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# *Codivisorial and Divisorial Modules over Completely Integrally Closed Domains (II)*

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## Introduction

In our paper [5], we have introduced an operation on modules over a completely integrally closed domain, which we called "*divisorial envelope*", and we have studied some basic properties of the divisorial envelope of a codivisorial module and also developed a theory of codivisorial and divisorial modules which shows us that the intrinsic nature of codivisorial and divisorial modules over a Krull domain is similar to that of modules over a Dedekind domain.

The fundamental theorem of finitely generated abelian groups is based on the fact that the ring of rational integers is a principal ideal domain, in other words, a ring in which every ideal is free. It is well known that the above theorem is generalized to finitely generated modules over a Dedekind domain which is characterized by the property that any ideal is projective. It seems plausibe to the authors that the theorem can be formulated for modules over a Krull domain as far as we are concerned with codivisorial and divisorial modules. In fact, in [3], N. Bourbaki dealt with the case of noetherian Krull domains. The main purpose of this Part II is to introduce the notion of an essentially finite module over a Krull domain and develop a theory of invariants by making use of the divisorial envelope.

## §3. Divisorial equivalence

Throughout this §, $A$ is always a strongly integrally closed domain, unless otherwise specified.

PROPOSITION 30. *Let $f\colon M\to N$ be a homomorphism of $A$-domules and $p\colon M\to M/\tilde{M}$, $q\colon N\to N/\tilde{N}$ be the canonical projections.*

(i) *There is a unique homomorphism $f_*\colon M/\tilde{M}\to N/\tilde{N}$ such that $f_*p=qf$.*

(ii) *If $f$ is pseudo-injective, then $f_*$ is injective, and if $f$ is pseudo-isomorphic, then so is $f_*$.*

(iii) *If $f$ is pseudo-isomorphic and $M$ is divisorial, then $f_*$ is an isomorphism.*

PROOF. The existence of $f_*$ follows from Prop. 3 and the uniqueness is clear.

Suppose first that $f$ is pseudo-injective. Since $\tilde{M}$ is contained in $f^{-1}(\tilde{N})$, we have the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow f^{-1}(\tilde{N}) \longrightarrow \tilde{N}.$$

This implies, by Prop. 6 (ii), that $f^{-1}(\tilde{N})$ is pseudo-null; therefore $\tilde{M}=f^{-1}(\tilde{N})$. Thus $f_*$ must be injective. If, moreover, $f$ is pseudo-surjective, then Coker $(f)$ is pseudo-null; since the induced homomorphism of Coker $(f)$ to Coker $(f_*)$ is surjective, Coker $(f_*)$ must be pseudo-null. This completes the proof of (ii).

Finally, suppose that $M$ is divisorial. Then $M \cong \tilde{M} \oplus M/\tilde{M}$ by Coroll. 2 to Prop. 15, and therefore $M/\tilde{M}$ is also divisorial. The assertion (iii) follows from Coroll. 1 to Prop. 11.

PROPOSITION 31. *Let $A$ be a completely integrally closed domain and $M$, $N$ be $A$-modules. Let $i$ be the canonical injection of $M$ to $D(M)$. If $N$ is codivisorial, then*

$$\mathrm{Hom}_A(i, D(N))\colon \mathrm{Hom}_A(D(M), D(N)) \longrightarrow \mathrm{Hom}_A(M, D(N))$$

*is an isomorphism.*

PROOF. Since $N$ is codivisorial, so is $D(N)$ by Prop. 4. On the other hand, $D(M)/M$ is pseudo-null by the definition of a divisorial envelope $D$. Therefore $\mathrm{Hom}_A(D(M)/M, D(N))=0$, which implies that $\mathrm{Hom}_A(i, D(N))$ is an injection. By Prop. 8, we can see that $\mathrm{Hom}_A(i, D(N))$ is a surjection.

COROLLARY. *Let $f\colon M \to N$ be a homomorphism of modules over a strongly integrally closed domain $A$. Then there exists a unique homomorphism $f_{**}$ of $D(M/\tilde{M})$ to $D(N/\tilde{N})$ such that $f_{**}i=jf$, where $i$ (resp. $j$) is the canonical homomorphism of $M$ (resp. $N$) to $D(M/\tilde{M})$ (resp. $D(N/\tilde{N})$). Moreover, if $f$ is a pseudo-isomorphism, then $f_{**}$ is an isomorphism.*

PROOF. The homomorphism induces the homomorphism $f_*$ of $M/\tilde{M}$ to $N/\tilde{N}$ by Prop. 30. Applying Prop. 31 to $f_*$, we can obtain a homomorphism $f_{**}$ of $D(M/\tilde{M})$ to $D(N/\tilde{N})$ such that $f_{**}i=jf$.

It is easy to see that, similarly to the proof of Prop. 31, $\mathrm{Hom}(i, D(N/\tilde{N}))$ is an injection. This shows the uniqueness of $f_{**}$.

