Hence

$$v(\omega) = \int_\omega v\, d\sigma_{fu}+\int_\omega f\, d\sigma_{uv}-\int_\omega uv\, d\sigma_f,$$

which is equal to zero by virtue of Lemma 2. (Note that $uv-ug\in\mathscr{P}_B(\omega)$, and hence $uv\in\mathscr{B}_E(\omega)$.)

LEMMA 6. *Let ω be a PC-domain, $u\in\mathscr{H}_{BE}(\omega)$ and $g\in\mathscr{P}_B(\omega)$. If $u^2\in\mathscr{B}_E(\omega)$, then*

$$\delta_{[u^2,g]}(\omega) = 2\int_\omega u\, d\delta_{[u,g]}.$$

PROOF. Let $v=2\delta_{[u^2,g]}-4u\delta_{[u,g]}$. We have

$$v = g\sigma_{u^2}-u^2\sigma_g-\sigma_{u^2g}+2u\sigma_{ug}+u^2g\pi.$$

[2, Proposition 2.4] implies

$$\sigma_{u^2g}(\omega) = \int_\omega u\, d\sigma_{ug}+\int_\omega u^2g\, d\pi.$$

On the other hand, by Lemma 2,

$$\int_\omega u\, d\sigma_{ug} = \int_\omega u^2 d\sigma_g-\int_\omega g\, d\sigma_{u^2}.$$

Hence $v(\omega)=0$.

LEMMA 7. *Let ω be a small PC-domain* (see [2, § 3.2] for a small domain), *$u\in\mathscr{H}_{BE}(\omega)$ and $g\in\mathscr{B}_E(\omega)$. If u^2, $ug\in\mathscr{B}_E(\omega)$, then*

$$\delta_{[u^2,g]}(\omega) = 2\int_\omega u\, d\delta_{[u,g]}.$$

PROOF. Let $g=v+g_0$ with $v \in \mathscr{H}_{BE}(\omega)$ and $g_0 \in \mathscr{P}_B(\omega)$. Let $\tilde{u}=(I-G_\omega)^{-1}u$ and $\tilde{v}=(I-G_\omega)^{-1}v$, where G_ω is the operator defined in [2, §3.1]. Put $p\equiv G_\omega\tilde{u}=U_\omega^{\tilde{u}\pi}$ and $q\equiv G_\omega\tilde{v}=U_\omega^{\tilde{v}\pi}$. Then $u=\tilde{u}-p$ and $v=\tilde{v}-q$ and $p, q \in \mathscr{P}_B(\omega)$. Now

$$\delta_{[u^2,g]} = \delta_{[\tilde{u}^2,\tilde{v}]}+\delta_{[u^2,g-\tilde{v}]}-2\delta_{[up,\tilde{v}]}-\delta_{[p^2,\tilde{v}]}.$$

Since $\tilde{u}$, $\tilde{v} \in \mathscr{H}^{\sim}(\omega)$ by [2, Proposition 3.5], Lemma 1 shows that $\delta_{[\tilde{u}^2,\tilde{v}]}=2\tilde{u}\delta_{[\tilde{u},\tilde{v}]}$. The previous lemma implies

$$\delta_{[u^2,g-\tilde{v}]}(\omega) = 2\int_\omega u\, d\delta_{[u,g-\tilde{v}]},$$

since $g-\tilde{v}=g_0-q \in \mathscr{P}_B(\omega)$. Since $u\tilde{v}=ug+uq$ and $uq \in \mathscr{P}_B(\omega)$, we see that $u\tilde{v} \in \mathscr{B}_E(\omega)$. Hence, By Lemma 5,

$$\delta_{[up,\tilde{v}]}(\omega) = \int_\omega u\, d\delta_{[p,\tilde{v}]}+\int_\omega p\, d\delta_{[u,\tilde{v}]}.$$

Finally, by Lemma 4, we have

$$\delta_{[p^2,\tilde{v}]}(\omega) = 2\int_\omega p\, d\delta_{[p,\tilde{v}]}.$$

Therefore

$$\begin{aligned}
&\delta_{[u^2,g]}(\omega)\\
&= 2\int_\omega \tilde{u}\, d\delta_{[\tilde{u},\tilde{v}]}+2\int_\omega u\, d\delta_{[u,g-\tilde{v}]}-2\int_\omega u\, d\delta_{[p,\tilde{v}]}\\
&\qquad -2\int_\omega p\, d\delta_{[u,\tilde{v}]}-2\int_\omega p\, d\delta_{[p,\tilde{v}]}\\
&= 2\int_\omega u\, d\delta_{[u,g]}.
\end{aligned}$$

Now we are ready to prove the theorem:

