HIROSHIMA MATH. J.
5 (1975), 407–438

The Stable Homotopy Groups of Spheres III

Shichirô OKA*)

(Received April 25, 1975)

Introduction

The present paper is the third part of the series [8]. §§1–8 are contained in Part I, §§9–17 are contained in Part II, and this part consists of §§18–23. We shall use all notations and notions defined in the previous parts.

Throughout this paper, we shall always assume that the fixed prime p is greater than 3, and not treat the 3-primary component. Our results in the previous parts are summarized in [8–II; Th. A], where we determined the group ${}_p\pi_k(S)$, the p-primary component of the k-th stable homotopy group of spheres, for $k \leqq (p^2+3p+1)q-6$, $q=2(p-1)$, and the first unsolved problem was to determine the composition $\alpha_1\beta_1\beta_{p+1}$. In [10] we have obtained the relations $\alpha_1\beta_1\beta_{p+s}=0$, $1 \leqq s \leqq p-3$, which enable us to extend our calculations.

In this part, we shall determine the group ${}_p\pi_k(S)$ for $(p^2+3p+1)q-5 \leqq k \leqq (2p^2+p)q-4$. In this range, there appear the following new generators:

$$\kappa_s = \{\beta_1\beta_{p+s}, \alpha_1, \alpha_1\}^{**)} \quad \textit{of degree} \quad (p^2+(s+2)p+s+1)q-5,\ 1 \leqq s \leqq p-3,$$

$$\lambda' = \{\beta_1^p, \varepsilon', \alpha_1\} \quad \textit{of degree} \quad (2p^2+1)q-5,$$

$$\lambda_1 = \{\varepsilon_1, \beta_1^p, \alpha_1\} \quad \textit{of degree} \quad (2p^2+1)q-4,$$

$$\lambda_i = \{\lambda_{i-1}, p\iota, \alpha_1\} \quad \textit{of degree} \quad (2p^2+i)q-4, \quad 2 \leqq i \leqq p-3,$$

$$\mu \in \{\lambda_{p-3} - y\beta_1\beta_{2p-2}, \alpha_1, \alpha_1\} \quad \textit{of degree} \quad (2p^2+p-1)q-5,$$

where $y \in Z_p$ is the coefficient in the relation $\alpha_1\lambda_{p-3} = y\alpha_1\beta_1\beta_{2p-2}$ and the orders of κ_s, λ', λ_i and μ are p, p, p and p^2, respectively. These elements together with the *α-families* $\{\alpha_r\}$, $\{\alpha'_{rp}\}$, $\{\alpha''_{rp^2}\}$ ([1], [9; §4], [14–IV]) and the *β-family* $\{\beta_r\}$ ([12], [17]) form a multiplicatively generating set for ${}_p\pi_k(S)$ in the cited range of k. Here, the orders of α_r, α'_{rp}, α''_{rp^2} and β_r are p, p^2, p^3 and p, respectively, and $\deg\alpha_r = rq-1$, $\deg\alpha'_{rp} = rpq-1$, $\deg\alpha''_{rp^2} = rp^2q-1$ and $\deg\beta_r = (rp+r-1)q-2$.

Our main results for ${}_p\pi_k(S)$ are Theorems 19.9, 21.6, 22.2 and 22.3, which are summarized in the following

---

*) This work was partially supported by the Sakkokai Foundation.

**) $\alpha = \{\beta, \gamma, \delta\}$ means that a secondary composition $\{\beta, \gamma, \delta\}$ consists of a single element α.

THEOREM C ($p\geqq 5$). *The group* ${}_p\pi_k(S)$ *for* $(p^2+3p+1)q-5\leqq k\leqq(2p^2+p)q-4$, $q=2(p-1)$, *is the direct sum of the cyclic groups generated by the following elements of degree* k(*r and s satisfy* $0\leqq r<p$, $s\not\equiv 0 \bmod p$ *and* $s\geqq 1$):

$$\alpha_s,\quad \alpha'_{sp},\quad \alpha''_{sp^2},\quad \beta_1^t(p+4\leqq t\leqq 2p+1)^{*)},\quad \beta_1^r\beta_s\ (s\geqq 2),$$

$$\alpha_1\beta_1^r\beta_s\ (2\leqq s<p),\quad \beta_1^r\beta_2\beta_{p-1},\quad \alpha_1\beta_1^r\beta_2\beta_{p-1}\ (r\leqq p-2),$$

$$\alpha_1\beta_s\ (p+1\leqq s\leqq 2p-2),\quad \alpha_1\beta_1\beta_{2p-2},\quad \beta_1^r\kappa_s\ (s\leqq p-3),$$

$$\lambda',\ \lambda_s\ (s\leqq p-3),\quad \alpha_1\lambda_s\ (s\leqq p-4),\quad \mu.$$

