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On Extremal Sets of Parallel Slits

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Introduction

Let W be a region in the extended z-plane containing the point ∞ and let $\{W_n\}_{n=1}^{\infty}$ be a regular exhaustion of W containing the point ∞, i.e., let W_n be regions such that $\infty \in W_n$, $\overline{W_n} \subset W_{n+1}$, $\cup W_n = W$ and the boundary of each W_n consists of a finite number of disjoint analytic Jordan curves. Let P_n be the unique vertical slit mapping of W_n with the following expansion about ∞:

$$P_n(z) = z + \frac{a_{1,n}}{z} + \cdots .$$

D. Hilbert, P. Koebe and R. Courant showed that P_n converges uniformly on compact subsets of W to a vertical slit mapping P_W, i.e., every component of the boundary of $P_W(W)$ is either a point or a line segment parallel to the imaginary axis. Let $\mathfrak{F}$ be the family of univalent meromorphic functions F on W with the expansion

$$(*) \qquad F(z) = z + \frac{a_1(F)}{z} + \cdots \qquad \text{about } \infty .$$

Then P_W is the unique function minimizing $\operatorname{Re} a_1(F)$ in $\mathfrak{F}$.

P. Koebe [4] showed that the complement $(P_W(W))^c$ of $P_W(W)$ has vanishing area. Therefore, for a region of infinite connectivity, the uniqueness of vertical slit mapping with the expansion (*) does not always hold. In 1918, P. Koebe [5] called $P_W(W)$ the minimal vertical slits region. For an arbitrary plane region W containing ∞, the univalent meromorphic mapping of W with the expansion (*) onto a minimal vertical slits region is uniquely determined. In the present paper we shall study the complements of minimal vertical slits regions. We call them extremal sets of vertical slits and denote their class by $\mathscr{E}$. P. Koebe [5] obtained the following results:

(I) E is a set of class $\mathscr{E}$ if and only if E is a bounded closed set such that $\int_{E^c} \partial f/\partial y \, dxdy = 0$ for every $f \in \mathbf{M}(E^c)$ which vanishes identically on a neighborhood of ∞, where $\mathbf{M}(E^c)$ denotes the class of Royden functions on E^c (see § 2).

(i) Every set of class $\mathscr{E}$ has vanishing area.

(ii) If the projection into the real axis of a bounded closed set E has vanishing linear measure, then E is of class $\mathscr{E}$.

P. Koebe [5] conjectured that the converse of (ii) is true, but H. Grötzsch [2] established a new characterization of extremal sets of vertical slits and constructed an example of extremal set of vertical slits such that the projection into the real axis is an interval. Grötzsch's characterization is expressed by using extremal length as follows:

(II) E is a set of class $\mathscr{E}$ if and only if E is a bounded closed set such that $\lambda(\Gamma_{R-E})=\lambda(\Gamma_R)$ for some open rectangle $R\supset E$ with horizontal and vertical sides, where Γ_R (resp. Γ_{R-E}) denotes the family of locally rectifiable curves joining horizontal sides of R in R (resp. $R-E$).

From (II) we see (cf. L. Sario and K. Oikawa [8, Theorem IX 4A]) that

(iii) If E is a set of class $\mathscr{E}$, then every two points z_1, $z_2\in E^c$ with $\mathrm{Re}\, z_1=\mathrm{Re}\, z_2$ can be joined in E^c by a curve whose length is arbitrarily close to $|z_1-z_2|$.

Now we present the properties of the class $\mathscr{E}$:

(a) If E is a set of class $\mathscr{E}$, then so is any closed subset of E.

(b) If $E_1,\dots, E_n$ are mutually disjoint sets of class $\mathscr{E}$, then so is their union.

(c) If E is a set of class $\mathscr{E}$, then so is its image under any affine transformation $x+iy\to ax+iby+c$ with real b and Re $ab\neq 0$.

All of these properties follow from (I) immediately.

L. Sario and K. Oikawa [8] posed the following question: Weaken the assumptions of the properties (b) and (c).

In the present paper we shall be concerned with this problem. It is known that

(d) E and $iE=\{iz|z\in E\}$ are sets of class $\mathscr{E}$ if and only if E is of class N_D, i.e., E is removable with respect to analytic functions with finite Dirichlet integral.

If E_n, $n=1, 2,\dots$ are sets of class N_D and if the union $\cup E_n$ is bounded and closed, then $\cup E_n$ is of class N_D (cf. L. Sario and M. Nakai [7, pp. 371–372]). Therefore it is plausible that the same is true for sets of class $\mathscr{E}$. But, by constructing examples, we shall show that if the assumption of finiteness or disjointness is removed in (b), then the conclusion does not necessarily hold. In the last section § 6, we deal with the property (c) and improve the result obtained in [6].

§ 1. Union of a countable number of bounded closed sets

Let $\mathscr{S}$ be a class of bounded closed sets in the extended z-plane satisfying the following conditions:

($\mathscr{S}$. 1) If E is a set of class $\mathscr{S}$, then so is any closed subset of E.

($\mathscr{S}$. 2) Let E be a bounded closed set and let S_1 and S_2 be open squares with horizontal and vertical sides such that $S_1 \cap S_2 \neq \phi$. If $E \cap \overline{S_j} \in \mathscr{S}$, $j=1, 2$, then $E \cap \overline{S_1 \cup S_2} \in \mathscr{S}$.

It is easy to show that if $E_1,\dots, E_n$ are mutually disjoint sets of class $\mathscr{S}$, then $\cup_{j=1}^n E_j \in \mathscr{S}$. For any bounded closed set E in the extended z-plane we define the closed subset $k_{\mathscr{S}}(E)$ of E by

$$k_{\mathscr{S}}(E) = \{z \in E | E \cap \overline{S(z, r)} \notin \mathscr{S} \quad \text{for every positive number } r\},$$

where $S(z, r)$ denotes the open square with horizontal and vertical sides of length r and center at z. Then

(k. i) $E_1 \subset E_2$ implies $k_{\mathscr{S}}(E_1) \subset k_{\mathscr{S}}(E_2)$.

(k, ii) $k_{\mathscr{S}}(E) = \phi$ if and only if $E \in \mathscr{S}$.

THEOREM 1.1. *The following four conditions are equivalent*:

(i) $k_{\mathscr{S}}(k_{\mathscr{S}}(E)) = k_{\mathscr{S}}(E)$ *for any bounded closed set* E.

(ii) *If* $E_n \in \mathscr{S}$, $n=1, 2,\dots$ *and if* $\cup_{n=1}^{\infty} E_n$ *is a bounded closed set, then* $\cup_{n=1}^{\infty} E_n \in \mathscr{S}$.

