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On Flat Extensions of Krull Domains

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Let A and B be Krull domains with A contained in B. We say that the condition "*no blowing up*", abbreviated to NBU, is satisfied if $ht(\mathfrak{P} \cap A) \leqq 1$ for every divisorial prime ideal $\mathfrak{P}$ of B. The main purpose of this paper is to give a criterion of the condition NBU by making use of the notion of divisorial modules, which was introduced in [5]. That is, the condition NBU is satisfied for Krull domains A and B if and only if B is divisorial as an A-module (Theorem 1). As an immediate consequence of the above criterion, we can obtain the well-known theorem: If B is flat over A, then the condition NBU is satisfied.

We shall also investigate the behavior of divisorial envelope under flat extensions of Krull domains. The main result is stated as follows: If, in addition to flatness, B is integral over A, $M \otimes B$ is a divisorial B-module for any codivisorial and divisorial A-module M.

We shall use freely the notation and the terminologies of [5] and [6].

§1. Flat modules over a Krull domain

In this section, we understand that A is always a Krull domain and K is the quotient field of A.

It is known that an A-lattice M is divisorial if and only if every regular A-sequence of length two is a regular M-sequence (cf. [4], Chap. I, §5, Coroll. 5.5. (f)). This result is valid for any torsion free divisorial module and to prove this, a similar method can be applied. Namely we have

PROPOSITION 1. *Let M be a torsion-free A-module. Then M is divisorial if and only if every regular A-sequence of length two is a regular M-sequence.*

The following corollary is a direct consequence of Prop. 1.

COROLLARY. *If M is a flat A-module, then M is divisorial.*

PROPOSITION 2. *Let M be an A-module and N be a flat A-module. Then we have:*

(i) *If M is codivisorial, then so is $M \otimes_A N$.*

(ii) $\tilde{M} \otimes_A N = \widetilde{M \otimes_A N}$.

(iii) *If M is codivisorial, then $D(M\otimes_A N)=D(M)\otimes_A N$.*

PROOF. (i): Since N is flat, $t(M)\otimes N=t(M\otimes N)$. Hence we may assume that M is a torsion module. Furthermore, since $M\otimes N\subseteq D(M)\otimes N$, we can replace M by $D(M)$. Thus we may assume that M is a codivisorial and divisorial torsion module. By [5], Th. 4, $M=\oplus M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the primes of $\mathrm{Ass}_A(M)$. Each $M_{\mathfrak{p}}\otimes N$ is an $A_{\mathfrak{p}}$-module and hence it is a codivisorial and divisorial A-module by [5], Prop. 16 and Coroll. to Prop. 23. Therefore $M\otimes N$ is codivisorial and divisorial by [5], Coroll. 1 to Prop. 12 and Coroll. 4 to Th. 3.

(ii): It is obvious that $\tilde{M}\otimes N\subseteq \widetilde{M\otimes N}$ by [5], Coroll. to Prop. 5. Therefore, by [5], Prop. 3, it suffices to show that if M is codivisorial, then so is $M\otimes N$. This is done in (i).

(iii): It follows from the above facts (i) and (ii) that the exact sequence $0\to M\otimes N\to D(M)\otimes N$ is an essentially isomorphic extension. Therefore it suffices to show that $D(M)\otimes N$ is divisorial. To do this we can assume that M is a torsion module or torsion-free by [6], Coroll. 3 to Th. 5 and Prop. 36. The case of a torsion module has already been done in the proof of (i). Suppose now that M is torsion-free. Then $E(M)=E(D(M))=M\otimes K$. Therefore $E(M)\otimes N$ is a divisorial A-module by [5], Coroll. to Prop. 23. On the other hand, $(E(D(M))/D(M))\otimes N$ is codivisorial by (i); hence $D(M)\otimes N$ is divisorial in $E(D(M))\otimes N$. Now the conclusion follows from [5], Coroll. 1 to Prop. 6.

§2. A flat extension of a Krull domain

In this section, A and B are always krull domains with A contained in B. We denote by $Q(A)$ (resp. $Q(B)$) the quotient field of A (resp. B).

1. The condition that, for every prime ideal $\mathfrak{P}\in Ht_1(B)$, height $(\mathfrak{P}\cap A)\leqq 1$ is known as the condition NBU. Here we give some criteria for the condition NBU.

