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Remarks on Extendible Vector Bundles over Lens Spaces and Real Projective Spaces

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§1. Introduction

Let K be a CW-complex and L be its subcomplex. A real (resp. complex) vector bundle ζ over L is said to be *extendible* to K if ζ is equivalent to the restriction of a real (resp. complex) vector bundle over K.

R. L. E. Schwarzenberger ([2], [6]) studied the extendibility of vector bundles over CP^n (resp. RP^n) to CP^m (resp. RP^m), $m>n$, where CP^n (resp. RP^n) is the complex (resp. real) projective n-space.

The purpose of this paper is to establish some results concerning the extendible real vector bundles over the standard lens space $L^n(p)=S^{2n+1}/Z_p$ and the real projective space by the somewhat different methods. Our main results are as follows.

Theorem 1.1. *Let p be an odd prime and ζ be a real t-plane bundle over $L^n(p)$. Assume that there is a positive integer l satisfying the following properties*:

(i) *ζ is stably equivalent to a sum of $[t/2]+l$ non-trivial real 2-plane bundles.*

(ii) $p^{[n/(p-1)]} > [t/2]+l$.

Then $n<2[t/2]+2l$ and ζ is not extendible to $L^m(p)$ for each $m\geqq 2[t/2]+2l$.

Theorem 1.2. *Let p be any integer >1. The tangent bundle $\tau(L^n(p))$ of $L^n(p)$ is extendible to $L^{n+1}(p)$ if and only if $n=0$, 1 or 3.*

We also obtain the results (Theorems 6.2 and 6.6) for RP^n corresponding to the above theorems.

In §2, we recall the structure of K-ring of $L^n(p)$ according to T. Kambe [3], which is useful in §3 for the proof of Theorem 1.1. In §4, we have sufficient conditions for the existence of the extension of a real vector bundle over $L^n(p)$ (Theorems 4.2 and 4.3) and give an example of a real t-plane bundle over $L^n(p)$ which is extendible to $L^{m-1}(p)$ but not to $L^m(p)$ $(m=2[t/2]+2l)$. The proof of Theorem 1.2 is carried out in §5. Also, we give an example of an extendible vector bundle over $L^n(p)$ which shows that the condition (ii) of Theorem 1.1 cannot be removed. In §6, we obtain the corresponding results for RP^n and give another proof of Corollary to Theorem 3 in [6].

§2. The structure of $\tilde{\mathbf{K}}(\mathbf{L}^n(\mathbf{p}))$

Let $L^n(p)=S^{2n+1}/Z_p$ be the $(2n+1)$-dimensional standard lens space mod p, where S^{2n+1} is the unit $(2n+1)$-sphere in the complex space C^{n+1} and Z_p is the cyclic group of order p. Let $[z_0,\dots, z_n]\in L^n(p)$ denote the class of $(z_0,\dots, z_n)\in S^{2n+1}$. The space $L^n(p)$ is naturally embedded in $L^{n+1}(p)$ by identifying $[z_0,\dots, z_n]$ with $[z_0,\dots, z_n, 0]$. Consider the subspace

$$L_0^k(p) = \{[z_0,\dots, z_k]\in L^k(p)|z_k \text{ is a non-negative real number}\}.$$

Then, it is well known that $L^n(p)$ has a structure of a CW-complex in which $L^k(p)-L_0^k(p)$ and $L_0^k(p)-L^{k-1}(p)$ are $(2k+1)$- and $2k$-cells for $k\leqq n$.

Let η_n be the canonical complex line bundle over the lens space $L^n(p)$, and set $\sigma_n=\eta_n-1$. For a prime p, the structure of the reduced K-ring $\tilde{K}(L^n(p))$ of $L^n(p)$ is determined by T. Kambe [3, Theorem 1] as follows.

(2.1) *Let p be a prime and let $n=s(p-1)+r$ $(0\leqq r<p-1)$. Then*

$$\tilde{K}(L^n(p)) = (Z_{p^{s+1}})^r+(Z_{p^s})^{p-r-1}$$

and $\sigma_n^1,\dots, \sigma_n^r$ generate additively the first r factors and $\sigma_n^{r+1},\dots, \sigma_n^{p-1}$ the last $p-r-1$ factors. Moreover, the ring structure of $\tilde{K}(L^n(p))$ is given by

$$\sigma_n^p = -\sum_{i=1}^{p-1}\binom{p}{i}\sigma_n^i, \qquad \sigma_n^{n+1}=0.$$

(Here, G^k denotes the direct sum of k-copies of an abelian group G).