(iii) *Let* E_0 *be a set of class* $\mathscr{S}$ *and let* $\{W_n\}_{n=1}^{\infty}$ *be a regular exhaustion of the complement* E_0^c *of* E_0, *i.e., let* W_n *be open sets such that* $\overline{W_n} \subset W_{n+1}$, $\cup W_n = E_0^c$ *and the boundary* ∂W_n *of each* W_n *consists of a finite number of disjoint analytic Jordan curves. If* E *is a bounded closed set such that* $E \cap \overline{W_n} \in \mathscr{S}$, $n=1, 2,\dots$, *then* $E \in \mathscr{S}$.

(iv) (1) *Let* E_0 *be a set of class* $\mathscr{S}$ *and let* $\{W_n\}_{n=1}^{\infty}$ *be a regular exhaustion of* E_0^c. *If* E_n, $n=1, 2,\dots$ *are sets of class* $\mathscr{S}$ *such that* $E_n \subset W_n - \overline{W_{n-1}}$, *then* $\cup_{n=1}^{\infty} E_n \in \mathscr{S}$.

Furthermore,

(2) *Let* E_0 *be a set of class* $\mathscr{S}$ *and let* $\{W_n\}_{n=1}^{\infty}$ *be a regular exhaustion of* E_0^c. *If* E *is a bounded closed set such that* $\bar{c} \cap E_0 \neq \phi$ *for each component* c *of* $E - E_0$ *and* $E \cap \overline{W_n} \in \mathscr{S}$, $n=1, 2,\dots$, *then* $E \in \mathscr{S}$.

PROOF. It is trivial that (ii) implies (iii) and (iii) implies (iv). To prove that (i) implies (ii), let E_n, $n=1, 2,\dots$ be sets of class $\mathscr{S}$ such that $\cup_{n=1}^{\infty} E_n$ is bounded and closed. Set $K = k_{\mathscr{S}}(\cup E_n)$ and $K_n = E_n \cap K$. Assume that $K \neq \phi$.

By the Baire category theorem there is at least one K_n which contains a point z such that $K \cap S(z, r) \subset K_n$ for a positive number r. Since $K_n \in \mathscr{S}$, we have $z \notin k_{\mathscr{S}}(K)$, so that

$$k_{\mathscr{S}}(k_{\mathscr{S}}(\cup E_n)) = k_{\mathscr{S}}(K) \subsetneqq K = k_{\mathscr{S}}(\cup E_n).$$

Therefore (i) implies $\cup E_n \in \mathscr{S}$.

To prove that (iii) implies (i), assume that (iii) is true and there is a bounded closed set E such that $k_{\mathscr{S}}(k_{\mathscr{S}}(E)) \subsetneqq k_{\mathscr{S}}(E)$. Let $z \in k_{\mathscr{S}}(E) - k_{\mathscr{S}}(k_{\mathscr{S}}(E))$, let $S(z, r)$ be a square such that $k_{\mathscr{S}}(E) \cap \overline{S(z, r)} \in \mathscr{S}$ and let $F = E \cap \overline{S(z, r)}$. Then $F \cap k_{\mathscr{S}}(E) = k_{\mathscr{S}}(E) \cap \overline{S(z, r)} \in \mathscr{S}$ and $F - (F \cap k_{\mathscr{S}}(E)) \subset F - k_{\mathscr{S}}(F)$. Therefore (iii) implies

$$F = (F \cap k_{\mathscr{S}}(E)) \cup (F - (F \cap k_{\mathscr{S}}(E))) \in \mathscr{S},$$

so that $z \notin k_{\mathscr{S}}(E)$. This is a contradiction.

To prove that (iv) implies (iii), let E_0 be a set of class $\mathscr{S}$ and let $\{W_n\}$ be a regular exhaustion of E_0^c. Assume that E is a bounded closed set such that $E \cap \overline{W}_n \in \mathscr{S}$, $n = 1, 2, \dots$ and let C be the union of components c of $E - E_0$ such that $\bar{c} \cap E_0 \neq \phi$. Then (2) of (iv) implies that the bounded closed set $E_0' = E_0 \cup C$ is of class $\mathscr{S}$. Since each component c of $E - E_0'$ satisfies $c = \bar{c}$ and $c \cap E_0' = \phi$, there exists a regular exhaustion $\{W_n'\}$ of E_0' such that $(E - E_0') \subset \cup_{n=1}^{\infty} (W_n' - \overline{W_{n-1}'})$. Hence, by (1) of (iv), we have $E \in \mathscr{S}$.

We now give examples of classes of bounded closed sets satisfying ($\mathscr{S}$. 1) and ($\mathscr{S}$. 2).

EXAMPLE 1.2. Let μ^* be a Carathéodory outer measure and let $\mathscr{S}$ be a class of bounded closed sets E such that $\mu^*(E) = 0$. Then ($\mathscr{S}$. 1) and (ii) of Theorem 1.1 are satisfied. In particular, the classes of bounded closed sets of Hausdorff h-measure zero and the classes of bounded closed sets of generalized capacity zero satisfy ($\mathscr{S}$. 1) and (ii) of Theorem 1.1 (cf. L. Carleson [1]).

EXAMPLE 1.3. The class of totally disconnected bounded closed sets satisfies ($\mathscr{S}$. 1) and (iv) of Theorem 1.1. The class of sets of vertical slits, i.e., the class of bounded closed sets E such that each component of E is either a point or a line segment parallel to the imaginary axis satisfies also ($\mathscr{S}$. 1) and (iv) of Theorem 1.1.

EXAMPLE 1.4. The classes N_B and N_D satisfy ($\mathscr{S}$. 1) and (iv) of Theorem 1.1 (cf. L. Sario and M. Nakai [7, pp. 371-372]).

EXAMPLE 1.5. The classes N_{SD} and N_p satisfy ($\mathscr{S}$. 1) and ($\mathscr{S}$. 2), but do not satisfy (iv) of Theorem 1.1 (cf. N. Suita [9] and D. A. Hejhal [3]).

In this paper, we shall denote by $\mathscr{L}$ the class of bounded closed sets E such that the projection of each E into the real axis has vanishing linear measure. This class $\mathscr{L}$ satisfies ($\mathscr{S}$. 1) and (ii) of Theorem 1.1.

§ 2. Extremal sets of vertical slits

Let $\mathbf{M}(W)$ be the class of Royden functions on a plane region W (cf. L. Sario and M. Nakai [7, Chap. III]), i.e., let $\mathbf{M}(W)$ be the class of functions f on W satisfying the following conditions:

(**M**. 1) f is bounded on W.
(**M**. 2) f is a continuous Tonelli function on W.
(**M**. 3) The Dirichlet integral $D_W(f)$ of f over W is finite.

