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A Note on Subloops of a Homogeneous Lie Loop and Subsystems of its Lie Triple Algebra

Michihiko KIKKAWA

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Introduction

In our previous paper [3], we have introduced a concept of a geodesic homogeneous Lie loop G which is a generalization of the concept of Lie groups, and shown that the tangent space $\mathfrak{G}$ at the identity of G forms a Lie triple algebra under the operations defined by the torsion and curvature tensors of the canonical connection of G, and that $\mathfrak{G}$ characterizes locally the homogeneous Lie loop G (cf. [3, Definitions 3.1, 3.5 and Theorems 7.2, 7.3, 7.8]).

In this paper, we observe the correspondence between the set of Lie subloops of G and the set of subsystems[1)] of $\mathfrak{G}$, and show the following main theorem:

THEOREM 1. *Let G be a connected geodesic homogeneous Lie loop and $\mathfrak{G}$ its Lie triple algebra. Then, for any connected left invariant Lie subloop H of G, the Lie triple algebra $\mathfrak{H}$ of H is a left invariant subsystem of $\mathfrak{G}$.*

Conversely, for any left invariant subsystem $\mathfrak{H}$ of $\mathfrak{G}$, there exists a unique connected left invariant Lie subloop H of G whose Lie triple algebra is $\mathfrak{H}$.

Here, we call a subloop H of G (resp. subsystem $\mathfrak{H}$ of $\mathfrak{G}$) *left invariant* if it is invariant under the left inner mapping group $L_0(G)$ of G (resp. the group $dL_0(G)$ of linear transformations of $\mathfrak{G}$ induced from $L_0(G)$).

It should be noted that, when G is reduced to a Lie group, the above theorem is reduced to the well known theorem of the correspondence of Lie subgroups of G and Lie subalgebras of the Lie algebra $\mathfrak{G}$ of G.

The notations and terminologies used in this paper are all refered to [3].

§1. Local subloops of a homogeneous Lie loop

To study local subloops of a geodesic local Lie loop in a locally reductive space, we consider its auto-parallel submanifolds. Let M be a differentiable manifold with a linear connection ∇. A submanifold N of M is called *autoparallel* if, for each vector X tangent to N at any x and for each piecewise differentiable

1) By a *subsystem* of a Lie triple algebra $\mathfrak{G}$, we mean a subalgebra of $\mathfrak{G}$ which is closed under the ternary operation of $\mathfrak{G}$.

curve γ starting from x and contained in N, the parallel displacement of X along γ (w.r.t. ∇) yields a vector tangent to N. Auto-parallel submanifolds have been treated in [4, Ch. VII § 8]. We recall here some results about them (cf. loc. cit. Propositions 8.2–8.6). A submanifold N of M is auto-parallel if and only if the vector field $\nabla_X Y$ is tangent to N at each point of N, for any vector fields X, Y on N, and so a linear connection ∇' on N is naturally induced from ∇ by $\nabla'_X Y = \nabla_X Y$. Moreover, the torsion tensor S', curvature tensor R' and their successive covariant derivatives of ∇' are obtained by the natural restriction of those of ∇ to N, respectively. Especially, if M is locally reductive, that is, the torsion S and curvature R of ∇ are both parallel, then so is N (w.r.t. ∇'). Every auto-parallel submanifold of M is totally geodesic. Conversely, if the torsion S of M vanishes identically, then every totally geodesic submanifold of M is auto-parallel.

Let (U, μ) be a local Lie loop with the identity e (cf. [3, Definition 4.2]). A submanifold V of U through e will be called a *local Lie subloop* of (U, μ) if the restriction μ_V of μ to the intersection of $V \times V$ and the domain of μ forms a local Lie loop in V.

PROPOSITION 1. *Let (U, μ) be a geodesic local Lie loop at e in a locally reductive space* [3, Definition 4.1]. *Any auto-parallel submanifold of U through e has a neighborhood V of e which is a local Lie subloop of (U, μ) and which coincides with a geodesic local Lie loop with respect to the induced connection ∇' on V.*

Conversely, any local Lie subloop V of (U, μ) is an auto-parallel submanifold of U.

Moreover, the Lie triple algebra of any local Lie subloop V of (U, μ) is a subsystem of the Lie triple algebra of U at e (cf. [3, Theorem 7.2]).