PROOF OF THEOREM 1. It is enough to prove the case $f=g$. Let ω_1 be any relatively compact small domain such that $\bar{\omega}_1 \subset \omega_0$ and let ω be another domain contained in ω_1. Then ω is a small PC-domain. We can write $f|\omega = u+f_0$ with $u \in \mathscr{H}_{BE}(\omega)$ and $f_0 \in \mathscr{P}_B(\omega)$. Since $f^2, f\phi \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$, we see by [2, Lemma 2.8] that u^2, $u(\phi|\omega) \in \mathscr{B}_E(\omega)$. Therefore, by Lemmas 4, 5 and 7, we have

$$\delta_{[f^2,\phi]}(\omega) = 2\int_\omega f\, d\delta_{[f,\phi]}. \tag{4}$$

It follows that (4) holds for any open set ω in ω_1, and hence

$$\delta_{[f^2,\phi]} = 2f\delta_{[f,\phi]}$$

holds on ω_1. Since such ω_1's cover ω_0, this equality holds on ω_0.

§3. Some auxiliary results on functions in $\mathscr{D}_{loc}(\omega_0)$.

LEMMA 8. *Let ω be a PB-domain. If $f \in \mathscr{D}_0(\omega)$ and μ is a non-negative measure such that U_ω^μ is bounded, then*

$$\int_\omega f^2 d\mu \leqq (\sup U_\omega^\mu)\|f\|_{I,\omega}^2.$$

PROOF. This is easily seen from [2, Theorem 1.2 and Lemma 1.3]. Cf. the proof of [2, Theorem 6.3].

LEMMA 9. *Let ω_0 be an open set and f be an extended real valued function on ω_0. If for each $x \in \omega_0$ there is an open neighborhood V_x of x such that $f|V_x \in \mathscr{D}_{loc}(V_x)$, then $f \in \mathscr{D}_{loc}(\omega_0)$.*

PROOF. If $V_x \cap V_{x'} \neq \emptyset$, then [2, Lemma 7.3] shows that $\delta_{f|V_x} = \delta_{f|V_{x'}}$ on $V_x \cap V_{x'}$. It follows that there is a non-negative measure δ_f^* on ω_0 such that $\delta_f^*|V_x = \delta_{f|V_x}$ for each $x \in \omega_0$. Similarly, given a PC-domain ω with $\bar{\omega} \subset \omega_0$ and $g \in \mathscr{D}_0(\omega)$, there is a signed measure $\delta_{[f,g]}^*$ on ω such that $\delta_{[f,g]}^*|V_x \cap \omega = \delta_{[f|V_x \cap \omega, g|V_x \cap \omega]}$ for each $x \in \omega_0$ with $V_x \cap \omega \neq \emptyset$. We can cover ω by a finite number of V_x's, which we write $\omega_1, \dots, \omega_k$. Then

$$\begin{aligned}
\delta_{[f,g]}^*(\omega) &= \sum_{j=1}^k \delta_{[f,g]}^*(\omega_j \cap \omega - \cup_{i=1}^{j-1} \omega_i) \\
&\leqq \sum_{j=1}^k \delta_{f|\omega_j}(\omega_j \cap \omega)^{1/2} \delta_g(\omega_j \cap \omega)^{1/2} \\
&\leqq k\delta_f^*(\omega)^{1/2}\delta_g(\omega)^{1/2}.
\end{aligned}$$

Also,

$$\left|\int_\omega fg\, d\pi\right| \leqq \left(\int_\omega f^2 d|\pi|\right)^{1/2} \left(\int_\omega g^2 d|\pi|\right)^{1/2}.$$

Since $\delta_f^*(\omega) < \infty$ and $\int_\omega f^2 d|\pi| < \infty$, it follows from [2, Theorem 6.3] and the above Lemma 8 that the mapping

$$g \longrightarrow \delta_{[f,g]}^*(\omega) + \int_\omega fg\, d\pi$$

is continuous on $\mathscr{D}_0(\omega)$. Obviously, this is linear. Since the mapping $(g_1, g_2)\to \delta_{[g_1,g_2]}(\omega)+\int_\omega g_1 g_2 d\pi$ gives the inner product for the Hiblert space $\mathscr{D}_0(\omega)$ (see [2, (6.4) in Theorem 6.3]), there is $f_0\in\mathscr{D}_0(\omega)$ such that

$$\delta^*_{[f,g]}(\omega)+\int_\omega fg\,d\pi = \delta_{[f_0,g]}(\omega)+\int_\omega f_0 g\,d\pi \tag{5}$$

for all $g\in\mathscr{D}_0(\omega)$. For any $x\in\omega$, choose a domain ω' such that $x\in\omega'\subset\bar{\omega}'\subset V_x\cap\omega$. If $\phi\in\mathscr{D}_0(\omega')$, then its extension to ω by 0 on $\omega-\omega'$ belongs to $\mathscr{D}_0(\omega)$ by [2, Lemma 6.7]. Hence, by (5), we obtain