Let r: $K(X)\to KO(X)$ and c: $KO(X)\to K(X)$ be the real restriction and the complexification respectively. Then, as is well known,

(2.2) $$r c\alpha = 2\alpha \quad \textit{and} \quad c r\alpha = \alpha+(\textit{the conjugation of } \alpha).$$

These operators are natural with respect to maps, and c is a ring homomorphism.

For the $2n$-skeleton $L_0^n(p)$ of $L^n(p)$, we have (cf. [3, (2.5) and (3.5)]

(2.3) $$\tilde{K}(L^n(p)) = \tilde{K}(L_0^n(p)).$$
$$r\tilde{K}(L_0^n(p)) = \widetilde{KO}(L_0^n(p)), \qquad \textit{if} \quad p \textit{ is odd}.$$

§3. Proof of Theorem 1.1

First, we prepare a lemma.

LEMMA 3.1. *Let p be an odd prime and ζ be a real t-plane bundle over $L^n(p)$. If l is a positive integer with $2[t/2]+2l\leqq n$, then ζ is not stably equivalent to a sum of $[t/2]+l$ non-trivial real 2-plane bundles.*

PROOF. We know that a real plane bundle over $L^n(p)$ is orientable for $n>0$, that an orientable real 2-plane bundle is a real restriction of a complex line bundle, and that the complex line bundles are classified by the first Chern classes. Therefore a non-trivial real 2-plane bundle over $L^n(p)$ is expressed as $r\eta_n^i$ for some i with $0<i<p$, since $\eta_n^p=1$.

Suppose that ζ is stably equivalent to a sum of $[t/2]+l$ non-trivial real 2-plane bundles. Then there exist non-negative integers c_i $(0<i<p)$ such that

$$\zeta-t = \sum_{i=1}^{p-1}c_i(r\eta_n^i-2),\quad \sum_{i=1}^{p-1}c_i = [t/2]+l.$$

The total Pontrjagin class $p(\zeta)$ of ζ is determined by the equalities (cf. [4, Lemma 2.3]):

$$p(\zeta) = \prod_{i=1}^{p-1}(1+i^2x^2)^{c_i},$$

where x is the generator of $H^2(L^n(p);Z)$. Thus

$$p_{[t/2]+l}(\zeta) = \prod_{i=1}^{p-1}i^{2c_i}x^{2[t/2]+2l}.$$

Since $n\geqq 2[t/2]+2l$, we have $x^{2[t/2]+2l}\neq 0$, and so $p_{[t/2]+l}(\zeta)\neq 0$.

On the other hand, the vector bundle ζ is t-dimensional and hence we have $p_{[t/2]+l}(\zeta)=0$ for $l>0$. This is a contradiction. q.e.d.

PROOF OF THEOREM 1.1. It follows from Lemma 3.1 and the assumption (i) that $n<2[t/2]+2l$.

To prove the second part, suppose that ζ is extendible to $L^m(p)$ $(m>n)$. Then there exists a real t-plane bundle α over $L^m(p)$ such that $i^!\alpha\cong\zeta$, where $i^!\alpha$ denotes the induced bundle of α by the inclusion map $i\colon L^n(p)\to L^m(p)$. Let $j\colon L_0^m(p)\to L^m(p)$ and $k\colon L^n(p)\to L_0^m(p)$ be the inclusions. Then $i=jk$. According to (2.3), $j^!(\alpha-t)\in\widetilde{KO}(L_0^m(p))=r\widetilde{K}(L^m(p))$. Thus, by (2.1), we have

$$j^!(\alpha-t) = r\sum_{i=1}^{p-1}a_i\sigma_m^i = r\sum_{i=1}^{p-1}b_i(\eta_m^i-1),$$

where a_i and b_i are some integers. We can take these integers sufficiently large. Now

$$i^!(\alpha-t) = k^!j^!(\alpha-t) = r\sum_{i=1}^{p-1}b_i(k^!\eta_m^i-1) = r\sum_{i=1}^{p-1}b_i(\eta_n^i-1).$$