Let U be a regular region in the extended $(z=x+iy)$-plane, i.e., let U be a region whose boundary ∂U of U consists of a finite number of analytic Jordan curves and let E be a bounded closed set contained in U. We denote by $\mathbf{M}(\overline{U}-E, y)$ the class of functions f such that $f\in\mathbf{M}(V_f-E)$ for some region $V_f\supset\overline{U}$ and $f|\partial U=y$. Let

$$d(U-E)=\inf_{f\in\mathbf{M}(\overline{U}-E,y)} D_{U-E}(f).$$

It is known that there is a unique function $f_0\in\mathbf{M}(\overline{U}-E, y)$ such that $d(U-E)=D_{U-E}(f_0)$ and $f_0=0$ on bounded components of E^c. The function f_0 is harmonic on $U-E$. We denote it by $L_{0(U-E)}(y)$.

Let $\mathscr{E}$ be the class of extremal sets of vertical slits. From the condition (I), the next lemma immediately follows:

LEMMA 2.1. *Let E be a bounded closed set in the extended $(z=x+iy)$-plane. Then the following conditions are equivalent:*

(i) *E is of class $\mathscr{E}$.*

(ii) *$\int_{E^c}\partial f/\partial y\,dxdy=0$ for every bounded C^1-function f on E^c with finite Dirichlet integral which vanishes identically on a neighborhood of ∞.*

(iii) *$\int_{E^c}\partial f/\partial y\,dxdy=0$ for every $f\in\mathbf{M}(E^c)$ which vanishes identically on a neighborhood of ∞.*

(iv) *$L_{0(U-E)}(y)=y$ for some regular region U containing E.*

(v) *$d(U-E)=d(U)$ for some regular region U containing E.*

The property (a) implies that $\mathscr{E}$ satisfies ($\mathscr{S}$. 1). To see that $\mathscr{E}$ satisfies ($\mathscr{S}$. 2), it is sufficient to show the lemmas below.

LEMMA 2.2. *Let E be a set in the $(z=x+iy)$-plane such that $E^+=E\cap\{z|x\geqq 0\}\in\mathscr{E}$ and $E^-=E\cap\{z|x\leqq 0\}\in\mathscr{E}$. Then $E\in\mathscr{E}$.*

PROOF. For any $\varepsilon>0$, let ω_ε be a C^∞-function on the space $\mathbf{R}$ of real numbers such that $0\leqq\omega_\varepsilon\leqq 1$ on $\mathbf{R}$, $\omega_\varepsilon(t)=0(t\leqq 0)$ and $\omega_\varepsilon(t)=1(t\geqq\varepsilon)$. Let f be a Royden function on E^c vanishing identically on a neighborhood of ∞. Set $f_\varepsilon(z)=f(z)\omega_\varepsilon(x)+f(z)\omega_\varepsilon(-x)$. Then $f(z)\omega_\varepsilon(x)\in\mathbf{M}(E^{+c})$ and $f(z)\omega_\varepsilon(-x)\in\mathbf{M}(E^{-c})$. Since

$$\int_{E^c}\frac{\partial f_\varepsilon}{\partial y}dxdy=\int_{E^{+c}}\frac{\partial(f(z)\omega_\varepsilon(x))}{\partial y}dxdy+\int_{E^{-c}}\frac{\partial(f(z)\omega_\varepsilon(-x))}{\partial y}dxdy$$
$$=0$$

and

$$\left|\int_{E^c}\frac{\partial f}{\partial y}dxdy-\int_{E^c}\frac{\partial f_\varepsilon}{\partial y}dxdy\right|^2$$
$$\leqq\int_{E^c\cap\{-\varepsilon\leqq x\leqq\varepsilon\}}\left(\frac{\partial f}{\partial y}\right)^2dxdy\int_{S(f)\cap\{-\varepsilon\leqq x\leqq\varepsilon\}}dxdy\longrightarrow 0\qquad(\varepsilon\to 0),$$

where $S(f)$ denotes the support of f, we deduce

$$\int_{E^c}\frac{\partial f}{\partial y}dxdy=0.$$

Therefore $E\in\mathscr{E}$.

LEMMA 2.3. *Let E be a set on the $(z=x+iy)$-plane such that $E_a^+=E\cap\{z|y\geqq -a\}\in\mathscr{E}$ and $E_a^-=E\cap\{z|y\leqq a\}\in\mathscr{E}$ for a positive number a. Then $E\in\mathscr{E}$.*

PROOF. Let f be a Royden function on E^c vanishing identically on a neighborhood of ∞ and let ω_a be a C^∞-function defined in the proof of Lemma 2.2. Then

$$\int_{E^c}\frac{\partial f}{\partial y}dxdy=\int_{(E_a^+)^c}\frac{\partial f(z)\omega_a(y)}{\partial y}dxdy+\int_{(E_a^-)^c}\frac{\partial f(z)(1-\omega_a(y))}{\partial y}dxdy$$
$$=0,$$

and hence $E\in\mathscr{E}$.

§3. Examples

In this section we construct examples of bounded closed sets which are countable unions of sets of class $\mathscr{E}$, yet are not of class $\mathscr{E}$.

EXAMPLE 3.1. Let $e=e(\{a_k\},\{n_k\})=\bigcap_{k=0}^{\infty}e_k$ be a generalized Cantor set contained in the interval [0, 1] on the real axis which has positive length and is a set of class $\mathscr{E}$ (cf. L. Sario and K. Oikawa [8, pp. 229–235]). We denote by x_k the length of $e_{k-1}-e_k$, where $k\geqq 1$. Since e has positive length, we have $\sum_{k=1}^{\infty}x_k<1$. Let T_k, $k=0,1,\dots$ be the sets of end points of e_k and let y_k be numbers such that $y_0=1$, $0<y_k\leqq 1$, $k=1,2,\dots$ and $y_k\to 0(k\to\infty)$. Set $E_0=\{z=x+iy|x\in T_0, 0\leqq y\leqq 1\}$ and $E_k=\{z=x+iy|x\in T_k-T_{k-1}, 0\leqq y\leqq y_k\}$. Then $E=e\cup\bigcup_{k=0}^{\infty}E_k$ is bounded and closed. In the following we shall show that if $\{y_k\}$ satisfies $\sum_{k=1}^{\infty}(x_k/y_k)<1$, then E is not of class $\mathscr{E}$. Let $U=\{z||z-(1+i)/2|<2\}$ and define a function f as follows:

$$(**)\qquad f(z)=\begin{cases} y, & z=x+iy\in U-\overline{S((1+i)/2,1)} \\ y/y_k, & x\in e_{k-1}-e_k,\ 0\leqq y\leqq y_k,\ k=1,2,\dots \\ 1 & \text{elsewhere}. \end{cases}$$

Then $f\in\mathbf{M}(\bar{U}-E, y)$ and $D_{U-E}(y)-D_{U-E}(f)=1-\sum_{k=1}^{\infty}(x_k/y_k)>0$. Hence $d(U-E)<d(U)$, and so $E\notin\mathscr{E}$.