Suppose now that $f$ is a pseudo-isomorphism. Then, by Prop. 30, $f_*$ is a pseudo-isomorphism ($f_*$ is necessarily injective). Since the canonical injection of $M/\tilde{M}$ to $D(M/\tilde{M})$ is an essential extension, $f_{**}$ must be an injection. Since both $f_*$ and the canonical injection of $N/\tilde{N}$ to $D(N/\tilde{N})$ are pseudo-surjective, so is the composition of them by Coroll. 2 to Prop. 6. We can conclude from this fact that $f_{**}$ is a pseudo-surjection. Since a pseudo-isomorphism of codivisorial and divisorial modules is an isomorphism by Coroll. 1 to Prop. 11, $f_{**}$ must be an isomorphism.

In Lemma 2 ([5]), we have shown that, for non-zero fractional ideals $\mathfrak{a}$, $\mathfrak{b}$ of a Krull domain $A$, $\widetilde{\mathfrak{a}:\mathfrak{b}}=\tilde{\mathfrak{a}}:\tilde{\mathfrak{b}}$, namely $D(\mathrm{Hom}_A(\mathfrak{b}:\mathfrak{a}))=\mathrm{Hom}_A(D(\mathfrak{b}), D(\mathfrak{a}))$. More generally, for A-lattices $M$, $N$, if $N$ is divisorial, then $D(N: M)=N: M=N: D(M)$ i.e., $D(\mathrm{Hom}_A(M, N))=\mathrm{Hom}_A(D(M), N)$ (See H. Bass [1], Coroll. 8.4, p. 151). Here we shall generalize the above fact for codivisorial modules over a strongly integrally closed domain.

PROPOSITION 32. *Let $M$ and $N$ be codivisorial $A$-modules. If $M$ is a submodule of a finitely generated $A$-module $L$, then we have*

$$D(\mathrm{Hom}_A(M, N)) \cong \mathrm{Hom}_A(D(N), D(M)).$$

PROOF. By Prop. 31, we have only to prove

$$D(\mathrm{Hom}_A(M, N)) \cong \mathrm{Hom}_A(M, D(N)).$$

Consider the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(M, D(N)) \longrightarrow \mathrm{Hom}_A(M, D(N)/N).$$

Since $N$ is codivisorial, so is $D(N)$; therefore, by Coroll. to Prop. 7, $\mathrm{Hom}_A(M, N)$ and $\mathrm{Hom}_A(M, D(N))$ are codivisorial. Also, by Cor. 3 to Prop. 8, $\mathrm{Hom}_A(M, D(N))$ is divisorial. Since a pseudo-isomorphism of codivisorial modules is an essential extension, it suffices to show that $\mathrm{Hom}_A(M, D(N)/N)$ is pseudo-null.

Generally, for a submodule $M_1$ of a finitely generated $A$-module $M_2$ and a pseudo-null $A$-module $N_1$, we shall show that $\mathrm{Hom}_A(M_1, N_1)$ is pseudo-null. Put $N_2=E(N_1)$. Then $N_2$ is pseudo-null by Th. 2. Let $\{x_1,\dots, x_n\}$ be a system of generators of $M_2$ and $f$ be a homomorphism of $M_2$ to $N_2$. Then $O(f)=O(f(x_1)) \cap\cdots\cap O(f(x_n))$. Since each $O(f(x_i))$ is equivalent to $A$, so is $O(f)$ by Coroll. 1 to Th. 1. Hence $\mathrm{Hom}_A(M_2, N_2)$ is pseudo-null. Therefore, $\mathrm{Hom}_A(M_1, N_2)$ is pseudo-null, because it is a homomorphic image of $\mathrm{Hom}_A(M_2, N_2)$, and $\mathrm{Hom}_A(M_1, N_1)$ must be pseudo-null because it is isomorphic to a submodule of $\mathrm{Hom}_A(M_1, N_2)$.

REMARK 8. Let $\varphi$ be the canonical homomorphism of $\mathrm{Hom}_A(M, N)$ to $\mathrm{Hom}_A(M, D(N))$. $\varphi$ is not necessarily pseudo-isomorphic.

EXAMPLE 3. Let $(A, \mathfrak{m})$ be a noetherian normal local domain of Krull dimension $\geqq 2$. Put $N=\oplus_{n=1}^{\infty}\mathfrak{m}^n$, $M=D(N)=\oplus D(\mathfrak{m}^n)=\oplus A$ (See Coroll. 4 to Th. 3). Let $p$ be the canonical projection of $\mathrm{Hom}_A(D(N), D(N))$ to $\mathrm{Coker}(\varphi)$. Then $O(p(1_{D(N)}))=\{a\in A; al_{D(N)}(D(N))\subset N\}=\mathrm{Ann}_A(D(N)/N)=\cap\,\mathfrak{m}^n=0$. Therefore $\mathrm{Coker}(\varphi)$ is not pseudo-null.