As in the proof of Lemma 3.1, the assumption (i) implies that there exist non-negative integers c_i $(0<i<p)$ such that

$$\zeta-t = \sum_{i=1}^{p-1}c_i(r\eta_n^i-2),\qquad \sum_{i=1}^{p-1}c_i = [t/2]+l. \tag{1}$$

Thus we have

$$\sum_{i=1}^{p-1} b_i r(\eta_n^i - 1) = i^!(\alpha - t) = \sum_{i=1}^{p-1} c_i r(\eta_n^i - 1).$$

It follows from (2.2) that $cr\eta_n^i = \eta_n^i + \eta_n^{p-i}$, and hence

$$(2) \qquad \sum_{i=1}^{p-1}(b_i + b_{p-i} - c_i - c_{p-i})(\eta_n^i - 1) = 0.$$

The identities $\eta_n^i - 1 = \sum_{j=1}^{i}\binom{i}{j}\sigma_n^j$ $(0<i<p)$ show that the above equality (2) becomes as follows.

$$\sum_{i=1}^{p-1}\{\sum_{s=i}^{p-1}\binom{s}{i}(b_s + b_{p-s} - c_s - c_{p-s})\}\sigma_n^i = 0.$$

Noting that $n \geqq p-1$ by (ii) and that $\sigma_n^1, \dots, \sigma_n^{p-1}$ are generators of $\tilde{K}(L^n(p))$ (cf. (2.1)), it holds that

$$\sum_{s=i}^{p-1}\binom{s}{i}(b_s + b_{p-s} - c_s - c_{p-s}) \equiv 0 \qquad \bmod p^{1+[(n-i)/(p-1)]}$$

for $0<i<p$. Hence, by induction on $p-i$, we obtain

$$b_i + b_{p-i} - c_i - c_{p-i} \equiv 0 \qquad \bmod p^{[n/(p-1)]} \quad (0<i<p).$$

So there are positive integers k_i $(0<i<p)$ such that

$$(3) \qquad b_i + b_{p-i} = k_i p^{[n/(p-1)]} + c_i + c_{p-i}.$$

Since the t-plane bundle $j^!\alpha$ is stably equivalent to $\sum_{i=1}^{p-1} b_i r\eta_m^i$, the total Pontrjagin class $p(j^!\alpha)$ of $j^!\alpha$ is determined by

$$p(j^!\alpha) = \prod_{i=1}^{p-1}(1+i^2x^2)^{b_i} = \prod_{i=1}^{(p-1)/2}(1+i^2x^2)^{b_i+b_{p-i}},$$

where x is the generator of $H^2(L_0^m(p); Z)$. Thus

$$p_{[t/2]+l}(j^!\alpha) = \sum_{j_1+\cdots+j_{(p-1)/2}=[t/2]+l}\prod_{i=1}^{(p-1)/2}\binom{b_i+b_{p-i}}{j_i} i^{2j_i} x^{2[t/2]+2l}.$$

By the assumptions (ii) and (1), we have $c_i + c_{p-i} \leqq [t/2]+l < p^{[n/(p-1)]}$. Therefore, by (3), for $1 \leqq i \leqq (p-1)/2$,

$$\binom{b_i+b_{p-i}}{j} \equiv \begin{cases} 0 \quad \bmod p & \text{if } \ c_i + c_{p-i} < j < p^{[n/(p-1)]}, \\ 1 \quad \bmod p & \text{if } \ j = c_i + c_{p-i}. \end{cases}$$

Thus we obtain

$$p_{[t/2]+l}(j^!\alpha) = \prod_{i=1}^{(p-1)/2} i^{2(c_i+c_{p-i})} x^{2[t/2]+2l}.$$

Since $m\geqq 2[t/2]+2l$, we have $x^{2[t/2]+2l}\neq 0$, and consequently, $p_{[t/2]+l}(j^!\alpha)\neq 0$.

On the other hand, the vector bundle $j^!\alpha$ is t-dimensional and so we see $p_{[t/2]+l}(j^!\alpha)=0$ for $l>0$. This is a contradiction. q.e.d.