$$\delta_{[(f-f_0)|\omega',\phi]}(\omega')+\int_{\omega'}(f-f_0)\phi\,d\pi = 0$$

for all $\phi\in\mathscr{D}_0(\omega')$. Using [2, Theorem 6.4] and modifying the values of f_0 on a polar set, we see that $f|\omega=f_0+u$ with $u\in\mathscr{H}(\omega)$. Thus $f|\omega\in\mathscr{D}_{\mathrm{loc}}(\omega)$ and $\delta_{f|\omega}=\delta^*_{f|\omega}$. Then by [2, Proposition 7.2], we conclude that $f|\omega\in\mathscr{D}_0(\omega)+\mathscr{H}_E(\omega)$. Therefore $f\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$.

LEMMA 10. *If ω is a PB-domain such that* $\sup U_\omega^{\pi^-}<1/4$, *then*

$$\|f\|^2_{I,\omega}+\|u\|^2_{E,\omega}\leqq c_\omega\{\delta_{f+u}(\omega)+\int_\omega(f+u)^2 d|\pi|\}$$

for any $f\in\mathscr{D}_0(\omega)$ and $u\in\mathscr{H}_E(\omega)$, where $c_\omega^{-1}=1-2(\sup U_\omega^{\pi^-})^{1/2}$.

PROOF. Put $\alpha=\sup U_\omega^{\pi^-}$. Using [2, Theorem 6.3] and Lemma 8 above, we have

$$\begin{aligned}
&\delta_{f+u}(\omega)+\int_\omega(f+u)^2 d|\pi|\\
&=\delta_f(\omega)+\int_\omega f^2 d|\pi|+2(\delta_{[f,u]}(\omega)+\int_\omega fu\,d|\pi|)+\delta_u(\omega)+\int_\omega u^2 d|\pi|\\
&\geqq \|f\|^2_{I,\omega}+\|u\|^2_{E,\omega}+4\int_\omega fu\,d\pi^-\\
&\geqq \|f\|^2_{I,\omega}+\|u\|^2_{E,\omega}-4\left(\int_\omega f^2 d\pi^-\right)^{1/2}\left(\int_\omega u^2 d\pi^-\right)^{1/2}\\
&\geqq \|f\|^2_{I,\omega}+\|u\|^2_{E,\omega}-4\alpha^{1/2}\|f\|_{I,\omega}\|u\|_{E,\omega}\\
&\geqq (1-2\alpha^{1/2})(\|f\|^2_{I,\omega}+\|u\|^2_{E,\omega}).
\end{aligned}$$

PROPOSITION 1. *If $\{f_n\}$ is a sequence in $\mathscr{D}_{\mathrm{loc}}(\omega_0)$ such that $f_n\to f$ q.e. on ω_0, $\delta_{f_n-f_m}(K)\to 0$ and $\int_K(f_n-f_m)^2 d|\pi|\to 0$ as $n, m\to\infty$ for any compact set*

K in ω_0, then $f\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$ and $\delta_{f_n-f}(K)\to 0$ as $n\to\infty$ for any compact set K in ω_0.

PROOF. Let ω be a PC-domain such that $\bar{\omega}\subset\omega_0$ and $U_\omega^{\pi^-}<1/4$. Let $f_n|\omega=g_n+u_n$ with $g_n\in\mathscr{D}_0(\omega)$ and $u_n\in\mathscr{H}_E(\omega)$. By the previous lemma, we see that $\|g_n-g_m\|_{I,\omega}\to 0$ and $\|u_n-u_m\|_{E,\omega}\to 0$ $(n, m\to\infty)$. Then it follows from [2, Theorems 6.1 and 5.3] (in case $\pi|\omega=0$, [1, Theorem 3.3] instead of [2, Theorem 5.3]) that $f|\omega=g+u$ with $g\in\mathscr{D}_0(\omega)$ and $u\in\mathscr{H}_E(\omega)$ and that $\|g_n-g\|_{I,\omega}\to 0$ and $\|u_n-u\|_{E,\omega}\to 0$ $(n\to\infty)$. Since each point has a neighborhood V_x which is a PC-domain and for which $\sup U_{V_x}^{\pi^-}<1/4$, Lemma 9 implies that $f\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$. Also, in the above argument, $\delta_{g_n-g}(\omega)\to 0$ and $\delta_{u_n-u}(\omega)\to 0$ $(n\to\infty)$. Hence $\delta_{f_n-f}(\omega)\to 0$ $(n\to\infty)$, and thus the last assertion of the proposition follows.

§4. Locally bounded Dirichlet functions.