For every non-negative number δ, we define an open set Δ_δ by

$$\Delta_\delta=\{z||z|<1\}-\{z=x+iy|x=0, |y|\geqq\delta\}.$$

To construct Example 3.3 below we prepare the following lemma:

LEMMA 3.2. *For any $f\in\mathbf{M}(\Delta_0)$ and for any positive number ε_1 and ε_2 there are a positive number $\delta=\delta(f,\varepsilon_1,\varepsilon_2)$ and a function $g\in\mathbf{M}(\Delta_\delta)$ such that $D_{\Delta_0}(f-g)<\varepsilon_1$ and $g=f$ on $\Delta_0\cap\{z||z|>\varepsilon_2\}$.*

PROOF.*) We define a Royden function $\omega_{\delta,\varepsilon_2}$ on $\{z||z|<1\}$ by

$$\omega_{\delta,\varepsilon_2}(z)=\begin{cases} 0, & |z|<\delta \\ \log(|z|/\delta)/\log(\varepsilon_2/\delta), & \delta\leqq|z|\leqq\varepsilon_2 \\ 1, & \varepsilon_2<|z|<1. \end{cases}$$

Then, for a sufficiently small number δ, $g=f\omega_{\delta,\varepsilon_2}$ is the required function.

EXAMPLE 3.3. Let $E=e\cup\bigcap_{k=0}^{\infty}E_k$ be the set defined in Example 3.1 satisfying $\sum_{k=1}^{\infty}(x_k/y_k)<1$ and let f be the function defined by (**). We may assume that $y_k\neq 1/2^j$, $k=1,2,\dots,j=1,2,\dots$. For a sequence $\{\delta_j\}_{j=1}^{\infty}$ of positive numbers

*) Author's proof was relatively long. This short proof was given by Mr. Y. Mizuta.

we set $F_1=E\cap\{z|y\geqq 1/2+\delta_1\}$, $F_j=E\cap\{z|1/2^j+\delta_j\leqq y\leqq 1/2^{j-1}-\delta_{j-1}\}$, $j=2, 3, \ldots$ and define $E(\{\delta_j\})$ by $E(\{\delta_j\})=e\cup\bigcup_{j=1}^{\infty}F_j$. Then $E(\{\delta_j\})$ is a countable union of mutually disjoint bounded sets of class $\mathscr{E}$. From Lemma 3.2 we know that there are a sequence $\{\delta_j\}$ and a function $g\in\mathbf{M}(\bar{U}-E(\{\delta_j\}), y)$ such that $D_{U-E}(g)<D_{U-E}(f)+(D_U(y)-D_{U-E}(f))/2$. Therefore $E(\{\delta_j\})$ is not of class $\mathscr{E}$ for such a sequence $\{\delta_j\}$.

§4. Characterizations of extremal sets of vertical slits

Let E be a bounded closed set and let U be a regular region containing E. For any $h\in C^1(\partial U)$, we denote by $\mathbf{M}(\bar{U}-E, h)$ the class of functions f such that $f\in\mathbf{M}(V_f-E)$ for some region $V_f\supset\bar{U}$ and $f|\partial U=h$. Let

$$i(U-E, h) = \inf_{f\in\mathbf{M}(\bar{U}-E,h)}\int_{U-E}\left(\frac{\partial f}{\partial y}\right)^2 dxdy\,.$$

A function $g\in\mathbf{M}(\bar{U}-E, h)$ such that

$$\int_{U-E}\left(\frac{\partial g}{\partial y}\right)^2 dxdy = i(U-E, h)$$

does not always exist and is not uniquely determined even if it exists. Since the operator $f\mapsto\partial f/\partial y$ is linear, $\partial\mathbf{M}(\bar{U}-E, h)/\partial y=\{\partial f/\partial y|f\in\mathbf{M}(\bar{U}-E, h)\}$ is convex in the space $L^2(U-E)$. Hence there is a unique function ϕ minimizing the L^2-norm in $\overline{\partial\mathbf{M}(\bar{U}-E, h)/\partial y}$. We call the function ϕ extremal and denote it by $L_{y(U-E)}(h)$. A function $\phi\in L^2(U-E)$ is extremal if and only if $\phi\in\overline{\partial\mathbf{M}(\bar{U}-E, h)/\partial y}$ and $\int_{U-E}\phi\partial f/\partial y\, dxdy=0$ for every $f\in\mathbf{M}(\bar{U}-E, 0)$.

The operator $L_{y(U-E)}$: $h\mapsto\phi$ has the following properties:

(L. i) $L_{y(U-E)}$ is a linear operator of $C^1(\partial U)$ into $L^2(U-E)$.

(L. ii) Let h_1, h_2 be C^1-functions on ∂U such that $h_1=h_2$ on $\partial U\cap\partial V$, where V is a component of $U\cap\{z=x+iy|a<x<b\}$. Then

$$L_{y(U-E)}(h_1)|V-E = L_{y(U-E)}(h_2)|V-E \quad \text{a.e. on } V-E\,.$$

(L. iii) Let V be a component of $U\cap\{z|a<x<b\}$ such that $\partial V\cap\{z=x+iy|x=c, a<c<b\}$ consists of two points whose distance is not less than $d>0$ for every c. Then

$$\int_{V-E}|L_{y(U-E)}(h)|^2dxdy \leqq \frac{4(b-a)}{d}\Big(\sup_{\zeta\in\partial U\cap\partial V}|h(\zeta)|\Big)^2 \qquad (h\in C^1(\partial U))\,.$$

(L. iv) Given $h\in C^1(\partial U)$, define a function l_h on U so that $l_h=h$ on ∂U

and $y \to l_h(c+iy)$ is linear on each component of $U \cap \{z|x=c\}$ for every c. Let V be a component of $U \cap \{z|a<x<b\}$ such that $V \cap \{z|x=c, a<c<b\}$ is connected for every c and $\bar{V} \cap E = \phi$. Then

$$L_{y(U-E)}(h)\,|\,V = L_{yU}(h)\,|\,V = \frac{\partial l_h}{\partial y}\,|\,V \quad \text{a.e.} \quad \text{on } V\,.$$

We shall prove (L. ii) and (L. iv); the other two properties are easily obtained. Let $f_{j,n}$, $j=1, 2$, $n=1, 2,\ldots$, be functions of class $\mathbf{M}(\bar{U}-E, h_j)$ such that

$$\int_{U-E}\left(\frac{\partial f_{j,n}}{\partial y}\right)^2 dxdy \longrightarrow \int_{U-E}\left(\frac{\partial L_{y(U-E)}(h_j)}{\partial y}\right)^2 dxdy \qquad (n\to\infty)\,.$$