§4. Extendible vector bundles over $L^n(p)$

LEMMA 4.1. *Let p be any integer >1. Suppose $t>2n+1$ and $l>0$. Let ζ be a real t-plane bundle over $L^n(p)$ which is stably equivalent to $r\eta_n^{a_1}\oplus\cdots\oplus r\eta_n^{a_{[t/2]+l}}$, where $0<a_i<p$ for $0<i\leqq[t/2]+l$. If there is an integer m $(>n)$ such that*

$$\mathrm{Span}\,(r\eta_m^{a_1}\oplus\cdots\oplus r\eta_m^{a_{[t/2]+l}}\oplus k)\geqq 2[t/2]+2l-t+k$$

for some $k\geqq 0$, then ζ is extendible to $L^m(p)$. (Here Span α *denotes the maximal number of linearly independent cross-sections of a vector bundle α).*

PROOF. By the assumptions, we have

$$\zeta\oplus(2[t/2]+2l-t)\cong r\eta_n^{a_1}\oplus\cdots\oplus r\eta_n^{a_{[t/2]+l}}$$

and

$$r\eta_m^{a_1}\oplus\cdots\oplus r\eta_m^{a_{[t/2]+l}}\oplus k=(2[t/2]+2l-t+k)\oplus\gamma$$

for some real t-plane bundle γ over $L^m(p)$. Let $i\colon L^n(p)\to L^m(p)$ be the inclusion. As $i^!\eta_m=\eta_n$, we get $\zeta\oplus(2[t/2]+2l-t+k)=i^!\gamma\oplus(2[t/2]+2l-t+k)$. Thus $\zeta\cong i^!\gamma$, since $t>2n+1$. q.e.d.

THEOREM 4.2. *Let p be an odd integer >1 and ζ be a real t-plane bundle over $L^n(p)$, where $t>2n+1$. Assume that there are positive integers l and m $(>n)$ satisfying the following properties:*

(i) *ζ is stably equivalent to a sum of $[t/2]+l$ non-trivial real 2-plane bundles.*

(ii) $t\geqq 2[m/2]+1$.

Then ζ is extendible to $L^m(p)$.

PROOF. Put

$$\beta_n=r\eta_n^{a_1}\oplus\cdots\oplus r\eta_n^{a_{[t/2]+l}},$$

where a_i $(0<i\leqq[t/2]+l)$ are positive integers with $a_i<p$. Consider the extension β_m of β_n to $L^m(p)$. Let $\pi_m\colon S^{2m+1}\to L^m(p)$ be the natural projection. Then $\pi_m^! r\eta_m^{a_i}$ is trivial and hence $\beta_m-(2[t/2]+2l)\in\widetilde{KO}(L^n(p))\cap\mathrm{Ker}\,\pi_m^!$. Thus it follows from [7, Theorem A] that

$$g\text{-}\dim(\beta_m-(2[t/2]+2l))\leqq 2[m/2]+1,$$

where g-$\dim\alpha$ denotes the *geometrical dimension* of a stable vector bundle α. (Note that Theorem A of [7] is true for any odd integer >1.) Using (ii) we have

$$\operatorname{Span}(\beta_m \oplus k) \geqq 2[t/2]+2l-t+k \quad \text{for} \quad k \geqq 2(m+1-[t/2]-l).$$

Therefore the result follows from Lemma 4.1. q. e. d.

THEOREM 4.3. *Let p be an odd integer >1 and ζ be a real t-plane bundle over $L^n(p)$, where $t=2s-1>2n+2$. Assume that there is a positive integer l satisfying the following properties*:

(i) *ζ is stably equivalent to $([t/2]+l)r\eta_n$.*

(ii) *$q>2l-1$ for any prime divisor q of p.*

(iii) $\binom{s+l-1}{i}\equiv 0 \bmod p$ *for any i with $s\leqq i<s+l-1$.*

Then ζ is extendible to $L^{2[t/2]+2l-1}(p)$.