We shall also determine the ring structure of ${}_p\pi_*(S)$ up to degree $(2p^2+p)q-3$, in Theorem 23.3. To put it briefly, the elements α's, β's, ε's, φ, κ's, λ', λ_1 and μ form a minimal generating set and any relation is a consequence of (22.3)–(22.9) and (23.1)–(23.10) given in §§22–23. We give hereupon some of these as follows ($x, y\not\equiv 0 \bmod p$):

$$\varepsilon_i\varepsilon_j=-\lambda_{i+j}\ (=x\beta_1^{p-1}\beta_2\beta_{p-1}\quad \textit{for } i+j=p-2,\ =0\ \textit{for } i+j\geqq p-1),$$

$$p\mu=-\varepsilon_{p-2}\varepsilon'=\alpha_1\varepsilon_i\varepsilon_{p-2-i}=\beta_1^p\varphi=-x\alpha_1\beta_1^{p-1}\beta_2\beta_{p-1},$$

$$\beta_{p-1}\varphi=-3x\alpha_1\beta_1\beta_{2p-2},\quad \alpha_1\lambda_{p-3}=y\alpha_1\beta_1\beta_{2p-2},\quad \alpha_1\beta_{2p-1}=0.$$

In § 18 and § 20, we shall calculate the cohomology group of the spectrum $\boldsymbol{K}_k$ for $(p^2+3p)q-3\leqq k\leqq(2p^2+p-3)q-4$, $q=2(p-1)$, as a module over the Steenrod algebra A^*. Our result is Theorem 18.2, which will be proved in § 20. This calculation is a continuation of Theorem 16.1 in Part II.

In § 19, we shall determine ${}_p\pi_k(S)$ for $k\leqq(2p^2+p-2)q-6$ from the results on $H^*(\boldsymbol{K}_k)$. But the group ${}_p\pi_{(2p^2+p-2)q-5}(S)$ can not be determined from our calculations for $H^*(\boldsymbol{K}_k)$, and so we can not continue the calculations by means of our method.

In § 21, we shall compare our results with Nakamura's on $\mathrm{Ext}_{A^*}(Z_p, Z_p)$ [5] via the Adams spectral sequence and obtain the results on ${}_p\pi_k(S)$ for $(2p^2+p-2)q-5\leqq k\leqq(2p^2+p)q-4$. In Theorems 21.6 and 22.3, we shall determine the group ${}_p\pi_k(S)$ in the cited range, from Nakamura's results.

In §§ 22–23, the ring structure of ${}_p\pi_*(S)$ will be discussed.

The author wishes to thank Mr. O. Nakamura for his valuable comments in preparing § 21.

§ 18. A^*-module structure of $H^*(\boldsymbol{K}_k)$

We recall the spectrum $\boldsymbol{K}_k=\{K_k(n)\}$ of (1.1) and its cohomology group

*) The superscript t indicates the power: $\beta_1^t=\beta_1\circ\cdots\circ\beta_1$ (t-times composition), $\beta_1^0=\iota$.

$H^*(\boldsymbol{K}_k)$ with the coefficient Z_p. (Here, we omit the coefficient Z_p). The space $K_k(n)$ is obtained from S^n by attaching cells such that the inclusion i: $S^n \to K_k(n)$ induces isomorphisms of $\pi_j(\ \ )$ for $j<n+k$ and that $\pi_j(K_k(n))=0$ for $j \geqq n+k$. There is the exact sequence of modules over A^*, the Steenrod algebra mod p,

$$(18.1)\quad \cdots \xrightarrow{j^*} H^t(\boldsymbol{K}_k) \xrightarrow{i^*} H^t(\boldsymbol{K}_{k+1}) \xrightarrow{\delta^*} H^{t-k}({}_p\pi_k(\boldsymbol{S})) \xrightarrow{j^*} H^{t+1}(\boldsymbol{K}_k) \xrightarrow{i^*} \cdots,$$

where $H^*({}_p\pi_k(\boldsymbol{S}))=H^*(\boldsymbol{K}({}_p\pi_k(\boldsymbol{S})))$ and $\boldsymbol{K}({}_p\pi_k(\boldsymbol{S}))=\{K({}_p\pi_k(\boldsymbol{S}), n)\}$ is an Eilenberg-MacLane spectrum (see (1.2)).