Let ω_0 be an open set in Ω. Besides $\mathscr{B}_{\mathrm{loc}}(\omega_0)$, we consider

$$\mathscr{B}_{C,\mathrm{loc}}(\omega_0)=\{f\in\mathscr{B}_{\mathrm{loc}}(\omega_0); U_\omega^{|\sigma_f|} \text{ is continuous for any PC-domain } \omega\}.$$

Functions in $\mathscr{B}_{C,\mathrm{loc}}(\omega_0)$ are continuous. We see from [2, Lemma 2.5] and the proof of [2, Proposition 2.1] that $\mathscr{B}_{C,\mathrm{loc}}(\omega_0)$ is also an algebra. Note that Axiom 5 states that $1\in\mathscr{B}_{C,\mathrm{loc}}(\Omega)$.

Now, let

$$\mathscr{D}_{B,\mathrm{loc}}(\omega_0)=\left\{g;\ \begin{array}{l}\text{there is a sequence } \{f_n\} \text{ in } \mathscr{B}_{\mathrm{loc}}(\omega_0) \text{ such}\\ \text{that } f_n\to g \text{ locally uniformly on } \omega_0 \text{ and}\\ \delta_{f_n-f_m}(K)\to 0\ (n, m\to\infty) \text{ for each compact } K\subset\omega_0\end{array}\right\}.$$

We similarly define $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$ replacing $\mathscr{B}_{\mathrm{loc}}(\omega_0)$ by $\mathscr{B}_{C,\mathrm{loc}}(\omega_0)$. Obviously, these are linear spaces and by Proposition 1

$$\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)\subset\mathscr{D}_{B,\mathrm{loc}}(\omega_0)\subset\mathscr{D}_{\mathrm{loc}}(\omega_0).$$

THEOREM 2. *$\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ and $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$ are algebras. For any $f, g\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ and $\phi\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$,*

$$\delta_{[fg,\phi]}=f\delta_{[g,\phi]}+g\delta_{[f,\phi]}.$$

PROOF. Let $f, g\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) and choose $\{f_n\}$ and $\{g_n\}$ in $\mathscr{B}_{\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{B}_{C,\mathrm{loc}}(\omega_0)$) such that $f_n\to f$ and $g_n\to g$ locally uniformly on ω_0 and $\delta_{f_n-f_m}(K)\to 0$ and $\delta_{g_n-g_m}(K)\to 0$ $(n, m\to\infty)$ for each compact set K in ω_0. By the corollary to Theorem 1, we have

$$\begin{aligned}\delta_{f_ng_n-f_mg_m}&\leqq 2(\delta_{(f_n-f_m)g_n}+\delta_{f_m(g_n-g_m)})\\&=2\{(f_n-f_m)^2\delta_{g_n}+2(f_n-f_m)g_n\delta_{[f_n-f_m,g_n]}+g_n^2\delta_{f_n-f_m}\\&\quad+f_m^2\delta_{g_n-g_m}+2f_m(g_n-g_m)\delta_{[f_m,g_n-g_m]}+(g_n-g_m)^2\delta_{f_m}\}.\end{aligned}$$

It follows that $\delta_{f_ng_n-f_mg_m}(K)\to 0$ $(n, m\to\infty)$ for any compact set K in ω_0. Obviously, $f_ng_n\to fg$ locally uniformly on ω_0. Hence $fg\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$). Therefore, $\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ and $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$ are algebras.

Next, let $\phi\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$. If ω is a PC-domain such that $\bar{\omega}\subset\omega_0$, then there is a sequence $\{\phi_n\}$ in $\mathscr{P}_{BC}(\omega)+\mathscr{H}_E(\omega)$ such that $\delta_{\phi_n-\phi}(\omega)\to 0$. By Theorem 1,

$$\delta_{[f_ng_n,\phi_m]}=f_n\delta_{[g_n,\phi_m]}+g_n\delta_{[f_n,\phi_m]}$$

on ω. Letting $m\to\infty$, we have

$$\delta_{[f_ng_n,\phi]}=f_n\delta_{[g_n,\phi]}+g_n\delta_{[f_n,\phi]}$$

on ω, and hence on ω_0. Let A be any relatively compact Borel set such that $\bar{A}\subset\omega_0$. Since $\delta_{f_ng_n-fg}(A)\to 0$ (Proposition 1),

$$\delta_{[f_ng_n,\phi]}(A)\to\delta_{[fg,\phi]}(A).$$

On the other hand, since $f_n\to f$, $g_n\to g$ uniformly on A and $\delta_{f_n-f}(A)\to 0$ and $\delta_{g_n-g}(A)\to 0$ (Proposition 1),

$$\int_A f_n\,d\delta_{[g_n,\phi]}\to\int_A f\,d\delta_{[g,\phi]}\quad\text{and}\quad\int_A g_n\,d\delta_{[f_n,\phi]}\to\int_A g\,d\delta_{[f,\phi]}.$$