THEOREM 1. *The following statements are equivalent*:

(i) *The condition NBU is satisfied for A and B.*

(ii) *Every codivisorial B-module is a codivisorial A-module.*

(iii) *B is divisorial as an A-module.*

PROOF. (i) implies (ii): Let M be a codivisorial B-module. Then, for any element x of M, the order ideal $O_B(x)$ is a divisorial ideal of B by [5], Prop. 5. Then there are prime ideals $\mathfrak{P}_1,\dots,\mathfrak{P}_r\in Ht_1(B)$ such that $O_B(x)=\mathfrak{P}_1^{(n_1)}\cap\cdots\cap\mathfrak{P}_r^{(n_r)}$, where $\mathfrak{P}_i^{(n_i)}$ means the n_i-th symbolic power of $\mathfrak{P}_i$. Hence $O_A(x)=O_B(x)\cap A=\cap(\mathfrak{P}_i^{(n_i)}\cap A)$. Thus, to see that $O_A(x)$ is a divisorial ideal of A, we must show that $\mathfrak{P}^{(n)}\cap A$ is a divisorial ideal of A for any $\mathfrak{P}\in Ht_1(B)$ and for any positive integer n. Put $\mathfrak{q}=\mathfrak{P}\cap A$. Then $\mathfrak{P}^{(n)}\cap A=(\mathfrak{P}^n B_{\mathfrak{P}}\cap A_{\mathfrak{q}})\cap A$. By the assumption,

$ht(\mathfrak{q}) \leqq 1$, which implies that $A_{\mathfrak{q}}$ is a field or principal valuation ring, and so $\mathfrak{P}^{(n)} \cap A$ is a divisorial ideal of A.

(ii) implies (iii): Clearly $Q(B)$ is a divisorial A-module. Since $Q(B)/B$ is a codivisorial B-module, it is a codivisorial A-module. This implies that B is divisorial in $Q(B)$ as A-modules. By [5], Coroll. 1 to Prop. 6 we can see that B is a divisorial A-module.

(iii) implies (i): By [5], Th. 4, $B = \cap B_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the primes of $Ht_1(A)$. We may consider $Ht_1(B_{\mathfrak{p}})$ as the subset of $Ht_1(B)$ which consists of the primes $\mathfrak{P} \in Ht_1(B)$ such that $\mathfrak{P} \cap A = 0$ or $\mathfrak{p}$. By [7], Th. 2.6 or [4], Prop. 3.15, $Ht_1(B) = \cup Ht_1(B_{\mathfrak{p}})$. This completes the proof.

COROLLARY 1. *We suppose that the conditions of Th.* 1 *are satisfied. If M is a divisorial torsion-free B-module, then M is divisorial as an A-module.*

PROOF. The assertion follows immediately from [5], Coroll. 1 to Prop. 6 and the fact that $E_B(M) = M \otimes Q(B)$ is a divisorial A-module.

Also, as a corollary to Th. 1, we can obtain the following well-known result (cf. [3], § 1, n°10, Prop. 15).

COROLLARY 2. *If B is flat over A, then the condition NBU is satisfied.*

PROPOSITION 3. *We suppose that the condition NBU is satisfied. If M is a pseudo-null A-module, then $M \otimes B$ is a pseudo-null B-module.*

PROOF. By [5], Prop. 18, we need to show that $M \otimes B_{\mathfrak{P}} = 0$ for every prime $\mathfrak{P} \in Ht_1(B)$. Put $\mathfrak{q} = \mathfrak{P} \cap A$. Then, by the assumption, $ht(\mathfrak{q}) \leqq 1$. Since M is a pseudo-null A-module, $M_{\mathfrak{q}} = 0$. Hence $M \otimes_A B_{\mathfrak{P}} = M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}} = 0$.

2. We understand, in the rest of this section, that B is always flat over A.

PROPOSITION 4. *If M is a codivisorial A-module, then $M \otimes B$ is a codivisorial B-module.*