Without loss of generality we may assume

$$\int_{V-E}\left(\frac{\partial f_{1,n}}{\partial y}\right)^2 dxdy \leqq \int_{V-E}\left(\frac{\partial f_{2,n}}{\partial y}\right)^2 dxdy,\ n=1, 2,\cdots.$$

Set $g_{n,\varepsilon}(z)=\{1-\omega_\varepsilon(a+\varepsilon-x)-\omega_\varepsilon(x-b+\varepsilon)\}f_{1,n}(z)+\{\omega_\varepsilon(a+\varepsilon-x)+\omega_\varepsilon(x-b+\varepsilon)\}\cdot f_{2,n}(z)$ on $V-E$ and $g_{n,\varepsilon}(z)=f_{2,n}(z)$ on $U-V-E$, $n=1, 2,\ldots$, where ε is a number such that $0<\varepsilon<(b-a)/2$ and ω_ε is the C^∞-function defined in the proof of Lemma 2.2. Then $g_{n,\varepsilon}\in\mathbf{M}(\bar{U}-E, h_2)$ and $\partial g_{n,\varepsilon}/\partial y=\{1-\omega_\varepsilon(a+\varepsilon-x)-\omega_\varepsilon(x-b+\varepsilon)\}\cdot \partial f_{1,n}/\partial y+\{\omega_\varepsilon(a+\varepsilon-x)+\omega_\varepsilon(x-b+\varepsilon)\}\partial f_{2,n}/\partial y$ on $V-E$. Hence for each n there is a number $\varepsilon=\varepsilon(n)$ such that

$$\int_{U-E}\left(\frac{\partial g_{n,\varepsilon}(z)}{\partial y}\right)^2 dxdy \leqq \int_{U-E}\left(\frac{\partial f_{2,n}}{\partial y}\right)^2 dxdy+\frac{1}{n}\,.$$

Therefore

$$L_{y(U-E)}(h_1)\,|\,V-E = \lim_n \frac{\partial g_{n,\varepsilon(n)}}{\partial y}\Big|\,V-E=L_{y(U-E)}(h_2)\,|\,V-E \text{ a.e. on } V-E\,.$$

Thus (L. ii) is proved.

Next we shall prove (L. iv). Choose a' and b' so that $a'<a<b<b'$ and the closure of $U \cap \{z|a'<x<b'\}$ is disjoint from E. Let h' be a function of $C^1(\partial U)$ which is equal to h on $\partial U \cap \partial V$ and to 0 outside of $\{z|a'<x<b'\}$. Then, for any $g\in\mathbf{M}(\bar{U}-E, 0)$ we have

$$\int_{U-E}\frac{\partial l_{h'}}{\partial y}\frac{\partial g}{\partial y}\,dxdy = \int_{U-E}\frac{\partial}{\partial y}\left(\frac{\partial l_{h'}}{\partial y}g\right)dxdy = 0\,.$$

From the characterization of $L_{y(U-E)}$ given above and (L. ii) we infer that

$$L_{y(U-E)}(h)\,|\,V=L_{y(U-E)}(h')\,|\,V=\frac{\partial l_{h'}}{\partial y}\Big|\,V=\frac{\partial l_h}{\partial y}\Big|\,V \quad \text{a.e.} \quad \text{on } V\,.$$

This proves (L. iv).

Now we are ready to show:

THEOREM 4.1. *Let E be a bounded closed set and let U be a regular region containing E. Then the following conditions are equivalent*:

(i) $E \in \mathscr{E}$.
(ii) $L_{y(U-E)}(y)=1$ *a.e. on* $U-E$.
(iii) $i(U-E, y)=i(U, y)$.
(iv) $L_{y(U-E)}(h)=L_{yU}(h)$ *a.e. on* $U-E$ *for every* $h \in C^1(\partial U)$.
(v) $i(U-E, h)=i(U, h)$ *for every* $h \in C^1(\partial U)$.

PROOF. By using the characterization of $L_{y(U-E)}(y)$ given after its definition we see that (iii) of Lemma 2.1 and the present (ii) are equivalent. Hence (i) and (ii) are equivalent. That (ii) and (iii) are equivalent and that (iv) and (v) are equivalent are trivial. The theorem will be proved if we show that (ii) implies (iv). Suppose that (ii) is valid. Let h be a C^1-function on ∂U and let V be a component of $U \cap \{z|a<x<b\}$ such that $\partial V \cap \{z|x=c, a<c<b\}$ consists of two points whose distance is not less than $d>0$ for every c. For any $\varepsilon>0$, let h_ε be a C^1-function on ∂U which satisfies $|h_\varepsilon-h|<\varepsilon$ on $\partial U \cap \partial V$ and is equal to $\alpha_j y+\beta_j$ on $\partial U \cap \partial V \cap \{z|a_j<x<b_j\}$, $j=1, 2,\dots, n$, where a_j and b_j are numbers such that $a=a_1<b_1<\cdots<a_n<b_n=b$ and $\sum_{j=1}^{n-1}(a_{j+1}-b_j)<\varepsilon$. Then, using ($L$. i) to ($L$. iv), we have

$$\int_{V-E}|L_{y(U-E)}(h_\varepsilon)-L_{y(U-E)}(h)|^2 dxdy \leqq \frac{4(b-a)}{d}\varepsilon^2,$$

$$\int_{V-E}|L_{yU}(h_\varepsilon)-L_{yU}(h)|^2 dxdy \leqq \frac{4(b-a)}{d}\varepsilon^2,$$

$$\int_{(V-E)\cap W}|L_{yU}(h_\varepsilon)-L_{y(U-E)}(h_\varepsilon)|^2 dxdy = 0$$

and

$$\int_{(V-E)\cap W^c}|L_{yU}(h_\varepsilon)-L_{y(U-E)}(h_\varepsilon)|^2 dxdy \leqq \frac{16\varepsilon}{d}\Big(\sup_{\partial U \cap \partial V}|h|+\varepsilon\Big)^2,$$

where $W=\cup\{z|a_j<x<b_j\}$. Letting $\varepsilon \to 0$, we have

$$L_{y(U-E)}(h)|V-E = L_{yU}(h)|V-E \quad \text{a.e. on} \quad V-E.$$

This together with (L. iv) gives (iv).