PROOF. We apply Lemma 4.1 to the case where $a_1=\cdots=a_{[t/2]+l}=1$ and $m=2[t/2]+2l-1$. Let $f\colon L^m(p)\to BSO(2[t/2]+2l+k)$ be the classifying map of the vector bundle $([t/2]+l)r\eta_m\oplus k=\alpha$, where $BSO(N)$ denotes the classifying space of oriented real N-plane bundles. Consider the lifting problem indicated in the diagram below:

$$\begin{tikzcd} & BSO(t) \arrow[d] \\ L^m(p) \arrow[r, "f"] \arrow[ur, dashed, "F"] & BSO(2[t/2]+2l+k) \end{tikzcd}$$

Put $k=2r+1$ $(r\geqq 0)$. According to [8, Theorem (1.7)], if $q>2l-1$ for any prime divisor q of p and $p_i(\alpha)=0$ for $s\leqq i\leqq [t/2]+l+r$, the lifting $F\colon L^m(p)\to BSO(t)$ of f exists, and so

$$\operatorname{Span}\alpha \geqq 2[t/2]+2l-t+k.$$

Then the conclusion follows from Lemma 4.1. But

$$p_i(\alpha)=\binom{[t/2]+l}{i}x^{2i}=\binom{s-1+l}{i}x^{2i},$$

where x is the generator of $H^2(L^m(p);Z)$, and $x^{m+1}=x^{2s+2l-2}=0$. Thus the condition (iii) implies $p_i(\alpha)=0$ for $s\leqq i\leqq [t/2]+l+r$, as desired. q. e. d.

REMARK 4.4. Combining Theorem 4.2 or 4.3 with Theorem 1.1, we have an example of a real t-plane bundle over $L^n(p)$ which is extendible to $L^{2[t/2]+2l-1}(p)$ but not extendible to $L^{2[t/2]+2l}(p)$. For example, we can take $n=2p-2$ (p odd prime), $t=2n+3$, $l=1$ and a bundle ζ such that $(n+2)r\eta_n=\zeta\oplus 1$.

§5. Proof of Theorem 1.2

The "only if" part of Theorem 1.2 is a consequence of the following

THEOREM 5.1. *Let p be any integer >1. If $n\neq 0$, 1 and 3, the tangent bundle $\tau(L^n(p))$ of $L^n(p)$ is not extendible to $L_0^{n+1}(p)$.*

PROOF. Suppose that $\tau(L^n(p))$ is extendible to $L_0^{n+1}(p)$. Then there exists a real $(2n+1)$-plane bundle α over $L_0^{n+1}(p)$ such that $i^!\alpha\cong\tau(L^n(p))$, where i: $L^n(p)\to L_0^{n+1}(p)$ is the natural inclusion. Let π_m: $S^{2m+1}\to L^m(p)$ be the natural projection. Since S^{2n+2} consists of elements $(z_0,\dots, z_n, z_{n+1})$ of S^{2n+3} such that z_{n+1} is a real number, the image of the restriction π of π_{n+1} to S^{2n+2} is contained in $L_0^{n+1}(p)$. Now the following diagram is commutative:

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{j} & S^{2n+2} \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi} \\ L^n(p) & \xrightarrow{i} & L_0^{n+1}(p), \end{array}$$

where j denotes the natural inclusion. Thus we see that $i\pi_n$ $(=\pi j)$ is homotopically trivial. Therefore, $\tau(S^{2n+1})=\pi^!(\tau(L^n(p))\cong\pi_n^!i^!\alpha=$trivial, and so $2n+1=1$, 3 or 7, as desired. q. e. d.

LEMMA 5.2. *Let K^n be the n-skeleton of a CW-complex K. If $n\geqq 3$, an oriented real 2-plane bundle over K^n is extendible to K^m for any $m>n$.*

PROOF. The equivalence class of an oriented real 2-plane bundle is determined by the homotopy class of a map f: $K^n\to BSO(2)$. Let m be any integer with $m>n$. The obstructions for extending f to K^m are contained in the groups

$$H^{r+1} = H^{r+1}(K^m, K^n; \pi_r(BSO(2))).$$

But $\pi_r(BSO(2))=\pi_{r-1}(SO(2))=0$ for $r>2$. Thus we have $H^{r+1}=0$ for all r, if $n\geqq 3$. q. e. d.