DEFINITION 7. *Let $M$ and $N$ be $A$-modules. We say that $M$ is divisorially equivalent to $N$ if there exists a pseudo-isomorphism of $D(M)$ to $D(N)$.*

PROPOSITION 33. (i) *$M$ is divisorially equivalent to $N$ if and only if $D(M/\tilde{M})$ is isomorphic to $D(N/\tilde{N})$. In particular, the "divisorial equivalence" is an equivalence relation.*

(ii) *If $f$ is pseudo-isomorphic to $N$, then $M$ is divisorially equivalent to $N$.*

PROOF. The "if" part follows from the facts that $D(M)\cong D(\tilde{M})\oplus D(M/\tilde{M})$, $D(N)\cong D(\tilde{N})\oplus D(N/\tilde{N})$ by Coroll. 2 to Prop. 15 and $D(\tilde{M})$, $D(\tilde{N})$ are pseudo-null by Th. 2. The "only if" part follows from Prop. 30.

The last assertion follows immediately from Coroll. to Prop. 31.

## §4. Codivisorial and divisorial modules over a Krull domain (continued)

**1.** From now on, $A$ is always a Krull domain and $K$ is the quotient field of $A$. Let $M$ be an $A$-module. We shall denote by $t_A(M)$, or simply $t(M)$ unless there is fear of confusion, the torsion part of $M$. In view of the fact that any module over a Dedekind domain is divisorial, the following theorem is a generalization of the well-known fact that the injective dimension of any module over a Dedekind domain is at most 1.

THEOREM 5. *Let $M$ be a divisorial torsion module. Then* $\operatorname{inj\,dim}_A(M) \leqq 1$.

PROOF. By Coroll. 2 to Prop. 15, $M\cong\tilde{M}\oplus M/\tilde{M}$ and $\tilde{M}$ is injecitve. Hence we may assume that $M$ is a codivisorial and divisorial torsion module. Therefore $M=\oplus M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the elements of $\operatorname{Ass}_A(M)$ by Th. 4. On the other hand, $E_A(M)\cong\oplus E_A(M_{\mathfrak{p}})$ by Coroll. 4 to Th. 3. Since $\operatorname{Ass}_A(E_A(M_{\mathfrak{p}}))=\operatorname{Ass}_A(M_{\mathfrak{p}})=\{\mathfrak{p}\}$ and $E_A(M_{\mathfrak{p}})=D(E_A(M_{\mathfrak{p}}))=E_A(M_{\mathfrak{p}})_{\mathfrak{p}}$ by Th. 4, $E_A(M_{\mathfrak{p}})=E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ by Prop. 26. Therefore $E_A(M)/M\cong\oplus E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})/M_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is a principal valuation ring, $E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})/M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-injective and therefore $A$-injective. Since each $E_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})/M_{\mathfrak{p}}$ is a codivisorial $A$-module by Coroll. to Prop. 23, $E_A(M)/M$ is an injective $A$-module by [2], Prop. 2.7, namely $\operatorname{inj\,dim}_A(M)\leqq 1$.

COROLLARY 1. *Let $M$ be a divisorial torsion $A$-module. Then $R^2\mathcal{N}(M)=0$.*

The assertion follows immediately from Th. 5 and the definition of $R^2\mathcal{N}$.

COROLLARY 2. *Let $N$ be a codivisorial and divisorial $A$-module and $M$ be a divisorial torsion submodule of $N$. Then $N/M$ is codivisorial and divisorial.*

PROOF. $N/M$ is codivisorial by Coroll. 1 to Prop. 11. Since $R^1\mathcal{N}(N)$

$\to R^1\mathcal{N}(N/M)\to R^2\mathcal{N}(M)$ is exact by Prop. 10 and $R^2\mathcal{N}(M)=0$ by the above corollary, we have $R^1\mathcal{N}(N/M)=0$ by noting that $N$ is divisorial and hence $R^1\mathcal{N}(N)=0$ by Prop. 11. Therefore $N/M$ is divisorial again by Prop. 11.

COROLLARY 3. *Let $M$ be a codivisorial $A$-module. Then $M$ is divisorial if and only if $t(M)$ and $M/t(M)$ are divisorial.*

PROOF. The "if" part follows immediately from Coroll. 1 to Prop. 11. Assume now that $M$ is divisorial. Since $t(M)$ is divisorial in $M$, $t(M)$ is divisorial by Coroll. 1 to Prop. 6. The above Cor. 2 leads to the last assertion.

**2.** I. Beck showed in [2] that a direct sum of codivisorial and injective modules over a Krull domain is still injective. The following result is a generalization of the above fact.