§5. Union of a countable number of extremal sets

We now show the following theorem:

THEOREM 5.1. *Let E_0 be a set of class $\mathscr{L}$ and let $\{W_n\}_{n=1}^{\infty}$ be a regular exhaustion of E_0^c. If E is a bounded closed set such that $E \cap \overline{W}_n \in \mathscr{E}$, $n=1, 2,\dots$, then $E \in \mathscr{E}$.*

PROOF. Let C be the union of components c of $E-E_0$ such that $\bar{c} \cap E_0 \neq \phi$. Then $E_0' = E_0 \cup C$ is of class $\mathscr{L}$. Let $\{W_n'\}_{n=1}^{\infty}$ be a regular exhaustion of $E_0'^c$ such that $(E-E_0') \subset \cup_{n=2}^{\infty}(W_n' - \overline{W_{n-1}'})$ and $E \cap \overline{W}_1' = \phi$. The theorem will be proved if we show

$$\int_{E^c} \frac{\partial f}{\partial y} dxdy = 0$$

for every bounded C^1-function f on E^c with finite Dirichlet integral which vanishes identically on a neighborhood of ∞. Let $U_n = W_n' - \overline{W_{n-1}'}$, $n=2, 3,\dots$. Then $E_n = (E-E_0') \cap U_n \in \mathscr{E}$. By virtue of Theorem 4.1, there are Royden functions f_n on U_n such that $f_n | \partial U_n = f$ and

$$\int_{U_n} \left(\frac{\partial f_n}{\partial y}\right)^2 dxdy < \int_{U_n - E_n} \left(\frac{\partial f}{\partial y}\right)^2 dxdy + \frac{1}{2^n}.$$

Let

$$g(z) = \begin{cases} f_n(z), & z \in \overline{U_n},\ n=2, 3,\dots \\ f(z), & z \in W_1' \end{cases}$$

and

$$g_n(z) = \begin{cases} f(z) - f_n(z), & z \in U_n - E_n \\ 0, & z \in U_n^c. \end{cases}$$

Then $f = g + \sum_{n=2}^{\infty} g_n$, $g_n \in \mathbf{M}(\overline{U}_n - E_n, 0)$, $n=2, 3,\dots$ and $\int_{E_0'^c} (\partial g/\partial y)^2 dxdy < +\infty$. Since $E_n \in \mathscr{E}$ and $E_0' \in \mathscr{L}$, we have

$$\int_{E^c} \frac{\partial f}{\partial y} dxdy = \int_{E_0'^c} \frac{\partial g}{\partial y} dxdy = 0.$$

COROLLARY 5.2. *Let E_1 and E_2 be sets of class $\mathscr{E}$. If $E_1 \cap E_2 \in \mathscr{L}$, then $E_1 \cup E_2 \in \mathscr{E}$.*

LEMMA 5.3. *Let E be a set of class $\mathscr{E}$ contained in $\{z=x+iy | 0<x<a,$*

$h_1<y<h_1+l\}$ where a, h_1 and l are positive numbers. For any function $f\in\mathbf{M}(\bar{R}-E)$, where $R=\{z=x+iy|0<x<a, 0<y<h_1+l+h_2\}$ and $h_2>0$, such that $f(iy)$ and $f(a+iy)$ and linear functions in y, there is a function $g\in\mathbf{M}(\bar{R})$ such that $g=f$ on ∂R and

$$D_R(g) \leqq \left(1+\frac{h_1+l+h_2}{\min(h_1,h_2)}\right)D_{R-E}(f).$$

PROOF. Let f_n, $n=1, 2,\dots$ be bounded C^1-functions on $\bar{R}-E$ with finite Dirichlet integral such that $\sup_{R-E}|f_n-f|\to 0$ $(n\to\infty)$ and $D_{R-E}(f_n-f)\to 0$ $(n\to\infty)$. Then $h_{f_n}\to h_f$ $(n\to\infty)$ and $D_R(h_f)\leqq \underline{\lim}\, D_R(h_{f_n})$, where h_f (resp. h_{f_n}) denotes the function continuous on $\bar{R}$ and harmonic on R such that $h_f=f$ on ∂R (resp. $h_{f_n}=f_n$ on ∂R). Since $D_R(h_{f_n})\leqq D_R(g_n)$ for every $g_n\in\mathbf{M}(\bar{R})$ such that $g_n=f_n$ on ∂R, to show the lemma we may assume that f is a C^1-function on $\bar{R}-E$. Let y_1 be a number such that $0\leqq y_1\leqq h_1$ and

$$\int_0^a\left\{\left(\frac{\partial f}{\partial x}(x+iy_1)\right)^2+\left(\frac{\partial f}{\partial y}(x+iy_1)\right)^2\right\}dx$$

$$=\min_{0\leqq y\leqq h_1}\int_0^a\left\{\left(\frac{\partial f}{\partial x}(x+iy)\right)^2+\left(\frac{\partial f}{\partial y}(x+iy)\right)^2\right\}dx.$$

Then

$$\int_0^a\left(\frac{\partial f}{\partial x}(x+iy_1)\right)^2dx \leqq D_{R_1}(f)/h_1,$$

where $R_1=\{z=x+iy|0<x<a, 0<y<h_1\}$. Let y_2 be a number such that $h_1+l\leqq y_2\leqq h_1+l+h_2$ and

$$\int_0^a\left(\frac{\partial f}{\partial x}(x+iy_2)\right)^2dx \leqq D_{R_2}(f)/h_2,$$

where $R_2=\{z=x+iy|0<x<a, h_1+l<y<h_1+l+h_2\}$. Set

$$g(z)=\begin{cases} f(z), & 0<y<y_1 \quad\text{or}\quad y_2<y<h_1+l+h_2\\ \dfrac{y-y_1}{y_1-y_2}f(x+iy_2)+\dfrac{y_2-y}{y_2-y_1}f(x+iy_1), & y_1\leqq y\leqq y_2.\end{cases}$$

Then $g\in\mathbf{M}(\bar{R})$, $g=f$ on ∂R,

$$\left(\int_0^a\left(\frac{\partial g}{\partial x}(x+iy)\right)^2dx\right)^{1/2}$$

$$\leqq\frac{y-y_1}{y_2-y_1}\left(D_{R_2}(f)/h_2\right)^{1/2}+\frac{y_2-y}{y_2-y_1}\left(D_{R_1}(f)/h_1\right)^{1/2}$$

$$\leqq\left(D_{R-E}(f)/\min(h_1, h_2)\right)^{1/2},\quad y_1\leqq y\leqq y_2$$

and

$$\int_{R_0}\left(\frac{\partial g}{\partial x}\right)^2 dxdy \leqq \frac{h_1+l+h_2}{\min(h_1,\, h_2)} D_{R-E}(f)\,,$$

where $R_0=\{z=x+iy|0<x<a,\ y_1<y<y_2\}$. Since $E\in\mathscr{E}$, by Theorem 4.1, we have $i(R_0, g)=i(R_0-E, f)$. Hence (L. iv) implies that

$$\int_{R_0}\left(\frac{\partial g}{\partial y}\right)^2 dxdy \leqq \int_{R_0-E}\left(\frac{\partial f}{\partial y}\right)^2 dxdy \leqq D_{R_0-E}(f)\,.$$

Therefore

$$D_R(g) \leqq D_{R-E}(f)+\frac{h_1+l+h_2}{\min(h_1,\, h_2)} D_{R-E}(f)\,.$$

The following theorem will be used to construct Example 5.5.