Therefore

$$\delta_{[fg,\phi]}(A)=\int_A f\,d\delta_{[g,\phi]}+\int_A g\,d\delta_{[f,\phi]},$$

and hence

$$\delta_{[fg,\phi]}=f\delta_{[g,\phi]}+g\delta_{[f,\phi]}.$$

COROLLARY. *For $f, g\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$,*

$$\delta_{fg}=f^2\delta_g+2fg\delta_{[f,g]}+g^2\delta_f.$$

PROPOSITION 2. *If $\{f_n\}$ is a sequence in $\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) which converges locally uniformly to f on ω_0 and if $\delta_{f_n-f_m}(K)\to 0$ $(n, m\to\infty)$ for each compact set K in ω_0, then $f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) and $\delta_{f-f_n}(K)\to 0$ $(n\to\infty)$ for each compact set K in ω_0.*

PROOF. Let $\{\omega_n\}$ be an exhaustion of ω_0. By definition, there is $g_n\in\mathscr{B}_{\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{B}_{C,\mathrm{loc}}(\omega_0)$) such that $|g_n-f_n|<1/n$ on $\bar{\omega}_n$ and $\delta_{g_n-f_n}(\omega_n)<1/n$ for each n. Then $g_n\to f$ locally uniformly on ω_0 and $\delta_{g_n-g_m}(K)\to 0$ $(n, m\to\infty)$ for each compact set K in ω_0. Hence $f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) and

$$\delta_{f_n-f}(K)\leqq 2\{\delta_{f-g_n}(K)+\delta_{g_n-f_n}(K)\}\to 0\qquad(n\to\infty)$$

for each compact set K in ω_0.

THEOREM 3. *Let* $f_1,\dots,f_k\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) *and regard* $\boldsymbol{f}=(f_1,\dots,f_k)$ *as a mapping from* ω_0 *into* $\boldsymbol{R}^k$. *If* Ω' *is an open set in* $\boldsymbol{R}^k$ *containing* $\cup\{\overline{\boldsymbol{f}(K)};K\colon \text{compact}\subset\omega_0\}$ (resp. $\boldsymbol{f}(\omega_0)$) *and if* $\Phi\in\boldsymbol{C}^1(\Omega')$, *then* $\Phi\circ\boldsymbol{f}\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) *and*

$$\delta_{[\Phi\circ\boldsymbol{f},g]}=\sum_{j=1}^{k}\left(\frac{\partial\Phi}{\partial x_j}\circ\boldsymbol{f}\right)\delta_{[f_j,g]} \tag{6}$$

for any $g\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$.

PROOF. If $\Phi\equiv$const., then the both sides of (6) vanish. If $\Phi(x_1,\dots,x_k)=x_j$, then both sides of (6) are reduced to $\delta_{[f_j,g]}$. Now, suppose the conclusions are true for Φ_1, $\Phi_2\in\boldsymbol{C}^1(\Omega')$ and let $\Phi=\Phi_1\Phi_2$. By Theorem 2,

$$\Phi\circ\boldsymbol{f}=(\Phi_1\circ\boldsymbol{f})(\Phi_2\circ\boldsymbol{f})\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)\ (\text{resp. }\mathscr{D}_{BC,\mathrm{loc}}(\omega_0))$$

and

$$\begin{aligned}
&\delta_{[\Phi\circ\boldsymbol{f},g]}\\
&=\delta_{[(\Phi_1\circ\boldsymbol{f})(\Phi_2\circ\boldsymbol{f}),g]}\\
&=(\Phi_1\circ\boldsymbol{f})\delta_{[\Phi_2\circ\boldsymbol{f},g]}+(\Phi_2\circ\boldsymbol{f})\delta_{[\Phi_1\circ\boldsymbol{f},g]}\\
&=\sum_{j=1}^{k}\left\{(\Phi_1\circ\boldsymbol{f})\left(\frac{\partial\Phi_2}{\partial x_j}\circ\boldsymbol{f}\right)+(\Phi_2\circ\boldsymbol{f})\left(\frac{\partial\Phi_1}{\partial x_j}\circ\boldsymbol{f}\right)\right\}\delta_{[f_j,g]}\\
&=\sum_{j=1}^{k}\left(\frac{\partial\Phi}{\partial x_j}\circ\boldsymbol{f}\right)\delta_{[f_j,g]}.
\end{aligned}$$

It follows that the conclusion of the theorem holds for any polynomial Φ in k-variables. Now let $\Phi\in\boldsymbol{C}^1(\Omega')$. Then there is a sequence $\{P_n\}$ of polynomials in k-variables such that $P_n\to\Phi$ and $\partial P_n/\partial x_j\to\partial\Phi/\partial x_j$, $j=1,\dots,k$, all locally uniformly on Ω'. Then $P_n\circ\boldsymbol{f}\to\Phi\circ\boldsymbol{f}$ locally uniformly on ω_0, since the image $\boldsymbol{f}(K)$ of a compact set K in ω_0 is relatively compact in Ω'. We have seen that

$$\begin{aligned}
&\delta_{P_n\circ\boldsymbol{f}-P_m\circ\boldsymbol{f}}\\
&=\sum_{j,l=1}^{k}\left(\frac{\partial(P_n-P_m)}{\partial x_j}\circ\boldsymbol{f}\right)\left(\frac{\partial(P_n-P_m)}{\partial x_l}\circ\boldsymbol{f}\right)\delta_{[f_j,f_l]}.
\end{aligned}$$