PROPOSITION 34. *Let $\Lambda$ be a directed set and $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ be an inductive system of codivisorial $A$-modules. If each $M_\lambda$ is divisorial, then so is $\varinjlim M_\lambda$.*

PROOF. Consider the exact sequence

$$0 \longrightarrow \varinjlim t(M_\lambda) \longrightarrow \varinjlim M_\lambda \longrightarrow \varinjlim M_\lambda/t(M_\lambda) \longrightarrow 0.$$

Since $t(M_\lambda)$ and $M_\lambda/t(M_\lambda)$ are divisorial for any $\lambda$ by Coroll. 3 to Th. 5 and $\varinjlim M_\lambda$ is codivisorial by Prop. 29, we may assume that each $M_\lambda$ is a torsion module or a torsion-free module.

Case 1: Suppose that each $M_\lambda$ is a torsion module. Let $p$ be the canonical projection of $\bigoplus_{\lambda\in\Lambda} M_\lambda$ to $\varinjlim M_\lambda$. Put $N=\mathrm{Ker}(p)$. Then we have the exact sequence

$$0 \longrightarrow N \longrightarrow \oplus M_\lambda \longrightarrow \varinjlim M_\lambda \longrightarrow 0.$$

Since $\varinjlim M_\lambda$ is codivisorial by Prop. 29, $N$ is divisorial in $\oplus M_\lambda$. Since $\oplus M_\lambda$ is divisorial by Coroll. to Th. 3, $N$ is divisorial by Coroll. 1 to Prop. 6. Therefore $\varinjlim M_\lambda$ is divisorial by Coroll. 2 to Th. 5.

Case 2: Suppose that each $M_\lambda$ is torsion free. Then $E(M_\lambda)\cong M_\lambda\otimes_A K$. Since $M_\lambda$ is divisorial, $E(M_\lambda)/M_\lambda\cong M_\lambda\otimes_A K/M_\lambda$ is codivisorial. Therefore $\varinjlim (M_\lambda\otimes_A K)/\varinjlim M_\lambda\cong\varinjlim (M_\lambda\otimes_A K/M_\lambda)$ is codivisorial by Prop. 29. Namely, $\varinjlim M_\lambda$ is divisorial in $\varinjlim (M_\lambda\otimes_A K)$. Since $\varinjlim (M_\lambda\otimes_A K)\cong(\varinjlim M_\lambda)\otimes_A K$, $\varinjlim (M_\lambda\otimes_A K)$ is divisorial and hence $\varinjlim M_\lambda$ is divisorial by Coroll. 1 to Prop. 6.

COROLLARY 1. *Let $\Lambda$ be a directed set and $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ be an inductive system of codivisorial and injective $A$-modules. Then $\varinjlim M_\lambda$ is injecitve.*

The assertion follows from Prop. 34 and the fact that an inductive limit of divisible modules is divisible.

COROLLARY 2. *Let $\Lambda$ be a directed set and $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ be an inductive system of A-modules. If $R^1\mathcal{N}(M_\lambda)=0$ for every $\lambda$, then $R^1\mathcal{N}(\varinjlim M_\lambda)=0$.*

PROOF. By Remark 4 and Prop. 29, $\varinjlim M_\lambda/\widetilde{\varinjlim M_\lambda}\cong\varinjlim M_\lambda/\tilde{M}_\lambda$. Since each $M_\lambda/\tilde{M}_\lambda$ is codivisorial and divisorial by Prop. 3 and Coroll. 1 to Prop. 15, $\varinjlim M_\lambda/\widetilde{\varinjlim M_\lambda}$ is divisorial by Prop. 34 and hence $R^1\mathcal{N}(\varinjlim M_\lambda)=0$ again by Coroll. 1 to Prop. 15.

COROLLARY 3. *Let $\Lambda$ be a directed set and $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ be an inductive system of A-modules. Then*

$$D(\varinjlim M_\lambda/\widetilde{\varinjlim M_\lambda})\cong\varinjlim D(M_\lambda/\tilde{M}_\lambda).$$

PROOF. By Remark 4 and Prop. 29, $\varinjlim M_\lambda/\widetilde{\varinjlim M_\lambda}\cong\varinjlim M_\lambda/\tilde{M}_\lambda$ and hence we may assume that each $M_\lambda$ is codivisorial. By Prop. 31, there exists a unique homomorphism $g_{\lambda\mu}$ of $D(M_\lambda)$ to $D(M_\mu)$ for $\lambda\leq\mu$ such that $g_{\lambda\mu}i_\lambda=i_\mu f_{\lambda\mu}$ where $i_\lambda$ (resp. $i_\mu$) is the canonical injection of $M_\lambda$ (resp. $M_\mu$) to $D(M_\lambda)$ (resp. $D(M_\mu)$). Hence $\{D(M_\lambda), g_{\lambda,\mu}\}$ is an inductive system over $\Lambda$ and $\{i_\lambda\}$ is an morphism of $\{M_\lambda, f_{\lambda,\mu}\}$ to $\{D(M_\lambda), g_{\lambda,\mu}\}$. Since each $M_\lambda$ is codivisorial, each $D(M_\lambda)$ is codivisorial by Prop. 4 and hence $\varinjlim D(M_\lambda)$ is codivisorial by Prop. 29. Since $(\varinjlim D(M_\lambda))_{\mathfrak{p}}=\varinjlim D(M_\lambda)_{\mathfrak{p}}=\varinjlim M_{\lambda\mathfrak{p}}=(\varinjlim M_\lambda)_{\mathfrak{p}}$ by Coroll. 2 to Th. 3 and Coroll. to Prop. 23, $(\varinjlim i_\lambda, \varinjlim D(M_\lambda))$ is an essentially isomorphic extension of $\varinjlim M_\lambda$ by Coroll. to Prop. 18 and Coroll. to Prop. 20. Therefore $D(\varinjlim M_\lambda)\cong\varinjlim(D(M_\lambda))$ by Prop. 13 because $\varinjlim D(M_\lambda)$ is divisorial by Prop. 34.