The "if" part of Theorem 1.2 is a special case of the following more general result.

THEOREM 5.3. *Let p be an integer >1. If $n=0$, 1 or 3, $\tau(L^n(p))$ is extendible to $L^m(p)$ for all $m>n$.*[1)]

1) We notice that Theorem 5 in [5] is not true for $n\leqq 3$. One of the authors should take this opportunity of correcting the following errata in [5, p. 399]: Add $t\geqq 3$ to the assumption in line 3, replace $s>[t/2]$ in line 5 by $3^{[n/2]}>s>[t/2]$, and add $n>3$ to the assumption of Theorem 5 in line 9.

PROOF. Since $L^n(p)$ is parallelizable for $n=0$ or 1, obviously $\tau(L^n(p))$ ($n=0$ or 1) is extendible to $L^m(p)$ for any $m>n$. According to [9], $L^3(p)$ has a tangent 5-field. Namely, there is a real 2-plane bundle β such that $\tau(L^3(p)) =\beta\oplus 5$. By Lemma 5.2, β is extendible to $L^m(p)$ for any $m>3$, and so is $\tau(L^3(p))$.
q. e. d.

REMARK 5.4. Setting $n=3$, $p=3$, $l=1$ and $\zeta=\tau(L^3(3))$, we have an example which shows that the condition (ii) in Theorem 1.1 is necessary.

§6. Extendible vector bundles over $\mathbf{RP}^n$

In this section we discuss the extendibility of real vector bundles over the real projective n-space RP^n to RP^m. The following lemma corresponds to Lemma 3.1.

LEMMA 6.1. *Let ζ be a real t-plane bundle over RP^n. If l is a positive integer with $t+l\leqq n$, then ζ is not stably equivalent to $(t+l)\xi_n$, where ξ_n is the canonical line bundle over RP^n.*

PROOF. Suppose that ζ is stably equivalent to $(t+l)\xi_n$. Then the total Stiefel-Whitney class $w(\zeta)$ of ζ is given by $w(\zeta)=(1+y)^{t+l}$, where y is the generator of $H^1(RP^n; Z_2)$. Thus $w_{t+l}(\zeta)=y^{t+l}\neq 0$, as $t+l\leqq n$.

But, since ζ is t-dimensional, we have $w_{t+l}(\zeta)=0$ for $l>0$. This is a contradiction.
q. e. d.

Considering the structure of KO-ring of RP^n (cf. J. F. Adams [1, Theorem 7.4]), we may prove the following result corresponding to Theorem 1.1 in §1.

THEOREM 6.2. *Let ζ be a real t-plane bundle over RP^n. Assume that there is a positive integer l satisfying the following properties:*

(i) *ζ is stably equivalent to $(t+l)\xi_n$.*

(ii) *$2^{\phi(n)}>t+l$, where $\phi(n)$ is the number of integers s such that $0<s\leqq n$ and $s\equiv 0$, 1, 2 or 4 mod 8.*

Then $n<t+l$ and ζ is not extendible to RP^m for each $m\geqq t+l$.

PROOF. The fact that $n<t+l$ follows from Lemma 6.1 and the assumption (i).

To prove the second part, suppose that ζ is extendible to RP^m $(m>n)$. Then there exists a real t-plane bundle α over RP^m such that $i^!\alpha\cong\zeta$, where $i: RP^n \to RP^m$ is the inclusion. According to [1],

$$\alpha-t = a(\xi_m-1)\in \widetilde{KO}(RP^m),$$

for some integer a. We may take a sufficiently large. Thus we have $\alpha\oplus(a-t)$

$\cong a\xi_m$. Therefore the Stiefel-Whitney class $w(\alpha)$ of α is given by $w(\alpha)=(1+y)^a$, where y is the generator of $H^1(RP^m; Z_2)$.

Now $\zeta-t=i^!\alpha-t=a(i^!\xi_m-1)=a(\xi_n-1)$. On the other hand, by the assumption (i), it holds that $\zeta-t=(t+l)\xi_n-(t+l)$. Hence $(a-t-l)(\xi_n-1)=0$. Thus, by [1],

$$a-t-l \equiv 0 \mod 2^{\phi(n)},$$

and so there is a positive integer k such that $a=k2^{\phi(n)}+t+l$. Therefore $\binom{a}{t+l}\equiv 1 \mod 2$ by the assumption (ii).