PROOF. We can readily see that $t_A(M) \otimes B = t_B(M \otimes B)$; therefore we may assume that M is a codivisorial torsion module. By [5], Prop. 29, we may, furthermore, assume that M is finitely generated. Since $M \otimes B \subset D_A(M) \otimes B$, $M \otimes B$ can be considered as a submodule of a finite direct sum of B-modules of the type $A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} \otimes_A B$, where $\mathfrak{p}$ is a prime of $Ht_1(A)$, by [5], Th. 4 and by [6], Th. 7. Since $\mathfrak{p}^n A_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module, $\mathfrak{p}^n A_{\mathfrak{p}} \otimes_A B$ is a free $A_{\mathfrak{p}} \otimes_A B$-module and hence $\mathfrak{p}^n A_{\mathfrak{p}} \otimes_A B$ is a divisorial $A_{\mathfrak{p}} \otimes_A B$-module. Therefore $A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} \otimes_A B \cong A_{\mathfrak{p}} \otimes_B A/\mathfrak{p}^n A_{\mathfrak{p}} \otimes B$ is a codivisorial $A_{\mathfrak{p}} \otimes_A B$-module by [5], Coroll. 1 to Prop. 11. By noting that $A_{\mathfrak{p}} \otimes_A B$ is a localization of B, we can see that $A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} \otimes_A B$ is a codivisorial B-module by Th. 1 and Coroll. 2 to Th. 1. Thus $M \otimes_A B$ is a codivisorial B-module as a submodule of a direct sum of codivisorial B-modules.

COROLLARY. *Let M be an A-module. Then $\widetilde{N_A(M)\otimes_A B}=N_B(M\otimes_A B)$, where $N_A(M)=\tilde{M}$ as an A-module and $N_B(M\otimes_A B)=\widetilde{M\otimes_A B}$ as a B-module.*

PROOF. It is clear that $N_A(M)\otimes B\subset N_B(M\otimes B)$ by Prop. 3. Since $M/N_A(M)$ is a codivisorial A-module by [5], Prop. 3, $M\otimes_A B/N_A(M)\otimes B\cong M/N_A(M)\otimes B$ is a codivisorial B-module by Prop. 4. Therefore, $N_A(M)\otimes B\supset N_B(M\otimes_A B)$ by [5], Prop. 3.

PROPOSITION 5. *Let M be a codivisorial A-module. Then we have*

$$D_B(M\otimes_A B)=D_B(D_A(M)\otimes_A B).$$

PROOF. By [5], Prop. 4, $D_A(M)$ is a codivisorial A-module and hence $D_A(M)\otimes B$ is a codivisorial B-module by Prop. 4. Therefore, by [5], Prop. 13, Coroll. 1 to Prop. 18 and Prop. 20, it suffices to show that $(M\otimes B)_{\mathfrak{P}}=D_A(M)\otimes B)_{\mathfrak{P}}$ for every $\mathfrak{P}\in Ht_1(B)$. Put $\mathfrak{q}=A\cap\mathfrak{P}$. Then $ht(\mathfrak{q})\leqq 1$ by Th. 1 and Coroll. 2 to Th. 1. By [5], Coroll. 2 to Th. 3, $(D_A(M)\otimes_A B)_{\mathfrak{P}}=D_A(M)_{\mathfrak{q}}\otimes_{A_{\mathfrak{q}}}B_{\mathfrak{P}}=M_{\mathfrak{q}}\otimes_{A_{\mathfrak{q}}}B_{\mathfrak{P}}$ $=(M\otimes_A B)_{\mathfrak{P}}$. This completes the proof.

PROPOSITION 6. *If M is a divisorial torsion-free A-module, then $M\otimes_A B$ is a divisorial B-module.*

PROOF. Since M is torsion-free, $E_A(M)\cong M\otimes Q(A)$. Hence $M\otimes Q(A)/M$ is codivisorial because M is divisorial. Thus, $M\otimes Q(A)\otimes B/M\otimes B\cong(M\otimes Q(A)/M)\otimes B$ is a codivisorial B-module by Prop. 4, i.e., $M\otimes B$ is divisorial in $M\otimes Q(A)\otimes B$. On the other hand, $M\otimes Q(A)\otimes B$ is isomorphic to a direct sum of copies of $Q(A)\otimes B$ and, since $Q(A)\otimes B$ is a localization of B, $Q(A)\otimes B$ is a divisorial B-module by [5], Prop. 23. This implies that $M\otimes Q(A)\otimes B$ is a divisorial B-module as a direct sum of divisorial B-modules. Combining this fact with [5], Coroll. 1 to Prop. 6, we can see that $M\otimes B$ is a divisorial B-module.

COROLLARY. *Let M be a torsion-free A-module. Then we have*

$$D_B(M\otimes_A B)=D_A(M)\otimes_A B.$$

The assertion follows immediately from Prop. 5 and Prop. 6.