LEMMA 3. *Let B be a noetherian ring and $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$, $\{N_\lambda, g_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ be inductive systems of B-modules over a directed set $\Lambda$ and $\{i_\lambda\}$ be a morphism of $\{M_\lambda, f_{\lambda,\mu}\}$ to $\{N_\lambda, g_{\lambda,\mu}\}$. If $i_\lambda$ is an essential extension for any $\lambda$, then so is $\varinjlim i_\lambda$.*

PROOF. Take a non-zero element $x$ of $\varinjlim N_\lambda$. Then there exists an element $\lambda_0$ of $\Lambda$ and an element $x_{\lambda_0}$ of $N_{\lambda_0}$ such that $g_{\lambda_0}(x_{\lambda_0})=x$ where $g_{\lambda_0}$ is the canonical homomorphism of $N_{\lambda_0}$ to $\varinjlim N_\lambda$. Let $\Lambda_0=\{\lambda\in\Lambda|\lambda\geqq\lambda_0\}$ and put $x_\lambda=g_{\lambda_0,\lambda}(x_{\lambda_0})$ for any $\lambda\in\Lambda_0$. Then $\Lambda_0$ is cofinal in $\Lambda$ and $g_\lambda(x_\lambda)=x$ for any $\lambda\in\Lambda_0$ where $g_\lambda$ is the canonical homomorphism of $N_\lambda$ to $\varinjlim N_\lambda$. Since $0(x_\lambda)\subseteq 0(x_{\lambda'})$ if $\lambda\leq\lambda'(\lambda,\lambda'\in\Lambda_0)$ and $\varinjlim 0(x_\lambda)=0(x)$, $0(x)=0(x_{\lambda_1})$ for some $\lambda_1\in\Lambda_0$ because $B$ is noetherian. Therefore $Bx_{\lambda_1}\cong Bx$. Since $i_{\lambda_1}$ is an essential extension, $Bx_{\lambda_1}\cap i_{\lambda_1}(M_{\lambda_1})\neq 0$ and hence $0\neq g_{\lambda_1}(Bx_{\lambda_1}\cap i_{\lambda_1}(M_{\lambda_1}))\subseteq g_{\lambda_1}(Bx_{\lambda_1})\cap g_{\lambda_1}i_{\lambda_1}(M_{\lambda_1})\subseteq Bx\cap(\varinjlim i_\lambda)(\varinjlim M_\lambda)$. This implies that $\varinjlim i_\lambda$ is an essential extension.

PROPOSITION 35. *Let $\{M_\lambda, f_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$, $\{N_\lambda, g_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ be inductive systems* *of codivisorial A-modules over a directed set* $\Lambda$ *and* $\{i_\lambda\}$ *be a morphism of* $\{M_\lambda, f_{\lambda,\mu}\}$ *to* $\{N_\lambda, g_{\lambda,\mu}\}$. *If* $i_\lambda$ *is an essential extension for any* $\lambda$, *then* $\varinjlim i_\lambda$ *is an essential extension.*

PROOF. By Coroll. to Prop. 20 and Prop. 29, it is sufficient to show that $\varinjlim i_{\lambda\mathfrak{p}}$ is an essential extension for any element $\mathfrak{p}$ of $Ht_1(A)$. Since $A_\mathfrak{p}$ is a principal valuation ring, the assertion follows from Lemma 3.