PROOF. Let V be an auto-parallel submanifold of U through e. Then any U-geodesic tangent to V must be a V-geodesic (a geodesic with respect to the induced connection ∇' in V). Since the U-parallel displacement of vectors tangent to V yields vectors tangent to V, along any V-geodesic, and since such a U-parallelism is also a V-parallelism, we see that there exists a V-geodesic local Lie loop defined in V at e, such that it is a local Lie subloop of the U-geodesic local Lie loop (U, μ).

Conversely, let (V, μ_V) be a local Lie subloop of (U, μ). By [3, Proposition 4.4] we know that there exists a local 1-parameter subgroup $x(t)$ of U which is a geodesic tangent to X at e, for each tangent vector X at e. Assume that X is tangent to V and consider the vector field $\tilde{X}$ on U defined by $\tilde{X}(x) = dL_x(X)$ $(x \in U)$. Since (V, μ_V) is supposed to be a local Lie subloop, we see that the restriction of $\tilde{X}$ to V is a differentiable vector field on a neighborhood of e in V.

Then, the local 1-parameter subgroup $x(t)$ becomes an integral curve of this vector field and so it must be a local 1-parameter subgroup of (V, μ_V). Thus we see that any geodesic $x(t)$ tangent to V at $e=x(0)$ is contained in V in a neighborhood of e. By definition, the left translation $L_{x(t)}$ induces the parallel displacement along the geodesic $x(t)$. We know also that any left translation L_x of (U, μ) is a local affine transformation [3, Lemma 4.2], and so we see that it commutes with the parallel displacements of vectors along any geodesic and along its L_x-image. Therefore, it follows that the tangent space $\mathfrak{V}_e$ to V at e is sent to $\mathfrak{V}_x$ tangent to V at x by the left translation L_x, and that the parallel displacement along a geodesic in V through x is obtained, locally, as an image of the parallel displacement along a geodesic in V through e, under L_x. From these facts we can conclude that the parallel displacement of a V-vector along any geodesic contained in V is still tangent to V. Hence $\nabla_X Y$ is tangent to V for any vector fields X, Y on V, that is, V is an auto-parallel submanifold of U.

Let ∇' be the induced connection on V. Then we see that μ_V is coincident, locally, with the local multiplication of a geodesic local Lie loop in V at e. Thus the Lie triple algebra $\mathfrak{V}=\mathfrak{V}_e$ of an arbitrarily given local Lie subloop (V, μ_V) is well defined as that of the underlying locally reductive space of the geodesic local Lie loop. Since the torsion and curvature of ∇' are obtained by the restriction of those of U, it is clear that $\mathfrak{V}$ is a subsystem of the Lie triple algebra of the geodesic local Lie loop (U, μ).

q.e.d.

§2. Germs of subloops of a geodesic homogeneous Lie loop

Let M be a differentiable manifold. Two local Lie loops (H_1, μ_1) and (H_2, μ_2) defined in M are *equivalent* if they have a common point e as their identities and a common neighborhood of e on which the local multiplication μ_1 coincides with μ_2. A *germ of local Lie loops* of M is an equivalence class of local Lie loops of M. On a locally reductive space M with a fixed point e, there is determined a unique germ of local Lie loops at e to which all geodesic local Lie loops at e belong. Moreover, from Proposition 1 it follows that any germ of local Lie subloops of a geodesic local Lie loop (U, μ) of M at e can be represented by a geodesic local Lie loop of an auto-parallel submanifold of M through e, and that there corresponds to each germ of local Lie subloops of (U, μ) a subsystem of its Lie triple algebra. In the following, we study the inverse of this correspondence for a geodesic homogeneous Lie loop G.

A homogeneous Lie loop G can be regarded as a reductive homogeneous space $A(G)/K(G)$, where $A(G)=G\times K(G)$ (semi-direct product) and $K(G)$ is the closure of the left inner mapping group $L_0(G)$ [3, Theorem 3.7]. If G is geodesic [3, Definition 5.1], then it belongs to the germ of local Lie loops determined by any geodesic local Lie loop at the identity e of G (with respect to the canonical connection of G which is known to be locally reductive [3, Theorem 5.7]).