THEOREM 5.4. *Let $\{a_n\}$ be a monotone decreasing sequence of positive numbers such that $a_0=3/2$, $a_1=1$, $a_n\to 0$ as $n\to\infty$ and*

$$\varliminf_n \frac{\min(a_{2n}-a_{2n+1},\, a_{2n-2}-a_{2n-1})}{a_{2n-2}-a_{2n+1}} > 0\,.$$

If E_n, $n=0, 1, 2,\ldots$ are sets of class $\mathscr{E}$ satisfying $E_0\subset\{z=x+iy|0\leqq x\leqq 1,\ y=0\}$ and $E_n\subset\{z=x+iy|0\leqq x\leqq 1,\ a_{2n}\leqq y\leqq a_{2n-1}\}$, $n=1, 2,\ldots$, and if $E=\cup E_n$ is bounded and closed, then $E\in\mathscr{E}$.

PROOF. It is sufficient to show that

$$\int_{E^c}\frac{\partial f}{\partial y}dxdy = 0$$

for every bounded C^1-function f on E^c with finite Dirichlet integral which vanishes identically on the complement of the square $S_0=\{z|-1<x<2,\ -1<y<2\}$. Let

$$R_n = \{z|-1 < x < 2,\ (a_{2n}+a_{2n+1})/2 < y < (a_{2n-2}+a_{2n-1})/2\}\,.$$

Then, by Lemma 5.3, there are functions $f_n\in\mathbf{M}(\overline{R_n})$ such that $f_n=f$ on ∂R_n and

$$D_{R_n}(f_n) \leqq \left(1+\frac{2(a_{2n-2}-a_{2n+1})}{\min(a_{2n}-a_{2n+1},\, a_{2n-2}-a_{2n-1})}\right) D_{R_n-E_n}(f)\,.$$

Set

$$g(z) = \begin{cases} f_n(z), & z\in R_n,\quad n = 1, 2,\ldots \\ f(z), & z\notin \cup R_n \end{cases}$$

and

$$g_n(z) = \begin{cases} f(z)-f_n(z), & z \in R_n - E_n \\ 0, & z \in R_n^c. \end{cases}$$

Then $f=g+\sum_{n=1}^{\infty}g_n$, $g_n \in \mathbf{M}(\overline{R_n}-E_n, 0)$, $n=1, 2,\dots$ and $g \in \mathbf{M}(\overline{S_0}-E_0, 0)$. Since $E_n \in \mathscr{E}$, $n=0, 1, 2,\dots$, we have

$$\int_{E^c} \frac{\partial f}{\partial y} dxdy = \int_{E_0^c} \frac{\partial g}{\partial y} dxdy + \sum_n \int_{E_n^c} \frac{\partial g_n}{\partial y} dxdy = 0.$$

EXAMPLE 5.5. Let F_j, $j=1, 2,\dots$ be the sets defined in Example 3.3. Set $E_1 = e \cup \bigcup_{k=0}^{\infty} F_{2k+1}$ and $E_2 = e \cup \bigcup_{k=1}^{\infty} F_{2k}$. Then $E_1 \cap E_2 = e$ and, from Theorem 5.4, we know that E_1 and E_2 are sets of class $\mathscr{E}$. But we have shown in Example 3.3 that $E_1 \cup E_2 = E(\{\delta_j\})$ is not always of class $\mathscr{E}$.

Finally, we give another sufficient condition for a countable union of sets of class $\mathscr{E}$ to be again of class $\mathscr{E}$.

LEMMA 5.6. *Let E be a closed set contained in $\{z \,|\, |z|<r, r<1\}$ and let Δ be the unit disc $\{z \,|\, |z|<1\}$. Then for any $f \in \mathbf{M}(\bar{\Delta}-E)$ there is a function $g \in \mathbf{M}(\bar{\Delta})$ such that $g=f$ on $\{z \,|\, |z|=1\}$ and*

$$D_\Delta(g) \leqq \left(1+\frac{3}{1-r}\right) D_{\Delta-E}(f).$$

PROOF. As in the proof of Lemma 5.3, we assume that f is a C^1-function on $\bar{\Delta}-E$. Let ρ_0 be a number such that $r \leqq \rho_0 \leqq 1$ and

$$\int_0^{2\pi} \left\{ \left(\frac{\partial f}{\partial \rho}(\rho_0 e^{i\theta})\right)^2 + \left(\frac{1}{\rho_0}\frac{\partial f}{\partial \theta}(\rho_0 e^{i\theta})\right)^2 \right\} \rho_0 d\theta$$
$$= \min_{r \leqq \rho \leqq 1} \int_0^{2\pi} \left\{ \left(\frac{\partial f}{\partial \rho}(\rho e^{i\theta})\right)^2 + \left(\frac{1}{\rho}\frac{\partial f}{\partial \theta}(\rho e^{i\theta})\right)^2 \right\} \rho d\theta.$$

Then

$$\int_0^{2\pi} \left(\frac{\partial f}{\partial \theta}(\rho_0 e^{i\theta})\right)^2 d\theta \leqq \frac{D_{\Delta-E}(f)}{1-r}$$

and

$$|f(\rho_0 e^{i\theta}) - f(\rho_0 e^{i\theta'})|^2 \leqq \left(\int_\theta^{\theta'} \left|\frac{\partial f}{\partial \theta}(\rho_0 e^{i\theta})\right| d\theta\right)^2$$
$$\leqq |\theta-\theta'| \frac{D_{\Delta-E}(f)}{1-r}.$$

Hence

$$|f(\rho_0 e^{i\theta})-a|^2 \leqq \frac{\pi}{4}\frac{D_{\Delta-E}(f)}{1-r}$$

for every θ, where $a=(\min_\theta f(\rho_0 e^{i\theta})+\max_\theta f(\rho_0 e^{i\theta}))/2$. Set

$$g(\rho e^{i\theta}) = \begin{cases} \frac{\rho}{\rho_0} f(\rho_0 e^{i\theta}) + \left(1-\frac{\rho}{\rho_0}\right)a, & 0 \leqq \rho \leqq \rho_0 \\ f(\rho e^{i\theta}), & \rho_0 < \rho \leqq 1. \end{cases}$$

Then $g \in \mathbf{M}(\bar{\Delta})$ and

$$\begin{aligned} D_{\{|z|\leqq\rho_0\}}(g) &= \frac{1}{\rho_0^2}\int_0^{\rho_0}\rho d\rho \int_0^{2\pi}\left\{(f(\rho_0 e^{i\theta})-a)^2+\left(\frac{\partial f}{\partial\theta}(\rho_0 e^{i\theta})\right)^2\right\}d\theta \\ &\leqq \frac{1}{2}\left(\frac{\pi^2}{2}+1\right)\frac{D_{\Delta-E}(f)}{1-r} \\ &\leqq \frac{3}{1-r}D_{\Delta-E}(f). \end{aligned}$$

Therefore

$$D_\Delta(g) \leqq \left(1+\frac{3}{1-r}\right)D_{\Delta-E}(f).$$

Let R be a doubly connected plane region. If R is conformally equivalent to an annulus $\{z|1<|z|<\mu\}$ we call μ the modulus of R and denote it by $\mu(R)$.