**3.** Now we study a relation between a divisorial envelope and the torsion part.

PROPOSITION 36. *Let M be an A-module. Then*

$$D(t(M)) \cong t(D(M)) \quad \textit{and} \quad D(M/t(M)) \cong D(M)/D(t(M)).$$

PROOF. First we shall show the assertion in the case that $M$ is codivisorial. Let $\mathfrak{p}$ be an element of $Ht_1(A)$. Then $t_A(D(M))_\mathfrak{p} = t_{A\mathfrak{p}}(D(M)_\mathfrak{p}) = t_{A\mathfrak{p}}(M_\mathfrak{p}) = t_A(M)_\mathfrak{p}$ by Coroll. 2 to Th. 3. Therefore $t(D(M))$ is an essentially isomorphic extension of $t(M)$ by Prop. 18 and Coroll. to Prop. 20. Hence $D(t(M)) \cong t(D(M))$ by Prop. 13 and Coroll. 3 to Th. 5. In what follows, we identify $D(t(M))$ with $t(D(M))$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t(M) & \longrightarrow & M & \xrightarrow{p} & M/t(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & & & \\
0 & \longrightarrow & D(t(M)) & \longrightarrow & D(M) & \xrightarrow{q} & D(M)/D(t(M)) & \longrightarrow & 0,
\end{array}$$

where $i$ is the canonical injection of $M$ to $D(M)$ and $p$ (resp. $q$) is the canonical projection of $M$ (resp. $D(M)$) to $M/t(M)$ (resp. $D(M)/D(t(M))$). Then there exists a homomorphism $f$ of $M/t(M)$ to $D(M)/D(t(M))$ such that $qi = fp$. Since $D(M)/D(t(M))$ is divisorial by Coroll. 3 to Th. 5, it is sufficient to show that $f$ is an essentially isomorphic extension by Prop. 13. $f$ is injective because $M \cap D(t(M)) = t(M)$. Hence we can consider $M/t(M)$ as a submodule of $D(M)/D(t(M))$ through $f$. Let $\mathfrak{p}$ be an element of $Ht_1(A)$. Then by Coroll. 2 to Th. 3, $(D_A(M)/D_A(t_A(M))_\mathfrak{p} = D_A(M)_\mathfrak{p}/D_A(t_A(M))_\mathfrak{p} = M_\mathfrak{p}/t_A(M)_\mathfrak{p} = (M/t_A(M))_\mathfrak{p}$. Therefore the assertion follows from Coroll. to Prop. 18 and Coroll. to Prop. 20.

Now we consider the general case. By Coroll. 2 to Prop. 15, $D(M) \cong D(\tilde{M}) \oplus D(M/\tilde{M})$ and $D(\tilde{M}) = \widetilde{D(M)}$. Hence $D(\tilde{M}) \subseteq t(D(M))$. Therefore $t(D(M)) \cong D(\tilde{M}) \oplus t(D(M/\tilde{M}))$. Since $M/\tilde{M}$ is codivisorial, $t(D(M/\tilde{M})) \cong D(t(M/\tilde{M}))$ and hence $t(D(M)) \cong D(\tilde{M}) \oplus D(t(M/\tilde{M}))$. On the other hand, since $\tilde{M} \subseteq t(M)$, $D(t(M)) \cong D(\tilde{M}) \oplus D(t(M)/\tilde{M})$ by Coroll. 2 to Prop. 15. It is easy to see that $t(M/\tilde{M}) = t(M)/\tilde{M}$. Therefore $D(t(M)) \cong D(\tilde{M}) \oplus D(t(M/\tilde{M}))$, namely $D(t(M)) \cong t(D(M))$. Since it is obvious that $M/t(M) \cong M/\tilde{M}/t(M/\tilde{M})$, $D(M/t(M)) \cong D(M/\tilde{M}/t(M/\tilde{M})) \cong D(M/\tilde{M})/D(t(M/\tilde{M}))$ because $M/\tilde{M}$ is codivisorial. On the other hand, $D(M)/$

$D(t(M)) \cong D(\tilde{M}) \oplus D(M/\tilde{M})/D(\tilde{M}) \oplus D(t(M/\tilde{M})) \cong D(M/\tilde{M})/D(t(M/\tilde{M}))$. Hence $D(M/t(M)) \cong D(M)/D(t(M))$.

## §5. A Theory of invariant factors over a Krull domain

**1.** Throughout this section A stands for a Krull domain and $K$ the quotient field of $A$.

DEFINITION 8. *Let $M$ be an $A$-moule. We say that $M$ is essentially finite if $M/t(M)$ is an $A$-lattice and $t(M)_{\mathfrak{p}}=0$ for almost all primes of $Ht_1(A)$ and $l_{\mathfrak{p}}(t(M)_{\mathfrak{p}})<\infty$ for any $\mathfrak{p}$ of $Ht_1(A)$, where $l_{\mathfrak{p}}(tM)_{\mathfrak{p}})$ is the length of the $A_{\mathfrak{p}}$-module $t(M)_{\mathfrak{p}}$.*

REMARK 9. It is easy to see that a finitely generated $A$-module is essentially finite and that an essentially finite module over a Dedekind domain is finitely generated.