PROPOSITION 7. *Let M and N be A-lattices. If N is divisorial, then we have*

$$(N: M)\otimes_A B=(N\otimes_A B):(M\otimes_A B).$$

PROOF. Let $\mathfrak{P}$ be a prime of $Ht_1(B)$ and put $\mathfrak{q}=\mathfrak{P}\cap A$. Then $ht(\mathfrak{q})\leqq 1$ by Th. 1 and Coroll. 2 to Th. 1. We have $(N: M)\otimes_A B_{\mathfrak{P}}=(N: M)_{\mathfrak{q}}\otimes_{A_{\mathfrak{q}}}B_{\mathfrak{P}}$.

By [1], Chap. III, §8, Coroll. 8.4, $(N: M)_{\mathfrak{q}} = N_{\mathfrak{q}} : M_{\mathfrak{q}}$; and hence $(N: M) \otimes B_{\mathfrak{P}} = (N_{\mathfrak{q}} : M_{\mathfrak{q}}) \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}}$. Since $M_{\mathfrak{q}}$ is a finitely generated free $A_{\mathfrak{q}}$-module, $(N_{\mathfrak{q}} : M_{\mathfrak{q}}) \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}} = (N_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}}) : (M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}}) = (N \otimes_A B_{\mathfrak{P}}) : (M \otimes_A B_{\mathfrak{P}}) = (N \otimes_A B \otimes_B B_{\mathfrak{P}}) : (M \otimes_A B \otimes_B B_{\mathfrak{P}})$. Since N is a divisorial A-lattice, $N: M$ is a divisorial A-lattice by [4], Prop. 2.6. Therefore $N \otimes_A B$ and $(N: M) \otimes_A B$ are divisorial B-lattices by Prop. 6. Hence, $(N \otimes_A B : M \otimes_A B) \otimes_B B_{\mathfrak{P}} = (N \otimes_A B \otimes_B B_{\mathfrak{P}}) : (M \otimes_A B \otimes B_{\mathfrak{P}})$ and our assertion follows from [5], Th. 4.

COROLLARY 1. *Let M and N be A-lattices. Then*

$$D_A(N: M) \otimes B = D_B(N \otimes B: M \otimes B).$$

PROOF. By [6], Prop. 32, $D_A(N: M) = D_A(N): D_A(M)$. Since $D_A(N)$ is a divisorial A-lattice, $(D_A(N): D_A(M)) \otimes B = D_A(N) \otimes B : D_A(M) \otimes B$ by Prop. 7. By Coroll. to Prop. 6, $D_A(N) \otimes B = D_B(N \otimes B)$ and $D_A(M) \otimes B = D_B(M \otimes B)$. Therefore, $D_A(N: M) \otimes B = D_B(N \otimes B): D_B(M \otimes B)$. Again, by [6], Prop. 32, $D_B(N \otimes B): D_B(M \otimes B) = D_B(N \otimes B: M \otimes B)$.

COROLLARY 2. *If B is a Dedekind domain and M, N are A-lattices, then* $(N: M) \otimes B = N \otimes B: M \otimes B$.

PROOF. By Coroll. to Prop. 6, $D_A(N: M) \otimes B = D_B((N: M) \otimes B)$. Since B is a Dedekind domain, $D_B((N: M) \otimes B) = (N: M) \otimes B$ by [5], Remark 3. Also, by Cor. 1, $D_A(N: M) \otimes B = D_B(N \otimes B: M \otimes B) = N \otimes B: M \otimes B$. Hence, we have $(N: M) \otimes B = N \otimes B: M \otimes B$.

REMARK. It is not necessarily true that $D_A(M) \otimes_A B = D_B(M \otimes_A B)$, even if M is a codivisorial A-module.

EXAMPLE. Put $A = Z$ and $B = Z[X]$, where X is an indeterminate. Let p be a prime number. Then $Z/(p)$ is codivisorial and divisorial as a Z-module. However, $Z/p \otimes Z[X] = Z[X]/pZ[X]$ is not a divisorial $Z[X]$-module. Otherwise, $Z[X]/pZ[X] = Z[X]/pZ[X] \otimes_{Z[X]} Z[X]_{pZ[X]} = Q(Z[X]/pZ[X])$ by [5], Th. 4, where $Q(Z[X]/pZ[X])$ is the quotient field of $Z[X]/pZ[X]$. Hence $pZ[X]$ must be a maximal ideal and this is a contradiction.

