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Parallelizability of Grassmann Manifolds

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§1. Introduction

Let $G_{n,m}$ be the Grassmann manifold of all m-planes through the origin of the Euclidean n-space R^n. A. Neifahs [3] proved that n is a power of 2 if $G_{n,m}$ is parallelizable.

In this note, we prove the following

THEOREM 1.1. *$G_{n,m}$ is parallelizable, i.e., the tangent bundle of $G_{n,m}$ is trivial, if and only if*

$$n = 2, 4 \text{ or } 8; \qquad m = 1 \text{ or } n-1.$$

To prove this theorem, we use the following theorem.

For a real vector bundle ξ, we denote by $Span\,\xi$ the maximum number of linearly independent cross-sections of ξ. Especially, we denote $Span\,M = Span\,\tau M$, where τM is the tangent bundle of a C^∞-manifold M.

THEOREM 1.2. *Let ξ_k be the canonical line bundle over the real projective k-space RP^k, and $n\xi_k$ the Whitney sum of n-copies of it.*

Then, $Span\,G_{n,m} \geqq k$ implies $Span\; nm\xi_{n-m} \geqq m^2 + k$.

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§2. Proof of Theorem 1.2

Let $\gamma_{n,m}$ be the canonical m-plane bundle over $G_{n,m}$, i.e., the total space of $\gamma_{n,m}$ be the subspace of $G_{n,m} \times R^n$ consisting of all pairs (x, v) where $x \in G_{n,m}$ and v is a vector in x. Then, by [2, Problem 5–B],

$$\tau G_{n,m} \cong \operatorname{Hom}(\gamma_{n,m}, \gamma_{n,m}^{\perp}), \tag{2.1}$$

where $\gamma_{n,m}^{\perp}$ denotes the orthogonal complement of $\gamma_{n,m}$ in the trivial bundle $G_{n,m} \times R^n \to G_{n,m}$.

Consider the Stiefel manifold $V_{n,m}$ of all orthonormal m-frames in R^n, which has the involution by sending each $(v_1, \dots, v_m)$ to $(-v_1, \dots, -v_m)$. By [5, Prop. 1], we see the following fact.

(2.2) There exists an equivariant map from $S^l = V_{l+1,1}$ to $V_{n,m}$ if and only if $Span\, n\xi_l \geqq m$, where ξ_k is the canonical line bundle over RP^k in Theorem 1.2.

PROOF OF THEOREM 1.2. Assume that $Span\, G_{n,m} \geqq k$. Then we have k linearly independent cross-sections $s_1, \ldots, s_k$ of $\mathrm{Hom}(\gamma_{n,m}, \gamma_{n,m}^{\perp})$ by (2.1).

For each $v = (v_1, \ldots, v_m) \in V_{n,m}$, we set

$$v^l = ((s_l(\tilde{v}))(v_1), (s_l(\tilde{v}))(v_2), \ldots, (s_l(\tilde{v}))(v_m)) \in (R^n)^m \qquad (1 \leqq l \leqq k),$$

where $\tilde{v}$ is the subspace of R^n spanned by v. Also, let f_i: $R^n \to (R^n)^m$ be the inclusion onto the i-th factor. Then, we see easily that

(2.3) $f_i(v_j)(1 \leqq i, j \leqq m)$, $v^l(1 \leqq l \leqq k)$ are linearly independent in $(R^n)^m$.

Therefore, we obtain a map φ: $V_{n,m} \to V_{nm,m^2+k}$, where $\varphi(v)$ is obtained from (2.3) by the orthonormalization. Also, this map φ is equivariant with respect to the involutions.

It is well known that $Span\, n\xi_{n-m} \geqq m$, and so there exists an equivariant map ψ: $S^{n-m} \to V_{n,m}$ by (2.2). Hence, we obtain an equivariant map $\varphi \circ \psi$: $S^{n-m} \to V_{nm,m^2+k}$, and so $Span\, nm\xi_{n-m} \geqq m^2 + k$ by (2.2). *q.e.d.*

§3. Proof of Theorem 1.1

As $G_{n,m}$ is diffeomorphic to $G_{n,n-m}$, it is sufficient to consider $G_{n,m}$ for $1 \leqq m \leqq n/2$.

LEMMA 3.1. *For even dimensional* $G_{n,m}$, $Span\, G_{n,m} = 0$.