PROPOSITION 37. *The following statements concerning an $A$-module $M$ are equivalent:*

(i) *$M$ is essentially finite.*
(ii) *$M/\tilde{M}$ is essentially finite.*
(iii) *$D(M)$ is essentially finite.*
(iv) *$D(M/\tilde{M})$ is essentially finite.*

PROPOSITION 38. *Let $S$ be a multiplicatively closed subset of $A$. If $M$ is an essentially finite $A$-module, then so is $S^{-1}M$ as an $S^{-1}A$-module.*

PROPOSITION 39. *Let $0\to L\to M\to N\to 0$ be an exact sequence of $A$-modules. Then $M$ is essentially finite if and only if $L$ and $N$ are essentially finite.*

PROOF. The assertion is obvious if $M$ is a torsion module. First we suppose that $M$ is essentially finite. Since $t(M)$ is essentially finite torsion module, $t(L)$ is also essentially finite. We put $L'=L/t(L)$, $M'=M/t(M)$ and $N'=N/t(N)$ respectively. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t(L) & \longrightarrow & t(M) & \xrightarrow[p_1]{} & t(N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \xrightarrow[p_2]{} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \xrightarrow[p_3]{} & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (*)$$

where each $p_i$ is the canonical homomorphism and the first two rows and all columns are exact. Since $0 \to L' \to M'$ is exact and $M'$ is an $A$-lattice, we can readily see that $L'$ is also an $A$-lattice. Thus $L$ is essentially finite. Next we shall prove that $N$ is essentially finite. Note first that, since $p_3$ is surjective and $M'$ is an $A$-lattice, $N'$ is also an $A$-lattice. To show that $t(N)$ is essentially finite, by applying Snake Lemma to the last two columns, we consider the exact sequence

$$0 \longrightarrow t(L) \longrightarrow L \xrightarrow{i} \mathrm{Ker}(p_3) \xrightarrow{\delta} \mathrm{Coker}(p_1) \longrightarrow 0 \qquad (**)$$

where $\delta$ is the connecting homomorphism. Since $M'$ is an $A$-lattice, $\mathrm{Ker}(p_3)$ is also an $A$-lattice; therefore $L' \cong \mathrm{Im}(i)$ is a sublattice of $\mathrm{Ker}(p_3)$, because Coker $(p_1)$ is a torsion module. By [4], Prop. 5.2, $L'_{\mathfrak{p}} = (\mathrm{Ker}(p_3))_{\mathfrak{p}}$ for almost all primes $\mathfrak{p}$ of $Ht_1(A)$ and, hence, $(\mathrm{Coker}(p_1))_{\mathfrak{p}} = 0$ for almost all primes $\mathfrak{p}$ of $Ht_1(A)$. It is also easy to see that $l_{\mathfrak{p}}(\mathrm{Coker}(p_1)_{\mathfrak{p}}) < \infty$ for any $\mathfrak{p} \in Ht_1(A)$. Thus $\mathrm{Coker}(p_1)$ is essentially finite. Now the conclusion follows immediately from this fact.

Conversely we suppose that $L$ and $N$ are essentially finite. We can readily see that $t(M)$ is essentially finite by observing the first row of the commutative diagram (*). We can obtain the following exact sequence from (**):

$$0 \longrightarrow L' \longrightarrow \mathrm{Ker}(p_3) \xrightarrow{\delta} \mathrm{Coker}(p_1) \longrightarrow 0.$$

Since $t(N)$ is essentially finite, $\mathrm{Coker}(p_1)$ is an essentially finite torsion module. Therefore $\mathrm{Coker}(p_1)_{\mathfrak{p}} = 0$ for almost all primes of $Ht_1(A)$, namely $L'_{\mathfrak{p}} = \mathrm{Ker}(p_3)_{\mathfrak{p}}$ for almost all primes $\mathfrak{p}$ of $Ht_1(A)$. Again by [4], Prop. 5.2, $\cap \mathrm{Ker}(p_3)_{\mathfrak{p}}$ is an $A$-lattice, because $L'$ is an $A$-lattice, where $\mathfrak{p}$ runs over the primes of $Ht_1(A)$. Since $\mathrm{Ker}(p_3)$ is contained in the above intersection, $\mathrm{Ker}(p_3)$ is an $A$-lattice. Now we consider the exact sequence:

$$0 \longrightarrow \mathrm{Ker}(p_3) \longrightarrow M' \longrightarrow N' \longrightarrow 0.$$

Let $F_1$ be a free submodule of $M'$ which has the same rank as that of $M'$. Put $F' = F \cap \mathrm{Ker}(p_3)$ and $F'' = p_3(F)$. Then it is easy to see that $F' \otimes K = \mathrm{Ker}(p_3) \otimes K$ and $F'' \otimes K = N' \otimes K$. Hence $\mathrm{rank}(F') = \mathrm{rank}(\mathrm{Ker}(p_3))$ and $\mathrm{rank}(F'') = \mathrm{rank}(N')$. Therefore $\mathrm{Ker}(p_3)_{\mathfrak{p}} = F'_{\mathfrak{p}}$ and $N'_{\mathfrak{p}} = F''_{\mathfrak{p}}$ for almost all primes $\mathfrak{p}$ of $Ht_1(A)$. This implies that $M'_{\mathfrak{p}} = F_{1\mathfrak{p}}$ for almost all $\mathfrak{p} \in Ht_1(A)$. By the preceding argument, we can see that $M'$ is an $A$-lattice. This completes the proof.