THEOREM 2. *For any codivisorial and divisorial A-module M,* $M \otimes_A B$ *is a divisorial B-module if and only if* $Q(A)/A_{\mathfrak{p}} \otimes_A B \cong Q(A) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}/B_{\mathfrak{p}}$ *is a divisorial* $B_{\mathfrak{p}}$*-module for every prime* $\mathfrak{p} \in Ht_1(A)$. *In particular, if B is integral over A, then the above condition is satisfied.*

PROOF. Since $Q(A)/A_{\mathfrak{p}}$ is a codivisorial and divisorial A-module by [5], Prop. 23, the "only if" part is clear.

Suppose therefore that $Q(A)/A_{\mathfrak{p}} \otimes B$ is a divisorial $B_{\mathfrak{p}}$-module for every

$\mathfrak{p} \in Ht_1(A)$. Let M be a codivisorial and divisorial A-module. By Prop. 6 and [6], Coroll. 3 to Th. 5, we may assume that M is a torsion module. By [2], Prop. 2.3, 2.4, 2.5 and 2.6, $E_A(M)$ is isomorphic to a direct sum of $Q(A)/A_{\mathfrak{p}}$, $\mathfrak{p} \in Ht_1(A)$. Since $Q(A)/A_{\mathfrak{p}} \otimes_A B$ is a codivisorial and divisorial B-module by Prop. 4 and [5], Prop. 23, $E_A(M) \otimes_A B$ is a divisorial B-module by [5], Coroll. 4 to Th. 3. Since $E_A(M)/M$ is a codivisorial A-module, $(E_A(M) \otimes B)/(M \otimes B) \cong E_A(M)/M \otimes B$ is a codivisorial B-module by Prop. 4. This implies that $M \otimes B$ is divisorial in $E_A(M) \otimes B$ as B-modules. Hence $M \otimes B$ is a divisorial B-module.

The last assertion follows from [5], Coroll. to Prop. 23 and the facts that a Krull domain of Krull dimension 1 is a Dedekind domain and every module over a Dedekind domain is divisorial.

PROPOSITION 8. *Let M be a divisorial B-module. Then M is a divisorial A-module.*

PROOF. By the assumption $E_B(M)/M$ is codivisorial B-module and hence is a codivisorial A-module by Th. 1. Therefore M is divisorial in $E_B(M)$ as A-modules. It is well known that any injective B-module is injective as an A-module, in case that B is flat over A. Hence $E_B(M)$ is an injective A-module and this implies that M is a divisorial A-module by [5], Coroll. 1 to Prop. 6.

3. PROPOSITION 9. *Let N be a codivisorial A-module and M be a submodule of N. If N is an essential extension of M, then $N \otimes B$ is an essential extension of $M \otimes B$ as B-modules.*

PROOF. It is easy to see that N is an essential extension of M if and only if $t(N)$ is an essential extension of $t(M)$ and $N/t(N)$ is an essential extension of $M/t(M)$. Therefore we may assume that N is a torsion module.

Since $N \otimes B$ is a codivisorial B-module by Prop. 4, it suffices to show that $(N \otimes B)_{\mathfrak{P}}$ is an essential extension of $(M \otimes B)_{\mathfrak{P}}$ as $B_{\mathfrak{P}}$-modules for every $\mathfrak{P} \in Ht_1(B)$ by [5], Coroll. to Prop. 20. Put $\mathfrak{q} = A \cap \mathfrak{P}$. Then $ht(\mathfrak{q}) \leqq 1$ by Th. 1. Since $(N \otimes B)_{\mathfrak{P}} = N_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}}$ and $(M \otimes B)_{\mathfrak{P}} = M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{P}}$, we may assume that B is a principal valuation ring and A is a principal valuation ring or a field. To show that $N \otimes B$ is an essential extension of $M \otimes B$, we may assume that N is finitely generated. Since A is a principal valuation ring or a field, $N = \oplus A y_i$ $(1 \leqq i \leqq n)$. Put $M' = \oplus(M \cap A y_i)$. Then $M' \subset M$ and N is an essential extension of M'. Since $N \otimes B = \oplus(A y_i \otimes B)$ and $M' \otimes B = \oplus((A y_i \cap M) \otimes B)$, we may assume that N is cyclic. Then $N \otimes B$ is also cyclic and hence $N \otimes B$ is a coirreducible B-module because B is a principal valuation ring. Therefore $N \otimes B$ is an essential extension of $M \otimes B$.