Hence, if K is a compact set in ω_0, then

$$\delta_{P_n\circ\boldsymbol{f}-P_m\circ\boldsymbol{f}}(K)\to 0\qquad(n,m\to\infty),$$

since $[\partial(P_n-P_m)/\partial x_j]\circ\boldsymbol{f}\to 0$ $(n,m\to\infty)$ uniformly on K for each j. Hence, by

Proposition 2, $\Phi\circ f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) and

$$\delta_{P_n\circ f-\Phi\circ f}(K)\to 0 \qquad (n\to\infty).$$

By an argument similar to the proof of Theorem 2, we see that (6) holds for the given Φ.

COROLLARY 1. *Let f_j $(j=1,\dots,k)$ and Φ be as in the above proposition. Then*

$$\delta_{\Phi\circ f}=\sum_{j,l=1}^{k}\left(\frac{\partial\Phi}{\partial x_j}\circ f\right)\left(\frac{\partial\Phi}{\partial x_l}\circ f\right)\delta_{[f_j,f_l]}.$$

COROLLARY 2. (a) *If $f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ and $\inf_K f>0$ for each compact set K in ω_0, then $1/f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ and*

$$\delta_{[1/f,g]}=-\frac{1}{f^2}\delta_{[f,g]} \quad \textit{for } g\in\mathscr{D}_{\mathrm{loc}}(\omega_0);\ \delta_{1/f}=\frac{1}{f^4}\delta_f. \tag{7}$$

(b) *If $f\in\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$ and $f>0$ on ω_0, then $1/f\in\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$ and (7) is valid.*

Next, we consider so-called Royden's algebras. For an open set ω_0 in Ω, set

$$\mathscr{D}_B(\omega_0)=\{f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0); f \text{ is bounded and } \delta_f(\omega_0)<\infty\}$$

and

$$\mathscr{D}_{BC}(\omega_0)=\mathscr{D}_B(\omega_0)\cap\mathscr{D}_{BC,\mathrm{loc}}(\omega_0).$$

For $f\in\mathscr{D}_B(\omega_0)$, let

$$\|f\|_{DB,\omega_0}=\delta_f(\omega_0)^{1/2}+\sup_{\omega_0}|f|.$$

THEOREM 4. *$\mathscr{D}_B(\omega_0)$ and $\mathscr{D}_{BC}(\omega_0)$ are Banach algebras with respect to the above norm.*

PROOF. By Theorem 2, we easily see that $\mathscr{D}_B(\omega_0)$ and $\mathscr{D}_{BC}(\omega_0)$ are algebras. Obviously, $\|\cdot\|_{DB,\omega_0}$ is a norm on these spaces. By the aid of the corollary to Theorem 2, we can easily verify that

$$\|fg\|_{DB,\omega_0}\leqq\|f\|_{DB,\omega_0}\|g\|_{DB,\omega_0}$$

for $f, g\in\mathscr{D}_B(\omega_0)$. The completeness of $\mathscr{D}_B(\omega_0)$ and $\mathscr{D}_{BC}(\omega_0)$ follows from Proposition 2.

REMARK 1. Using the algebra $\mathscr{D}_{BC}(\Omega)$ we may extend the classical theory involving Royden's algebra (see, e.g., [3, Chap. III]) to self-adjoint harmonic spaces.

§5. Self-adjoint harmonic space on a Euclidean domain.

We consider the special case where Ω is a domain in the Euclidean space $\boldsymbol{R}^k$ $(k\geqq 1)$.

THEOREM 5. *Let Ω be a domain in $\boldsymbol{R}^k$ and let $\mathfrak{H}$ be a self-adjoint harmonic structure on Ω satisfying Axioms $1\sim 5$. Furthermore we assume that the coordinate functions x_j $(j=1,\dots,k)$ all belong to $\mathscr{D}_{B,\mathrm{loc}}(\Omega)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\Omega)$). Then, for any open set $\omega_0\subset\Omega$, every $f\in\boldsymbol{C}^1(\omega_0)$ belongs to $\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) and its gradient measure is expressed as*

$$\delta_f=\sum_{i,j=1}^{k}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\nu_{ij},$$

where ν_{ij}, $i,j=1,\dots,k$, are signed measures on Ω having the following properties:

(a) $\nu_{ij}=\nu_{ji}$ $\quad(i,j=1,\dots,k)$;

(b) *For each $\xi=(\xi_1,\dots,\xi_k)\in\boldsymbol{R}^k$ with $\xi\neq 0$,*

$$\mu_\xi=\sum_{i,j=1}^{k}\xi_i\xi_j\nu_{ij}$$

is a positive measure whose support is equal to Ω.