**2.** It is well known that the torsion part of a finitely generated module over a Dedekind domain is a direct summand. N. Bourbaki showed, in [8], §4, $n$°4, Th. 4, that a finitely generated module $M$ over a noetherian Krull domain is pseudo-isomorphic to $t(M) \oplus M/t(M)$. However, it seems to the authors that the finiteness condition "noetherian" is rather unnatural. By noting that any module over a Dedekind domain is divisorial, we shall formulate a theorem in view of the principle stated in Part I.

THEOREM 6. *Let $M$ be an essentially finite $A$-module. Then* $D(M) = D(t(M)) \oplus D(M/t(M))$.

PROOF. By Coroll. 2 to Prop. 15, Prop. 36 and Prop. 37, we may assume that $M$ is codivisorial and divisorial. By Prop. 36, $t(M)$ is divisorial. Hence $t(M) = \oplus t(M)_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over the primes of $\mathrm{Ass}_A(t(M))$. Since $A_{\mathfrak{p}}$ is a principal valuation ring and $M_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module by Remark 9 and Prop. 38, $t(M)_{\mathfrak{p}}$ is a direct summand of $M_{\mathfrak{p}}$. Let $\varphi_{\mathfrak{p}}$ be the canonical projection of $M_{\mathfrak{p}}$ to $t(M)_{\mathfrak{p}}$ and $i_{\mathfrak{p}}$ be the canonical homomorphism of $M$ to $M_{\mathfrak{p}}$. Since $t(M)$ is codivisorial and essentially finite, $\mathrm{Ass}_A(t(M))$ is a finite set. Hence $\varphi = \oplus \varphi_{\mathfrak{p}} i_{\mathfrak{p}}$ is a homomorphism of $M$ to $t(M) = \oplus t(M)_{\mathfrak{p}}$. We can see that the restriction of $\varphi$ to $t(M)$ is the identity map. Therefore $t(M)$ is a direct summand of $M$.

The following theorem is also a generalization of the fact that a finitely generated module over a Dedekind domain can be decomposed to a direct sum of primary cyclic modules and a projective module uniquely up to isomorphisms.

THEOREM 7. *Let $M$ be an essentially finite $A$-module. Then $M$ is divisorially equivalent to $\oplus_{i \in I} A/\mathfrak{p}_i^{(n_i)} \oplus N$ where $\{\mathfrak{p}_i ; i \in I\}$ is a finite subset of $Ht_1(A)$, $N$ is a divisorial lattice and $\mathfrak{p}_i^{(n_i)}$ means the symbolic $n_i$th power of $\mathfrak{p}_i$. Furthermore the set of pairs $\{(n_i, \mathfrak{p}_i) ; i \in I\}$ is uniquely determined up to permutations and $N$ is uniquely determined up to isomorphisms.*

PROOF. By Prop. 33 and Prop. 37, we may assume that $M$ is codivisorial and divisorial. Then, by Th. 6, $M \cong t(M) \oplus M/t(M)$ and $M/t(M)$ is a divisorial lattice by Coroll. 3 to Th. 5. On the other hand, $t(M) = \oplus t(M)_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over the primes of $\mathrm{Ass}_A(t(M))$ and $\mathrm{Ass}_A(M)$ is a finite set of primes of height 1. Since $A_{\mathfrak{p}}$ is a principal valuation ring and $t(M)_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module by Remark 9, $t(M)_{\mathfrak{p}} \cong \oplus_{j \in I(\mathfrak{p})} A_{\mathfrak{p}}/\mathfrak{p}^{n_j} A_{\mathfrak{p}}$, where $I(\mathfrak{p})$ is a finite set. Furthermore it is well known that $(n_j)_{j \in I(\mathfrak{p})}$ is uniquely determined up to permutations. Since $D(A/\mathfrak{p}^{(n_j)}) \cong A_{\mathfrak{p}}/\mathfrak{p}^{n_j} A_{\mathfrak{p}}$ by Th. 4, $M$ is divisorially equivalent to $\oplus A/\mathfrak{p}^{(n_j)} \oplus M/t(M)$, where $\mathfrak{p}$ runs over the primes of $\mathrm{Ass}_A(t(M))$ and $j$ runs over the set $I(\mathfrak{p})$. The last assertion is clear.

The results stated for noetherian normal domains in Bourbaki [3], § 4, $n°5$ and $n°7$ can be generalized to the case of Krull domains by replacing "*pseudo-isomorphism*" by "*divisorial equivalence*" and "*finitely generated*" by "*essentially finite*".

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