THEOREM 5.7. *Let E_n, $n=0, 1,\ldots$ be mutually disjoint sets of class $\mathscr{E}$ such that $\cap_{m=1}^{\infty}\overline{\cup_{n=m}^{\infty}E_n}\subset E_0$. If there are two sequences $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ of simply connected regions such that $U_m \cap U_n=\phi (m\neq n)$, $E_n\subset \overline{V_n}\subset U_n$, $n=1, 2,\ldots$ and $\varliminf_n \mu(U_n-\overline{V_n})>1$, then $E=\cup_{n=0}^{\infty}E_n \in \mathscr{E}$.*

PROOF. For every $n\geqq 1$, let F_n be a conformal mapping of U_n onto $\{w||w|<1\}$ such that $\{w|r<|w|<1\}\subset F_n(U_n-\overline{V_n})$, where $r>0$ is independent of n (cf. L. Sario and K. Oikawa [8, pp. 201–204]). We denote by F_n^{-1} the continuous extension of the inverse function of F_n onto $\{w||w|\leqq 1\}$. Let f be a bounded C^1-function on E^c with finite Dirichlet integral vanishing identically on a neighborhood of ∞. Then, by Lemma 5.6, there are functions $f_n \in \mathbf{M}(\{w||w|\leqq 1\})$ such that $f_n=f\circ F_n^{-1}$ on $\{w||w|=1\}$ and

$$D_{\{|w|<1\}}(f_n) \leqq \left(1+\frac{3}{1-r}\right)D_{F_n(U_n-E_n)}(f\circ F_n^{-1})$$

$$=\left(1+\frac{3}{1-r}\right)D_{U_n-E_n}(f).$$

Set

$$g(z)=\begin{cases}(f_n\circ F_n)(z), & z\in U_n,\quad n=1,2,\dots\\ f(z), & z\in E_0^c-\cup_{n=1}^{\infty}U_n\end{cases}$$

and

$$g_n(z)=\begin{cases}f(z)-(f_n\circ F_n)(z), & z\in U_n-E_n\\ 0, & z\in U_n^c.\end{cases}$$

Then $f=g+\sum_{n=1}^{\infty}g_n$, $g_n\in\mathbf{M}(\overline{U_n}-E_n, 0)$, $n=1, 2,\dots$ and g is a Royden function on E_0^c which vanishes identically on a neighborhood of ∞. Since $E_n\in\mathscr{E}$, $n=0, 1, 2,\dots$, we have

$$\int_{E^c}\frac{\partial f}{\partial y}dxdy=\int_{E_0^c}\frac{\partial g}{\partial y}dxdy+\sum_n\int_{E_n^c}\frac{\partial g_n}{\partial y}dxdy=0.$$

Hence $E\in\mathscr{E}$.

§6. Subboundaries of the image regions under quasiconformal mappings

In this section, we shall be concerned with the property (c) of extremal sets of vertical slits and improve the result obtained in [6]. From Lemma 2.1 the next lemma immediately follows:

LEMMA 6.1. *A bounded closed set E is of class $\mathscr{E}$ if and only if*

$$\int_{E^c}\frac{\partial f}{\partial y}dxdy=0$$

for every bounded Tonelli function f on E^c with finite Dirichlet integral which vanishes a.e. on the complement of an arbitrary fixed bounded region containing E.

THEOREM 6.2. *Let E be a set of class $\mathscr{E}$ in the $(z=x+iy)$-plane and let U be a region containing E. Let $\phi=u+iv$ be a quasiconformal mapping of $U-E$ into the $(w=u+iv)$-plane such that*

(q. i) $\partial u/\partial y=0$ *a.e.* on $U-E$,

(q. ii) $\partial u/\partial x$ *and* $\partial v/\partial y$ *are bounded Tonelli functions on $U-E$ with finite Dirichlet integral,*

(q. iii) $\partial x/\partial u$ *and* $\partial y/\partial v$ *are bounded Tonelli functions on $\phi(U-E)$ with finite Dirichlet integral.*

Then $\partial(\phi(V-E))-\phi(\partial V)\in\mathscr{E}$, *where* V *is a subregion of* U *such that* $E\subset V\subset\bar{V}\subset U$.

PROOF. It is sufficient to show

$$\int_{\phi(V-E)}\frac{\partial f}{\partial v}dudv=0$$

for every bounded C^1-function f on $\phi(\bar{V}-E)$ with finite Dirichlet integral which vanishes on $\phi(\partial V)$. From the assumption we know that $\partial x/\partial v=0$ a.e. on $\phi(U-E)$, $\partial\{((\partial y/\partial v)\circ\phi)(\partial u/\partial x)(\partial v/\partial y)\}/\partial y=0$ a.e. on $U-E$ and $(f\circ\phi)((\partial y/\partial v)\circ\phi)(\partial u/\partial x)(\partial v/\partial y)$ is a bounded Tonelli function on $V-E$ with finite Dirichlet integral (cf. L. Sario and M. Nakai [7], Chap. III, § 3). We may assume $D(u,v)/D(x,y)>0$ a.e. on $U-E$. By virtue of Lemma 6.1 we have

$$\begin{aligned}\int_{\phi(V-E)}\frac{\partial f}{\partial v}dudv &= \int_{V-E}\left\{\frac{\partial(f\circ\phi)}{\partial x}\left(\frac{\partial x}{\partial v}\circ\phi\right)+\frac{\partial(f\circ\phi)}{\partial y}\left(\frac{\partial y}{\partial v}\circ\phi\right)\right\}\frac{D(u,v)}{D(x,y)}dxdy\\ &= \int_{V-E}\frac{\partial}{\partial y}\left\{(f\circ\phi)\left(\frac{\partial y}{\partial v}\circ\phi\right)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right\}dxdy\\ &= 0.\end{aligned}$$

This completes the proof.

COROLLARY 6.3 ([6]). *Let* E *be a set of class* $\mathscr{E}$ *in the* $(z=x+iy)$*-plane and let* U *be a region containing* E. *If* ϕ *is a diffeomorphism of class* C^2 *of* U *into the* $(w=u+iv)$*-plane such that* $\partial u/\partial y=0$ *on* U, *then the image* $\phi(E)$ *of* E *is of class* $\mathscr{E}$.

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