For $0<t\leqq k$, the group $H^t(\boldsymbol{K}_k)$ vanishes (see (1.3)), and for $t=k+1$, there is the epimorphism

$$(18.2)\qquad \phi\colon {}_p\pi_k(\boldsymbol{S}) \longrightarrow H^{k+1}(\boldsymbol{K}_k)\ \ (\approx {}_p\pi_k(\boldsymbol{S})\otimes Z_p),$$

which is essentially the projection to the quotient (see (2.1)).

For any $a\in H^t(\boldsymbol{K}_k)$ and $l>k$, we denote by *a in $\boldsymbol{K}_l$* or simply by a the element $i^*a\in H^t(\boldsymbol{K}_l)$ for the inclusion $i\colon \boldsymbol{K}_l\to \boldsymbol{K}_k$, and for any non-zero $a\in H^t(\boldsymbol{K}_k)$, $t>0$, we define

$$h(a) = \min\{\, l \mid \textit{there is } a'\in H^t(\boldsymbol{K}_l) \textit{ such that } a' = a \textit{ in } \boldsymbol{K}_k\}.$$

In [8], we have calculated the A^*-modules $H^*(\boldsymbol{K}_k)$ for $k\leqq(p^2+3p)q-4$. Here, we denote always $q=2(p-1)$, and also we put

$$R_k = (k+1)\mathscr{P}^1\Delta - k\Delta\mathscr{P}^1 \in A^{q+1},$$

$$W_k = (k+1)\mathscr{P}^p\mathscr{P}^1\Delta - k\mathscr{P}^{p+1}\Delta + (k-1)\Delta\mathscr{P}^{p+1} \in A^{(p+1)q+1}.$$

The following result is a part of Theorem 16.1.

(18.3) *Within the limits of degree less than $(2p^2+p)q-3$, $H^*=H^*(\boldsymbol{K}_{(p^2+3p)q-4})$ is generated over A^* by the following elements ($r=p^2+3p$):*

$$a_0\in H^0,\ a_r\in H^{rq},\ a'_r\in H^{rq+1},\ b^1_{p+1}\in H^{rq-3},$$

$$b^0_{p+2}\in H^{(r+1)q-1},\ c^{p+2}_1\in H^{(r+1)q-8},\ b^{p+3-s}_s\in H^{(r+s+2)q+2s-9}\ (4\leqq s\leqq p),$$

$$d_5\in H^{2p^2q-3},\ l_1\in H^{(r+1)q-3},\ l'_2\in H^{(r+p+2)q-2},$$

and in addition $d_7\in H^{(2p^2+p-2)q}$ for $p=5$. Also, the A^*-module structure of H^* is given by the following relations:

(i) $A^i a_0 = 0 \quad$ *for* $\quad 0<i<p^3q$;

(ii) $\Delta a_r = 0,\quad \Delta a'_r = 0,\quad \Delta\mathscr{P}^1 a_r - \mathscr{P}^1 a'_r = 0$;

(iii) $\mathscr{P}^1 b^1_{p+1} = x l_1 \qquad (x\in Z_p)$;

(iv) $\mathscr{P}^1 b^0_{p+2}=0,\quad W_2 b^0_{p+2}=0,\quad \mathscr{P}^1 b^{p+3-s}_s=0$;

(v) $\mathscr{P}^{p-2} c^{p+2}_1=0$;

(vi) $\mathscr{P}^1 l_1=0,\quad W_3 l_1=0,\quad \mathscr{P}^1 l'_2=0,\quad W_4 l'_2=0$;

(vii) $\mathscr{P}^1 d_5-\mathscr{P}^{p(p-3)} l_1=0,\quad \mathscr{P}^1 \Delta d_5-(1/2)\Delta\mathscr{P}^{p(p-3)} l_1=0$;

(viii) $(p=5)$ $\Delta d_7\equiv 0 \bmod A^* b^0_7,\quad \mathscr{P}^1 d_7\equiv 0 \bmod A^*\{l_1, b^0_7\}$*).