PROOF. In this case, it is well known that the i-dimensional homology group $H_i(G_{n,m}; Z)$ for odd i of $G_{n,m}$ with the integral coefficient Z does not contain the free part. Hence, the Euler characteristic of $G_{n,m}$ is positive, and so $Span\, G_{n,m} = 0$ by Hopf's theorem. *q.e.d.*

LEMMA 3.2. *If* $G_{n,m}$ *is parallelizable, then* $nm \equiv 0 \mod 2^{\varphi(n-m)}$, *where* $\varphi(n-m)$ *is the number of integers* s *such that* $0 < s \leqq n-m$ *and* $s \equiv 0$, 1, 2 *or* 4 mod 8.

PROOF. Since $Span\, G_{n,m} = m(n-m)$ by the assumption, we see $Span\, nm\xi_{n-m} = nm$ by Theorem 1.2. Thus, we have the desired result by [1, Th. 7.4]. *q.e.d.*

LEMMA 3.3. *If* $G_{n,m}(1 \leqq m \leqq n/2)$ *is parallelizable, then* $(n, m) = (2, 1), (4, 1), (8, 1)$ *or* $(8, 3)$.

PROOF. By the above two lemmas, the assumption implies that m is odd, n is even and $n \equiv 0 \mod 2^{\varphi(n-m)}$. Therefore, we have the lemma by noticing that

$n<2^{\varphi(n/2)}$ for even $n>16$ and by the straightforward calculations. *q.e.d.*

Now, we calculate the Stiefel-Whitney class of $G_{8,3}$ by using the following result, which is an immediate consequence of [4, Th. 1].

LEMMA 3.4. *Let* $\sigma_1,\dots,\sigma_r$ *denote the elementary symmetric functions in variables* $x_1,\dots,x_r$, *and set*

$$\Phi'_r(\sigma_1,\dots,\sigma_r)=\Pi^r_{i,j=1}(1+x_i+x_j),$$

in the polynomial ring (over the integers mod 2). *Then, for any r-plane bundle* η, *the total Stiefel-Whitney class* $w(\eta\otimes\eta)$ *is given by*

$$w(\eta\otimes\eta)=\Phi'_r(w_1(\eta),\dots,w_r(\eta)),$$

where $w(\eta)=1+w_1(\eta)+\cdots+w_r(\eta)$.

LEMMA 3.5. $w(\gamma_{8,3}\otimes\gamma_{8,3})=1+(w_1^4+w_2^2)+(w_1^2w_2^2+w_3^2)$, *where* w_i $(i=1,2,3)$ *is the i-th Stiefel-Whitney class of* $\gamma_{8,3}$.

PROOF. It is easy to see that

$$\Pi^3_{i,j=1}(1+x_i+x_j)=(1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3)^2=1+\sigma_1^4+\sigma_2^2+\sigma_1^2\sigma_2^2+\sigma_3^2.$$

Thus, the result follows from the above lemma. *q.e.d.*

LEMMA 3.6. $w_4(G_{8,3})$ *is not zero.*

PROOF. $\tau G_{8,3}\cong\mathrm{Hom}(\gamma_{8,3},\gamma_{8,3}^\perp)\cong\gamma_{8,3}^*\otimes\gamma_{8,3}^\perp\cong\gamma_{8,3}\otimes\gamma_{8,3}^\perp$ by (2.1), because the dual bundle $\gamma_{8,3}^*$ of $\gamma_{8,3}$ is isomorphic to $\gamma_{8,3}$ [2, Problem 3–D]. Also, $(\gamma_{8,3}\otimes\gamma_{8,3}^\perp)\oplus(\gamma_{8,3}\otimes\gamma_{8,3})\cong\gamma_{8,3}\otimes(\gamma_{8,3}^\perp\oplus\gamma_{8,3})\cong 8\gamma_{8,3}$. So, $w(G_{8,3})w(\gamma_{8,3}\otimes\gamma_{8,3})=w(8\gamma_{8,3})$ $=1+w_1^8$. Thus, we see that $w_4(G_{8,3})=w_1^4+w_2^2$ by the above lemma, which is not zero by [2, Problem 6–B and Th. 7.1]. *q.e.d.*

PROOF OF THEOREM 1.1. It is well known that $RP^n=G_{n+1,1}$ $(n=1,3,7)$ is parallelizable, and so the theorem follows immediately by Lemmas 3.3 and 3.6. *q.e.d.*

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