COROLLARY. *Let M be a codivisorial A-module. Then we have*

$$E_B(M \otimes B) = E_B(E_A(M) \otimes B).$$

THEOREM 3. *For every codivisorial and injective A-module M,* $M \otimes B$ *is an injective B-module if and only if* $Q(B) = Q(A) \otimes B$ *and* $B_{\mathfrak{p}}$ *is a Dedekind domain for any prime* $\mathfrak{p}$ *of* $Ht_1(A)$. *In particular, if B is integral over A, then the above condition is satisfied.*

PROOF. First we show the "only if" part. It is easy to see that $Q(B) = Q(A) \otimes B$. Since $Q(A)/A_{\mathfrak{p}}$ is a codivisorial and injective A-module for any $\mathfrak{p} \in Ht_1(A)$, $Q(A) \otimes B/A_{\mathfrak{p}} \otimes B = Q(B)/B_{\mathfrak{p}}$ is an injective B-module. In particular, $Q(B)/B_{\mathfrak{p}}$ is an injective $B_{\mathfrak{p}}$-module by Prop. 4 and [5], Coroll. 1 to Th. 3. Therefore $B_{\mathfrak{p}}$ is a Dedekind domain by [4], Chap. III, § 3, Th. 13.1 (d).

Next we show the "if" part. Let M be a codivisorial and injective A-module. Then M is isomorphic to a direct sum of $Q(A)$ and $Q(A)/A_{\mathfrak{p}}$, $\mathfrak{p} \in Ht_1(A)$ by [2], Prop. 2.3, 2.4, 2.5 and 2.6. By the assumption, $Q(A) \otimes B = Q(B)$ and $Q(A)/A_{\mathfrak{p}} \otimes B = Q(B)/B_{\mathfrak{p}}$ is an injective $B_{\mathfrak{p}}$-module because $B_{\mathfrak{p}}$ is a Dedekind domain. In particular, $Q(A) \otimes B$ and $Q(A)/A_{\mathfrak{p}} \otimes B$ are codivisorial and injective B-modules. Hence $M \otimes B$ is an injective B-module by [2], Prop. 2.7. The last assertion is clear.

4. From now on, we assume that B is always faithfully flat over A.

PROPOSITION 10. *Let M be an A-module.*

(i) *If* $M \otimes B$ *is a codivisorial B-module, then M is a codivisorial A-module.*

(ii) *If* $M \otimes B$ *is a codivisorial and divisorial B-module, then M is a divisorial A-module.*

(iii) *If* $M \otimes B$ *is a codivisorial and injective B-module, then M is an injective A-module.*

The assertions follow from Coroll. to Prop. 4, Prop. 5 and Coroll. to Prop. 9.

PROPOSITION 11. *Suppose that B is integral over A. Let M be a codivisorial A-module. Then*

(i) $D_B(M \otimes B) = D_A(M) \otimes B$. *In particular,* $M \otimes B$ *is a divisorial B-module if and only if M is a divisorial A-module.*

(ii) $E_B(M \otimes B) = E_A(M) \otimes B$. *In particular,* $M \otimes B$ *is an injective B-module if and only if M is an injective A-module.*

The assertions follow from Coroll. to Prop. 4, Prop. 5, [5], Coroll. to Prop. 19, Th. 2, Th. 3 and Prop. 10.

References

[1] H. BASS, Algebraic K-theory, Benjanin, New York, 1968.
[2] I. BECK, Injective modules over a Krull domain, J. Algebra, **17** (1971), 116–131.
[3] N. BOURBAKI, Éléments de mathématique, Algèbre commutative, Chapitre 7, Hermann, Paris, 1965.
[4] R. M. FOSSUM, The divisor class group of a Krull domain, Springer-Verlag, Berlin, Heidelberg-New York, 1973.
[5] M. NISHI and M. SHINAGAWA, Codivisorial and divisorial modules over completely integrally closed domains (I), Hiroshima Math. J., **5** (1975).
[6] M. NISHI and M. SHINAGAWA, Codivisorial and divisorial modules over completely integrally closed domains (II), Hiroshima Math. J., **6** (1975).
[7] O. ZARISKI and P. SAMUEL, Commutative algebra, Vol. II, Van Nostrand, 1960.

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