REMARK. The above relations (iii) and (v)–(vii) are slightly different from Table B9 in Theorem 16.1. The coefficient x_7 in Table B9 $(b-3)$ is denoted in the above (iii) by x. Theorem 16.1 is lacking the relation (v) (and the element c^{p+2}_1) (see the next corrections). In (15.4) (iii), we can take $C_1=(1/2)\Delta\mathscr{P}^{p(p-3)}$ and $C_2=-(1/2)\mathscr{P}^{p(p-3)}$ by easy calculations using the Adem relations, and so the above (vii) coincides with the relation B8$(d-1)$ [8–II; p. 142] with $b^2_{p-1}=b^{p+1}_1=b^1_p=l'_1=b^0_{p+1}=0$. In Theorem 16.1, we proved the last relation of (vi) above in a weak form: $W_4 l'_2\equiv 0 \bmod A^{a+2} l_1+A^{a+q+2} b^1_{p+1}+A^a b^0_{p+2}$, $a=(2p+2)q$ (Table B9$(l-1)$). From the relation $\alpha_1\beta_1\beta_{p+1}=0$ [10; Cor. 1] it follows that $x\neq 0$ in (iii) above (see Lemma 18.1 in the below), and hence we can omit the term $A^{a+2} l_1$. We have $A^{a+q+2}=Z_p\{\Delta\mathscr{P}^{2p+3}\Delta\}+\mathrm{Im}(\mathscr{P}^2)^*+\mathrm{Im}(W_3\mathscr{P}^1)^*+\mathrm{Im}(W_4)_*$**) and $A^a=Z_p\{\mathscr{P}^{2p+2}\}+\mathrm{Im}(\mathscr{P}^1)^*$, and so $W_4 l'_2=y\Delta\mathscr{P}^{2p+3}\Delta b^1_{p+1}+z\mathscr{P}^{2p+2} b^0_{p+2}$ for some $y, z\in Z_p$. Operating $\mathscr{P}^1$ and W_5 to this, we obtain $y=0$ and $z=0$, respectively. Thus, $W_4 l'_2=0$ as desired.

CORRECTIONS TO § 16. (i) There are lacks of the elements c^{p+2}_1 of $H^*(\boldsymbol{K}_k)$ and $\beta^3_1\varepsilon'$ of $\pi_*(S)$. In Table A9, the element c^{p+2}_1 satisfying

$$\deg c^{p+2}_1=(p^2+3p+1)q-8,\ h(c^{p+2}_1)=(p^2+3p-1)q-7,\ \delta^* c^{p+2}_1=\mathscr{P}^2 j^{*-1} b^{p+2}_1$$

should be added. In Table B9, the relation

$$\mathscr{P}^{p-2} c^{p+2}_1=0$$

should be added. Also, in the table of Theorem 16.2, the element $\beta^3_1\varepsilon'$ satisfying the following should be added:

$$\deg\beta^3_1\varepsilon'=(p^2+3p+1)q-9,\ \textit{order of}\ \beta^3_1\varepsilon'=p,\ \phi(\beta^3_1\varepsilon')=c^{p+2}_1.$$

In the 2nd and 9th lines of [8–II; p. 146], the element c^{p+2}_1 and the equality

*) We denote by $A^*\{d_1,\dots,d_n\}$ the (left) A^*-module generated by the elements $d_1,\dots,d_n$, and simply by $A^* d_1$ if $n=1$. Also, we denote by $Z_p\{d_1,\dots,d_n\}$ the linear space over Z_p with basis $d_1,\dots,d_n$.

**) For any $\alpha\in A^*$, α_*: $A^*\to A^*$ and α^*: $A^*\to A^*$ denote the right and the left translations by α, i.e., $\alpha_*(\beta)=\alpha\beta$ and $\alpha^*(\beta)=\beta\alpha$, respectively.

$\phi(\beta_1^3\varepsilon')=c_1^{p+2}$ should be listed.

(ii) Recently we withdrew a statement $\gamma_1=0$ and proved [10] an opposite result $\gamma_1\neq 0$. So, the 3rd line in the remark at the end of Theorem 16.2 and the foot-note to this remark [8–II; p. 147] should be cancelled.

Now we shall come back to (18.3).

LEMMA 18.1. *In* (18.3) (iii), *the coefficient x is non-zero, and hence the module structure of the submodule* $A^*b_{p+1}^1$ *of* $H^*(\boldsymbol{K}_{(p^2+3p)q-4})$ *is given by the following relations*:

$$\mathscr{P}^2 b_{p+1}^1 = 0, \quad W_3\mathscr{P}^1 b_{p+1}^1 = 0.$$

PROOF. Denote simply b_{p+1}^1 by b. By Theorem 16.2, we have $b=\phi(\beta_1\beta_{p+1})$. Applying Theorem 3.3 to this, we see that $x=0$ implies $\alpha_1\beta_1\beta_{p+1}\neq 0$. But we have proved $\alpha_1\beta_1\beta_{p+1}=0$ [10; Cor. 1], and so $x\neq 0$ as desired.