PROOF. Define

$$\nu_{ij}=\delta_{[x_i,x_j]},\qquad i,j=1,\dots,k.$$

By our assumption, these are well-defined signed measures on Ω. Property (a) is obvious. For $\xi\in\boldsymbol{R}^k$, if A is a Borel set in Ω, then

$$\mu_\xi(A)=\sum_{i,j}\xi_i\xi_j\delta_{[x_i,x_j]}(A)=\delta_{\Sigma\xi_ix_i}(A)\geqq 0.$$

Furthermore, if $\xi\neq 0$, then the function $f_\xi(x)=\Sigma\xi_ix_i$ is non-constant on any open set ω in Ω. Hence $\delta_{f_\xi}(\omega)>0$ by virtue of [2, Theorem 7.3]. Hence the support of μ_ξ is the whole space Ω. If $f\in\boldsymbol{C}^1(\omega_0)$, then Theorem 3 implies that $f\in\mathscr{D}_{B,\mathrm{loc}}(\omega_0)$ (resp. $\mathscr{D}_{BC,\mathrm{loc}}(\omega_0)$) and Corollary 1 to Theorem 3 shows that

$$\delta_f=\sum_{i,j=1}^{k}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\delta_{[x_i,x_j]}=\sum_{i,j=1}^{k}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\nu_{ij}.$$

REMARK 2. Under the assumptions of Theorem 5, if ω is a PB-domain, then $\boldsymbol{C}_0^1(\omega)\subset\mathscr{D}_0(\omega)$, where $\boldsymbol{C}_0^1(\omega)=\{f\in\boldsymbol{C}^1(\omega);\ \mathrm{supp}f$ is compact in $\omega\}$. Hence, it follows from [2, Theorem 6.3] that every $u \in \mathscr{H}(\omega_0)$ (ω_0: any open set in Ω) satisfies

$$\delta_{[u,\psi]}(\omega_0) + \int_{\omega_0} u\psi \, d\pi = 0$$

for all $\psi \in \boldsymbol{C}_0^1(\omega_0)$. In particular, if $u \in \mathscr{H}(\omega_0) \cap \boldsymbol{C}^1(\omega_0)$, then by the above theorem, it satisfies

$$\sum_{i,j=1}^{k} \int_{\omega_0} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j} d\nu_{ij} + \int_{\omega_0} u\psi \, d\pi = 0$$

for all $\psi \in \boldsymbol{C}_0^1(\omega_0)$. In this sense, we may say that every $u \in \mathscr{H}(\omega_0)$ is a "solution" of the formal differential equation

$$\sum_{i,j=1}^{k} \frac{\partial}{\partial x_i}\left(\nu_{ij} \frac{\partial u}{\partial x_j}\right) - \pi u = 0 .$$

§6. An application of Theorem 1.

Now, we return to the general case and let h be a positive continuous function on Ω. Then $\mathfrak{H}^{(h)} = \mathfrak{H}/h$ is a self-adjoint harmonic structure on Ω with a consistent system of Green functions $\{G_\omega^{(h)}(x, y)\}_{\omega: \text{P-domain}}$:

$$G_\omega^{(h)}(x, y) = \frac{G_\omega(x, y)}{h(x)h(y)} .$$

Obviously, for any open set ω_0 in Ω,

$$\mathscr{B}_{\text{loc}}^{(h)}(\omega_0) = \left\{\frac{f}{h} \,;\, f \in \mathscr{B}_{\text{loc}}(\omega_0)\right\}$$

and for $f \in \mathscr{B}_{\text{loc}}(\omega_0)$,

$$\sigma_{f/h}^{(h)} = h\sigma_f ,$$

where the index (h) means that the notion is considered with respect to $\mathfrak{H}^{(h)}$.

PROPOSITION 3. *If $h \in \mathscr{B}_{\text{loc}}(\Omega)$ is positive continuous, then $\mathscr{B}_{\text{loc}}^{(h)}(\omega_0) = \mathscr{B}_{\text{loc}}(\omega_0)$ for any open set ω_0; in particular $1 \in \mathscr{B}_{\text{loc}}^{(h)}(\Omega)$ and $1/h \in \mathscr{B}_{\text{loc}}(\Omega)$.*