We have proved in [2] the following result:

LEMMA [2, Theorem 4]. *Let $G=A/K$ be a reductive homogeneous space with the origin e, A acting effectively on G, and let $\mathfrak{H}$ be an arbitrary subsystem of the Lie triple algebra $\mathfrak{G}$ of the geodesic local Lie loop at e (w.r.t. the canonical connection). Then there exists an auto-parallel submanifold H of G tangent to $\mathfrak{H}$ at e.*

By using this lemma we show the following

THEOREM 2. *Let G be a geodesic homogeneous Lie loop and $\mathfrak{G}$ its Lie triple algebra. There exists a one-to-one correspondence between the set of all germs of local Lie subloops of G and the set of all subsystems of $\mathfrak{G}$.*

PROOF. To a representative H of an arbitrarily given germ of local Lie subloops of G we can assign the Lie triple algebra $\mathfrak{H}$ of H which is a subsystem of $\mathfrak{G}$, by Proposition 1. Then $\mathfrak{H}$ does not depend on the choice of the representative H of the germ. If the same subsystem $\mathfrak{H}$ is assigned to two germs with representatives H_1 and H_2, respectively, then by Proposition 1 H_i's are auto-parallel submanifolds tangent to each other at the identity e. Since the exponential mapping at e (w.r.t. the canonical connection) is a local diffeomorphism which sends a neighborhood of zero vector in $\mathfrak{H}$ to an auto-parallel submanifold of G, we see that H_1 and H_2 have a common neighborhood of e. Using Proposition 1 again, we can conclude that H_1 and H_2 are equivalent to a geodesic local Lie loop with respect to the induced connection. Thus the germ to which a given subsystem $\mathfrak{H}$ is assigned is unique, if it exists.

Now we apply the above lemma to our homogeneous Lie loop $G=A(G)/K(G)$. Then, given a subsystem $\mathfrak{H}$ of the Lie triple algebra $\mathfrak{G}$, we get an auto-parallel submanifold H tangent to $\mathfrak{H}$ at e. Since G is supposed to be geodesic, Proposition 1 shows that $\mathfrak{H}$ is the Lie triple algebra of a geodesic local Lie loop in H, which is a subsystem of $\mathfrak{G}$. *q.e.d.*

§3. Left invariant subloops

Let G be a homogeneous Lie loop. A submanifold H of G is called a *Lie subloop* of G if H is a subloop of G and if $\mu_H: H\times H\to H$ is differentiable, where μ_H is the restriction of the multiplication μ of G to $H\times H$.

PROPOSITION 2. *Every connected Lie subloop H of a geodesic homogeneous Lie loop G is itself geodesic homogeneous. Moreover, H is an auto-parallel submanifold of G and the canonical connection of H is coincident with* *the induced connection on H.*

PROOF. Let H be a connected Lie subloop of G. Then H is itself homogeneous since any abstract subloop of a homogeneous loop is homogeneous. By Proposition 1, there exists a neighborhood V of the identity e in H which is an autoparallel submanifold of G. Since any left translation of G is an affine transformation, by translating V under L_x $(x \in H)$, it can be shown that H is auto-parallel. Let $\mathfrak{H}$ be the tangent space to H at e. For any fixed $X_0 \in \mathfrak{H}$, consider an integral curve $x(t)$ $(x(0)=e)$ of the vector field $\bar{X}^H(x)=dR^H{}_x(X_0)$ $(x \in H)$ on H.[2] Then, by [3, Proposition 5.1], the curve $x(t)$ is a geodesic of H with respect to the canonical connection of H. Since H is a Lie subloop of G, $x(t)$ is also an integral curve of the vector field $\bar{X}(x)=dR_x(X_0)$ $(x \in G)$ on G. It follows that any H-geodesic through e is a G-geodesic. By considering the homogeneous structure [3, Definition 1.5] of H, we see that any H-geodesic is a geodesic of the induced connection in H, and vice versa. Moreover, since G is geodesic, the left translation $L_{x(t)}$ induces a parallel displacement along the curve $x(t)$ in a neighborhood of $e=x(0)$, and so, restricting it to H and taking account of the homogeneity of H, we can show that the canonical connection of H is coincident with the induced connection of H. The equality $L_{x(t),x(s)}=\mathrm{id}$ on G implies $L^H_{x(t),x(s)}=\mathrm{id}$ on H, which shows that H is geodesic. *q.e.d.*

In the rest of this paper, a homogeneous Lie loop G is always assumed to be geodesic. Then, by Theorem 2 and Propositino 2, the Lie triple algebra of any Lie subloop of G is a subsystem of the Lie triple algebra $\mathfrak{G}$ of G. Let $L_0(G)$ denote the left inner mapping group of G and $dL_0(G)$ the group of linear transformations of $\mathfrak{G}$ induced from $L_0(G)$. A subloop H of G will be called *left invariant* if H is invariant under $L_0(G)$. For instance, any normal subloop of G is left invariant and, when G is reduced to a Lie group, any subgroup of G is left invariant.