By replacing l_1 by $(1/x)\mathscr{P}^1 b$ in (18.3), the relations in $H^*(\boldsymbol{K}_{(p^2+3p)q-4})$, $*<(2p^2+p)q-3$, involving b are listed by the following

$$(*) \qquad \mathscr{P}^2 b = 0, \quad W_3\mathscr{P}^1 b = 0,$$

$$(**) \qquad x\mathscr{P}^1 d_5 - \mathscr{P}^{p(p-3)}\mathscr{P}^1 b = 0, \quad x\mathscr{P}^1\Delta d_5 - (1/2)\Delta\mathscr{P}^{p(p-3)}\mathscr{P}^1 b = 0.$$

The A^*-module structure of the submodule $A^*\{\mathscr{P}^1, \mathscr{P}^1\Delta\}$ of A^* is determined by the relations $-R_1\cdot\mathscr{P}^1+\mathscr{P}^1\cdot\mathscr{P}^1\Delta=0$, $-\Delta\mathscr{P}^1\Delta\cdot\mathscr{P}^1+\Delta\mathscr{P}^1\cdot\Delta\mathscr{P}^1=0$ and $\mathscr{P}^{p-1}\cdot\mathscr{P}^1=0$ (cf. Lemma 20.3 in the below). Hence we see that the relations in A^*b given from $(**)$ are $(2R_1-\mathscr{P}^1\Delta)\mathscr{P}^{p(p-3)}\mathscr{P}^1 b=R_2\mathscr{P}^{p(p-3)}\mathscr{P}^1 b=0$, $\Delta\mathscr{P}^1\Delta\mathscr{P}^{p(p-3)}\mathscr{P}^1 b=0$ and $\mathscr{P}^{p-1}\mathscr{P}^{p(p-3)}b=0$. But we have $\Delta\mathscr{P}^1\Delta=(1/3)\Delta R_2$ and $R_2\mathscr{P}^1\mathscr{P}^{p(p-3)}+\mathscr{P}^{p(p-4)}W_3$, $\mathscr{P}^{p-1}\mathscr{P}^{p(p-3)}\in\mathrm{Im}(\mathscr{P}^1)^*$. Therefore $(**)$ gives no new relations in A^*b, and so the structure of A^*b is given by $(*)$. *q.e.d.*

Now, our main results for $H^*(\boldsymbol{K}_k)$ are stated as follows.

THEOREM 18.2 ($p\geqq 5$). *Let* $(p^2+3p)q-3\leqq k\leqq(2p^2+p-3)q-4$, $q=2(p-1)$. *Then*, $H^*(\boldsymbol{K}_k)$ *is generated, over* A^*, *by the elements a in Table* A10 *below such that* $h(a)\geqq k$ *and by some elements of degree* $\geqq(2p^2+p-1)q-1$. *The* A^**-module structure of the submodule of* $H^*(\boldsymbol{K}_k)$ *generated by the elements in Table* A10 *is given by the relations in Table* B10 *below and by some relations of degree* $\geqq(2p^2+p-1)q$.

TABLE A 10 ($p\geqq 5$)

generator a	deg a	$h(a)$
a_0	0	1
a_r $(p^2+3p\leqq r\leqq 2p^2+p-2)$	rq	$(r-1)q$
a'_{sp} $(p+3\leqq s\leqq 2p)$	$spq+1$	$(sp-1)q$
b_1^{p+r} $(3\leqq r\leqq p)$	$(p^2+(r-1)p-1)q-2r-3$	$\begin{cases}(p^2+rp+1)q-2r-2 & \text{for } r<p\\ (2p^2+1)q-4 & \text{for } r=p\end{cases}$
b_s^r $((r,s)\in I)$	$((r+s)p+s-1)q-2r-1$	$\begin{cases}((r+s-1)p+s)q-2r & \text{for } (r,s)\in I_1\cup I_3\\ ((s-1)p+s-2)q-1 & \text{for } (r,s)\in I_2\\ ((r+s-1)p+s+1)q-2r & \text{for } (r,s)\in I_4\end{cases}$
c_1^{p+r} $(2\leqq r\leqq p-2)$	$(p^2+(r+1)p+1)q-2r-4$	$(p^2+(r+1)p-1)q-2r-3$
c_s^r $((r,s)\in I_1\cup I_2)$	$((r+s)p+s)q-2r-2$	$((r+s)p+s-1)q-2r-1$
d_5	$2p^2q-3$	$(p^2+p+1)q-2$
d_7	$(2p^2+p-2)q$	$(2p^2-2p-3)q$
$e'_i(2)$ $(1\leqq i\leqq p-3)$	$(2p^2+i)q-4$	$\begin{cases}(2p^2-2)q-3 & \text{for } i=1\\ (2p^2+i-1)q-3 & \text{for } i\geqq 2\end{cases}$
$e_i(2)$ $(1\leqq i\leqq p-3)$	$(2p^2+i)q-3$	$\begin{cases}(2p^2-2)q-3 & \text{for } i=1\\ (2p^2+i-1)q-3 & \text{for } i\geqq 2\end{cases}$
k_s^r $((r,s)\in J)$	$(p^2+(r+s+2)p+s+1)q-2r-4$	$(p^2+(r+s+2)p+s-1)q-2r-3$
l'_s $(2\leqq s\leqq p-2)$	$(p^2+(s+2)p+s)q-2$	$(p^2+sp+s)q-2$
l_s $(2\leqq s\leqq p-2)$	$(p^2+(s+2)p+s)q-3$	$(p^2+(s+1)p+s-2)q-3$