PROOF. If $f \in \mathscr{B}_{\text{loc}}(\omega_0)$, then $f = (fh)/h$ and $fh \in \mathscr{B}_{\text{loc}}(\omega_0)$. Hence $f \in \mathscr{B}_{\text{loc}}^{(h)}(\omega_0)$. In particular, since $1 \in \mathscr{B}_{\text{loc}}(\Omega)$ (Axiom 5), $1 \in \mathscr{B}_{\text{loc}}^{(h)}(\Omega)$. It follows that $\mathscr{B}_{\text{loc}}^{(h)}(\omega_0)$ is also an algebra (cf. the proof of [2, Proposition 2.1]; it requires only the assumption $1 \in \mathscr{B}_{\text{loc}}(\Omega)$). Since $1/h \in \mathscr{B}_{\text{loc}}^{(h)}(\Omega)$, $1/h^2 \in \mathscr{B}_{\text{loc}}^{(h)}(\Omega)$, and hence $1/h \in \mathscr{B}_{\text{loc}}(\Omega)$. If $f \in \mathscr{B}_{\text{loc}}^{(h)}(\omega_0)$, then $fh \in \mathscr{B}_{\text{loc}}(\omega_0)$. Hence $f = (fh)/h \in \mathscr{B}_{\text{loc}}(\omega_0)$.

COROLLARY. *For any open set ω_0 in Ω, if $f \in \mathscr{B}_{loc}(\omega_0)$ is continuous and does not vanish on ω_0, then $1/f \in \mathscr{B}_{loc}(\omega_0)$.*

LEMMA 11. *If $h \in \mathscr{B}_{C,loc}(\Omega)$ is positive, then $\mathfrak{H}^{(h)}$ satisfies Axiom 5.*

PROOF. By Proposition 3, $1 \in \mathscr{B}^{(h)}_{loc}(\Omega)$. Since $\sigma_1^{(h)} = h\sigma_h$, we have

$$\int_\omega G_\omega^{(h)}(\cdot,\ y)d|\sigma_1^{(h)}|(y) = \int_\omega G_\omega^{(h)}(\cdot, y)h(y)d|\sigma_h|(y) = \frac{1}{h}U_\omega^{|\sigma_h|}.$$

Hence, $\int_\omega G_\omega^{(h)}(\cdot, y)d|\sigma_1^{(h)}|(y)$ is continuous on ω for any PC-domain ω.

Thus, if h is a function as in this lemma, then we can consider the gradient measure $\delta_f^{(h)}$ for $f \in \mathscr{B}^{(h)}_{loc}(\omega_0) = \mathscr{B}_{loc}(\omega_0)$ with respect to the self-adjoint harmonic structure $\mathfrak{H}^{(h)}$. Then we have

LEMMA 12. *If $h \in \mathscr{B}_{C,loc}(\Omega)$ is positive then for $f \in \mathscr{B}_{loc}(\omega_0)$*

$$\delta_f^{(h)} = h^2\delta_f.$$

PROOF. Noting that $\sigma_g^{(h)} = h\sigma_{hg}$ for $g \in \mathscr{B}_{loc}(\omega_0)$, we have

$$\begin{aligned}\delta_f^{(h)} &= \frac{1}{2}(2f\sigma_f^{(h)} - \sigma_{f^2}^{(h)} - f^2\sigma_1^{(h)}) \\ &= \frac{1}{2}(2fh\sigma_{hf} - h\sigma_{hf^2} - f^2h\sigma_h) \\ &= \frac{h}{2}(2f\sigma_{hf} - \sigma_{hf^2} - f^2\sigma_h).\end{aligned}$$

Now, by Theorem 1, $\delta_{[f^2,h]} = 2f\delta_{[f,h]}$, which may be written as

$$f^2\sigma_h + h\sigma_{f^2} - \sigma_{hf^2} - f^2h\pi = 2f(f\sigma_h + h\sigma_f - \sigma_{hf} - fh\pi),$$

or

$$2f\sigma_{hf} - \sigma_{hf^2} - f^2\sigma_h = 2fh\sigma_f - h\sigma_{f^2} - f^2h\pi = 2h\delta_f.$$

Hence

$$\delta_f^{(h)} = \frac{h}{2}2h\delta_f = h^2\delta_f.$$

We can also consider the spaces $\mathscr{D}^{(h)}_{loc}(\omega_0)$, $\mathscr{D}^{(h)}_{B,loc}(\omega_0)$ and $\mathscr{D}^{(h)}_{BC,loc}(\omega_0)$ with respect to $\mathfrak{H}^{(h)}$. By Proposition 3 and Lemma 12, we can easily show

THEOREM 6. *Let h be a function as in Lemma 12. Then $\mathscr{D}^{(h)}_{B,loc}(\omega_0) = \mathscr{D}_{B,loc}(\omega_0)$ and $\mathscr{D}^{(h)}_{BC,loc}(\omega_0) = \mathscr{D}_{BC,loc}(\omega_0)$ for any open set ω_0; for $f \in \mathscr{D}_{B,loc}(\omega_0)$,*

$$\delta_f^{(h)} = h^2 \delta_f .$$

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