A subsystem $\mathfrak{H}$ of the Lie triple algebra $\mathfrak{G}$ of G will be called *left invariant* if the group $dL_0(G)$ leaves $\mathfrak{H}$ invariant.

PROPOSITION 3. *For any left invariant subsystem $\mathfrak{H}$ of $\mathfrak{G}$, the assignment $\Sigma\colon x \to \mathfrak{H}_x = dL_x(\mathfrak{H})$ $(x \in G)$ defines a differentiable distribution on G which is parallel with respect to the canonical connection.*

PROOF. For any fixed basis $\{X_1, X_2, \dots, X_m\}$ $(m=\dim \mathfrak{H})$ of the subspace $\mathfrak{H}$ of $\mathfrak{G}$, the differentiable vector fields $\tilde{X}_i$ $(i=1, 2, \dots, m)$ defined by $\tilde{X}_i(x)=dL_x(X_i)$ $(x \in G)$ form a basis of $\mathfrak{H}_x$ at each $x \in G$. Hence the distribution Σ is differentiable. We observe that Σ is invariant under any left translation $L_y^{(x)}$ of any transposed loop $G^{(x)}$ of G centered at x. In fact, by the definition [3, (1.5)] of the multiplication of $G^{(x)}$, we get

2) The superscript H denotes the corresponding argument in the homogeneous Lie loop H.

$$dL_y^{(x)}(\mathfrak{H}_x) = dL_x \circ dL_{x^{-1}y} \circ dL_x^{-1}(\mathfrak{H}_x)$$
$$= dL_y \circ dL_{x,x^{-1}y}(\mathfrak{H})$$
$$= dL_y(\mathfrak{H}) = \mathfrak{H}_y \qquad \text{for any} \quad x, y \in G.$$

Now we show that the distribution Σ is parallel, that is, for any $x, y \in G$ the parallel displacement τ_γ along any piecewise differentiable curve γ joining x to y sends $\mathfrak{H}_x$ to $\mathfrak{H}_y$. From the assumption that G is geodesic, it follows that every transposed loop $G^{(x)}$ of G centered at any $x \in G$ is also geodesic. Therefore, if γ is a geodesic segment, τ_γ is coincident with the linear map $dL_y^{(x)}$ and so it sends $\mathfrak{H}_x$ to $\mathfrak{H}_y$ as was shown above. For any piecewise differentiable curve γ: $t \to x(t)$ $(x(0)=x, x(1)=y)$, we can choose an ordered set $\{x_0=x, x_1, \ldots, x_k=y\}$ of points on γ such that each $x_i = x(t_i)$ $(0=t_0<t_1<\cdots<t_{i-1}<t_i<\cdots<t_k=1)$ is contained in a normal neighborhood U_i of x_{i-1}. Joining x_{i-1} to x_i by a geodesic segment γ_i in U_i, we see that the parallel displacement along the piecesise geodesic arc $\gamma_1\gamma_2\ldots\gamma_k$ is equal to the composition $dL_y^{(x_{k-1})} \circ \cdots \circ dL_{x_i}^{(x_{i-1})} \circ \cdots \circ dL_{x_1}^{(x)}$ of the linear isomorphisms. Since the parallel displacement τ_γ of a vector is given as a solution of the differential equation $\nabla_{\dot{x}} X = 0$ along $x(t)$, it can be regarded as the limit of a sequence of parallel displacements along piecewise geodesic arcs from x to y (as given above) converging to γ. As each of such parallel displacements sends $\mathfrak{H}_x$ to $\mathfrak{H}_y$, we have $\tau_\gamma(\mathfrak{H}_x) = \mathfrak{H}_y$. *q.e.d.*

In view of this proof we see the following

COROLLARY. *Every left invariant subsystem of the Lie triple algebra $\mathfrak{G}$ of G is invariant under the holonomy group of the canonical connection.*