If $p\geqq 7$ and the coefficient y_{p-4} in the relation $(l-3)$ of Table B10 *is zero, then the element d_7 is omitted. If $z=0$ (resp. $y=0$) in the relation $(b-9)$ (resp. $(c-4)$) of Table* B10, *then the following element b^0_{2p-1} (resp. b^p_p) is added. If $z\neq 0$ in $(b-9)$, the following element b' is added.*

b^0_{2p-1}	$(2p^2+p-2)q-1$	$(2p^2-3)q-1$
b^p_p	$(2p^2+p-2)q-3$	$(2p^2-1)q-2$
b'	$(2p^2+p-2)q$	$(2p^2-3)q-1$

In the above, the indexing sets $I, I_1,\dots, I_4$ and J are defined by

$$I = I_1 \cup I_2 \cup I_3 \cup I_4,$$

$$I_1 = \{(r,s)|1 \leqq r < p, 4 \leqq s \leqq p, r+s \geqq p+3\},$$

$$I_2 = \{(0,s)|p+2 \leqq s \leqq 2p-2\},$$

$$I_3 = \{(1,s)|p+2 \leqq s \leqq 2p-2\},$$

$$I_4 = \{(r,s)|2 \leqq r < p, p+1 \leqq s \leqq 2p-3, r+s \leqq 2p-1\},$$

$$J = \{(r,s)|r \geqq 0, 1 \leqq s \leqq p-3, r+s \leqq p-2\}.$$

TABLE B 10 ($p \geqq 5$)

(*a*–1) $\Delta a_0 = 0, \quad \mathscr{P}^1 a_0 = 0, \quad \mathscr{P}^p a_0 = 0, \quad \mathscr{P}^{p^2} a_0 = 0.$

(*a*–2) $R_r a_r = 0, \quad r \not\equiv 0 \bmod p, \quad \Delta a_{sp} = 0, \quad \Delta a'_{sp} = 0,$
$\Delta\mathscr{P}^1 a_{sp} - \mathscr{P}^1 a'_{sp} = 0, \quad \Delta\mathscr{P}^1\Delta a_{sp-1} = 0.$

(*b*–1) $\mathscr{P}^2 b_1^{p+r} = 0$ *for* $r \leqq p-2$.

(*b*–2) $\mathscr{P}^2 b_1^{2p-1} = d_5$, *and hence* $\mathscr{P}^3 b_1^{2p-1} = 0, \quad \mathscr{P}^3\Delta b_1^{2p-1} = 0.$

(*b*–3) $\mathscr{P}^1 b_s^r = 0$ *for* $(r,s) \in I_1$.

(*b*–4) $\mathscr{P}^1 b^0_{p+2} = 0, \quad \mathscr{P}^1 b^0_{p+s} - \overline{W}_s c^0_{p+s-1} \equiv 0 \bmod Z_p\{\Delta\mathscr{P}^1\Delta l_{s-1}\}$
for $3 \leqq s \leqq p-3$,
$\mathscr{P}^1 b^0_{2p-2} - \overline{W}_{p-2} c^0_{2p-3} \equiv 0 \bmod Z_p\{\Delta\mathscr{P}^1\Delta l_{p-3}, \mathscr{P}^p b_1^{2p-2}\}.$

(*b*–5) $\mathscr{P}^1 b^1_{p+s} = l_s$, *and hence* $\mathscr{P}^2 b^1_{p+s} = 0, \quad W_{s+2}\mathscr{P}^1 b^1_{p+s} = 0,$
for $2 \leqq s \leqq p-3$.