If $m\geqq t+l$, $y^{t+l}\neq 0$ and $w_{t+l}(\alpha)=\binom{a}{t+l}y^{t+l}\neq 0$. While $w_{t+l}(\alpha)=0$ for $l>0$, since α is t-dimensional. This is a contradiction. q.e.d.

LEMMA 6.3. *Suppose $t>n$ and $l\geqq 0$. Let ζ be a real t-plane bundle over RP^n which is stably equivalent to $(t+l)\xi_n$. If there is an integer m $(>n)$ such that*

$$\mathrm{Span}\,((t+l)\xi_m \oplus k) \geqq l+k$$

for some $k\geqq 0$, then ζ is extendible to RP^m.

PROOF. By the assumptions, we have $\zeta\oplus l\cong(t+l)\xi_n$ and $(t+l)\xi_m\oplus k=(l+k)\oplus\gamma$ for some real t-plane bundle γ over RP^m. Let $i\colon RP^n\to RP^m$ be the inclusion. Since $i^!\xi_m=\xi_n$, we get $i^!\gamma\cong\zeta$, as desired. q.e.d.

THEOREM 6.4. *In addition to the assumptions of Theorem 6.2, assume that $t>n$ and that $\mathrm{Span}\,\tau(RP^{t+l-1})\geqq l-1$. Then ζ is extendible to RP^{t+l-1} but not extendible to RP^{t+l}.*

PROOF. Set $m=t+l-1$. Then it follows from Lemma 6.3 that ζ is extendible to RP^{t+l-1}, for

$$\mathrm{Span}\,(t+l)\xi_{t+l-1} = \mathrm{Span}\,(\tau(RP^{t+l-1})\oplus 1) \geqq l$$

by our assumption. The latter part is a consequence of Theorem 6.2. q.e.d.

Write $t+l=(2a+1)2^b$ and $b=c+4d$, where a, b, c and d are integers and $0\leqq c\leqq 3$. Then

$$\mathrm{Span}\,\tau(RP^{t+l-1}) = \mathrm{Span}\,\tau(S^{t+l-1}) = 2^c+8d-1\,.$$

(cf. [1, Theorem 1.1]). Therefore we have many pairs (l, t) of l and t for which $\mathrm{Span}\,\tau(RP^{t+l-1})\geqq l-1$. For example: $l=1$, t any; $l=2$, $t\equiv 0 \mod 2$; $l=3$, $t\equiv 1 \mod 4$; etc.

THEOREM 6.5. *Let ζ be a real t-plane bundle over RP^n which is extendible to RP^N, where $N=2^{\phi(n)}-1$. Then ζ is stably equivalent to a sum of t real line bundles.*

PROOF. There is an integer l such that

$$\zeta-t=(t+l)(\xi_n-1)\in \widetilde{KO}(RP^n).$$

Since ξ_n-1 is of order $2^{\phi(n)}$, we have $0\leqq t+l<2^{\phi(n)}$. If $l>0$, ζ is not extendible to RP^m for any $m\geqq t+l$ by Theorem 6.2. This contradicts our assumption. Therefore $l\leqq 0$, as desired. q.e.d.

R. L. E. Schwarzenberger has proved in [6, Corollary to Theorem 3] that a real t-plane bundle over RP^n, which is extendible to RP^m for arbitrarily large m, is stably equivalent to a sum of t real line bundles. We obtain another proof of the result by Theorem 6.5.

The following result which corresponds to Theorem 1.2 may be well known.

THEOREM 6.6. *The tangent bundle $\tau(RP^n)$ of RP^n is extendible to RP^{n+1} if and only if $n=1$, 3 or 7.*

PROOF. If $n=1$, 3 or 7, RP^n is parallelizable, and hence $\tau(RP^n)$ is extendible to RP^m for all $m>n$. The "only if" part is an immediate consequence of Theorem 6.2, or is proved directly in the way similar to the proof of Theorem 5.1. q.e.d.

We see from this result that the condition (ii) of Theorem 6.2 cannot be removed.

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