REMARK. From [3, Theorem 7.7] and the above corollary, it follows that every left invariant subsystem $\mathfrak{H}$ of $\mathfrak{G}$ is sent into itself under the inner derivation algebra $\mathfrak{K}_0$ of $\mathfrak{G}$, that is, the subsystem $\mathfrak{H}$ satisfies

$$(*) \qquad [X, Y, \mathfrak{H}] \subset \mathfrak{H} \qquad \text{for any} \quad X, Y \in \mathfrak{G},$$

where the bracket denotes the ternary operation of $\mathfrak{G}$. Suppose that G is simply connected and the closure $K(G)$ of $L_0(G)$ is a simple Lie group. Then $K(G)$ coincides with the holonomy group of the canonical connection of G and so the subsystem $\mathfrak{H}$ of $\mathfrak{G}$ is left invariant if and only if $\mathfrak{H}$ satisfies the above condition (*). (Cf. [3, Theorem 7.3].)

§4. Proof of the main theorem

Now we prove Theorem 1 mentioned in the introduction. The first half of the theorem is clear from Theorem 2, Proposition 2 and from the definition of the left invariance of subloops and subsystems. Therefore it is sufficient to show the following

THEOREM 3. *Let G be a connected geodesic homogeneous Lie loop and $\mathfrak{G}$ the Lie triple algebra of G. For any left invariant subsystem $\mathfrak{H}$ of $\mathfrak{G}$, the distribution Σ given in* Proposition 3 *is completely integrable and the maximal integral manifold H through the identity e is a left invariant Lie subloop of G.*

In fact, if this theorem is proved, then by Theorem 2 and Proposition 2 H is an only Lie subloop of G tangent to $\mathfrak{H}$ at e such that H is itself a geodesic homogeneous Lie loop with $\mathfrak{H}$ as its Lie triple algebra.

PROOF. Let X, Y be any vector fields on G belonging to the distribution Σ. The value for X, Y of the torsion S of the canonical connection ∇ of G is given by

$$(**)\qquad S(X, Y) = [X, Y] - \nabla_X Y + \nabla_Y X .$$

By Proposition 3, Σ is parallel and so the vector fields $\nabla_X Y$ and $\nabla_Y X$ belong again to Σ. On the other hand, since the connection ∇ is locally reductive, S is parallel so that

$$\tau_\gamma(S_e(X_e, Y_e)) = S_x(\tau_\gamma(X_e), \tau_\gamma(Y_e))$$

holds for any point $x \in G$ and for any curve γ joining e to x. The bilinear operation of the Lie triple algebra $\mathfrak{G}$ is given, by definition, as the value at e of the torsion tensor S (cf. [3, Theorem 7.3]). Then we get $S_e(X_e, Y_e) \in \mathfrak{H}$ for any X_e, $Y_e \in \mathfrak{H}$ and so the preceding equality implies $S_x(X_x, Y_x) \in \mathfrak{H}_x$ for the vector field X, $Y \in \Sigma$, since Σ is parallel. Thus $[X, Y]_x \in \mathfrak{H}_x$ $(x \in G)$ is obtained in (**), which shows that Σ is completely integrable.

Now, let H be a maximal integral manifold of Σ through e. As was shown in the proof of Proposition 3, Σ is invariant under any left translation L_x $(x \in G)$. Hence xH is an integral manifold of Σ through x. If $x \in H$, then we have $xH \subset H$, and by the left inverse property of the loop G we get the equalities $xH = H = x^{-1}H$. It follows that H is an abstarct homogeneous subloop of G. It can be shown, by the same way as in the case of a connected Lie group, that the homogeneous Lie loop G is generated by any neighborhood of e. Hence G has a countable basis and so does H. Then it is shown that the restriction $\mu_H\colon H \times H \to H$ of the multiplication μ of G to $H \times H$ is differentiable. The proof of this fact goes similarly to that in the theory of Lie groups (cf., e.g., [1, p. 108]). Therefore, we see that H is a Lie subloop of G. For any x, $y \in G$, the left inner mapping $L_{x,y}$ leaves the distribution Σ invariant, and so does $L_{x,y}^{-1} = L_{y^{-1},x^{-1}}$ [3, Lemma 1.8]. It follows that the submanifold $L_{x,y}(H)$ coincides with H as a maximal integral manifold of Σ through e. Thus we proved that H is left invariant. *q.e.d.*

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Department of Mathematics,
Faculty of Literature and Science,
Shimane University