(*b*–6) $\mathscr{P}^1 b^1_{2p-2} = x l_{p-2}, \quad x \in Z_p$.

(*b*–7) $\mathscr{P}^2 b_s^r = 0$ *for* $(r,s) \in I_4$.

(*b*–8) $W_2 b^0_{p+2} = 0, \quad W_s b^0_{p+s} - A_s c^0_{p+s-1} \equiv 0 \bmod A^*\{a_{p^2+sp+s-2}, b^{p-1}_{s+1}\}$
for $3 \leqq s \leqq p-3$.

(*b*–9) $W_{p-2} b^0_{2p-2} - A_{p-2} c^0_{2p-3} \equiv z d_7 \bmod A^*\{a_{2p^2-p-4}, b^{p-1}_{p-1}, b_1^{2p-2}, l'_{p-3}\}$
($z = 0$ *if* d_7 *does not exist*).

(*b*–10) $\Delta b' \equiv 0 \bmod A^*\{b_p^{p-1}, a_{2p^2-3}\}$.

(*c*–1) $\mathscr{P}^{p-2} c_1^{p+r} = 0$ *for* $r \leqq p-2$.

(*c*–2) $\mathscr{P}^{p-1} c_s^r = 0$ *for* $r \leqq p-2$.

(*c*–3) $\mathscr{P}^{p-1} c_s^{p-1} = l'_{s-2}$, *and hence* $W_s\mathscr{P}^{p-1} c_s^{p-1} = 0$ *for* $4 \leqq s \leqq p-1$, *and also* $\Delta\mathscr{P}^{3p}\Delta\mathscr{P}^1\Delta c_{p-2}^{p-1} = 0$ *in* $\boldsymbol{K}_k$ *for* $k \geqq (2p^2-2p-4)q-4$ *if* $p \geqq 7$ *and* $y_{p-4} \neq 0$ *in* $(l-3)$.

(*c*–4) $\mathscr{P}^{p-1}c_p^{p-1} = yl'_{p-2}, \quad y \in Z_p$.

(*d*–1) $\mathscr{P}^1 d_5 = 0, \quad \mathscr{P}^1\Delta d_5 = 0$.

(*d*–2) $\Delta d_7 \equiv 0 \bmod A^*\{b_1^{2p-3}, b_{2p-3}^0\}, \quad \mathscr{P}^1 d_7 \equiv 0 \bmod A^*\{b_1^{2p-3}, b_{2p-3}^0\}$.

(*e*–1) $\mathscr{P}^{p-3}e'_1(2) = 0$.

(*e*–2) $\mathscr{P}^1 e_1(2) - R_3 e'_i(2) = 0, \quad \mathscr{P}^1 e_i(2) - R_1 e'_i(2) = 0$ *for* $2 \leqq i \leqq p-4$.

(*e*–3) $\mathscr{P}^1 e_{p-3}(2) - R_1 e'_{p-3}(2) = u_1 l_{p-2} + u_2 b_p^p, \quad u_i \in Z_p$
($u_2 = 0$ *if* $y \neq 0$ *in* $(c-4)$).

(*e*–4) $\mathscr{P}^1\Delta e_1(2) - 3\Delta\mathscr{P}^1\Delta e'_1(2) = 0, \quad \mathscr{P}^1\Delta e_i(2) - \Delta\mathscr{P}^1\Delta e'_i(2) = 0$
for $2 \leqq i \leqq p-4$.

(*e*–5) $\mathscr{P}^1\Delta e_{p-3}(2) - \Delta\mathscr{P}^1\Delta e'_{p-3}(2) = v_1\Delta l_{p-2} + v_2\Delta b_p^p + v_3 l'_{p-2}$,
$v_i \in Z_p$ ($v_2 = v_3 = 0$ *if* $y \neq 0$ *in* (*c*–4)).

(*k*–1) $\mathscr{P}^{p-2}k_s^r = 0$ *for* $r+s \leqq p-3$.

(*l*–1) $\mathscr{P}^1 l'_s = 0, \quad W_{s+2}l'_s = 0$ *for* $2 \leqq s \leqq p-3$.

(*l*–2) $\mathscr{P}^1 l_s - 2\overline{W}_{s+1}k_{s-1}^0 = 0$ *for* $2 \leqq s \leqq p-3$.

(*l*–3) $W_{s+2}l_s - y_s\mathscr{P}^{p+1}l'_s - 2A_{s+2}k_{s-1}^0 = 0, \quad y_s \in Z_p,$ *for* $2 \leqq s \leqq p-3$.

(*l*) $a'_{sp} = 0$ *in* $\boldsymbol{K}_k, \quad k \geqq spq$,
$a = 0$ *in* $\boldsymbol{K}_k, \quad k \geqq \deg a,$ *for any* $a\ (\neq a_0)$ *in Table* A10,

and hence

$d_5 = 0$ *in* $\boldsymbol{K}_k$ *for* $k \geqq (2p^2-2)q-3$,
$l'_s = 0$ *in* $\boldsymbol{K}_k, \quad 2 \leqq s \leqq p-3,$ *for* $k \geqq (p^2+(s+1)p+s+1)q-2$,
$l_s = 0$ *in* $\boldsymbol{K}_k, \quad 2 \leqq s \leqq p-3,$ *for* $k \geqq (p^2+(s+2)p+s+1)q-4$,
$l'_{p-2} = 0$ *in* $\boldsymbol{K}_k$ *for* $k \geqq (2p^2+p-1)q-4$ *if* $y \neq 0$ *in* $(c-4)$,
$d_7 = 0$ *in* $\boldsymbol{K}_k$ *for* $k \geqq (2p^2-3)q-1$ *if* $z \neq 0$ *in* $(b-9)$.

In the above, $\overline{W}_s$ *and* A_s *are elements of* A^* *such that* $\mathscr{P}^1 W_s = \overline{W}_s\mathscr{P}^1$ *and* $W_s W_{s-1} = A_s\mathscr{P}^1$.

For any generator $a\ (\neq a_0)$ *in Table* A10, *the image of* a *in* $\boldsymbol{K}_{h(a)}$ *by the homomorphism* $\delta^*: H^*(\boldsymbol{K}_{h(a)}) \to H^*({}_p\pi_{h(a)-1}(\mathbf{S}))$ *is given by the following equalities*:

$$\text{(18.4)} \qquad \delta^* a_r = R_{r-1}(j^{*-1}a_{r-1}) \quad \text{for} \quad r \not\equiv 1 \bmod p,$$

$$\delta^* a_{sp+1} = \Delta\mathscr{P}^1(j^{*-1}a_{sp}) - \mathscr{P}^1(j^{*-1}a'_{sp}),$$

$$\delta^* a'_{sp} = \Delta\mathscr{P}^1\Delta(j^{*-1}a_{sp-1}),$$

$$\delta^* b_1^{p+r} = \begin{cases} \mathscr{P}^{p-2}(j^{*-1}c_1^{p+r-1}) & \text{for} \quad r \leqq p-1, \\ \mathscr{P}^{p-3}(j^{*-1}e'_1(2)) & \text{for} \quad r = p, \end{cases}$$

$$\delta^* b_{p+s}^0 = W_{s-1}(j^{*-1}b_{p+s-1}^0) \quad \text{for} \quad s \leqq p-1,$$

$$\delta^* b_s^r = \begin{cases} \mathscr{P}^{p-1}(j^{*-1} c_s^{r-1}) & \textit{for} \quad (r, s) \in I_1 \cup I_3 \cup \{(p, p)\}, \\ \mathscr{P}^{p-2}(j^{*-1} k_{s-p}^{r-2}) & \textit{for} \quad (r, s) \in I_4, \end{cases}$$

$$\delta^* c_1^{p+r} = \mathscr{P}^2(j^{*-1} b_1^{p+r-1}), \quad \delta^* c_s^r = \mathscr{P}^1(j^{*-1} b_s^r),$$

$$\delta^* d_5 = \mathscr{P}^{p(p-2)} \mathscr{P}^{p-1}(j^{*-1} c_2^{p-1}), \quad \delta^* d_7 = \Delta \mathscr{P}^{3p} \Delta \mathscr{P}^1 \Delta(j^{*-1} c_{p-2}^{p-1}),$$

$$\delta^* e_i'(2) = \begin{cases} \mathscr{P}^3(j^{*-1} b_1^{2p-1}) & \textit{for} \quad i = 1, \\ \mathscr{P}^1(j^{*-1} e_{i-1}(2)) & \textit{for} \quad i \geqq 2, \end{cases}$$

$$\delta^* e_i(2) = \begin{cases} \mathscr{P}^3 \Delta(j^{*-1} b_1^{2p-1}) & \textit{for} \quad i = 1, \\ \mathscr{P}^1 \Delta(j^{*-1} e_{i-1}(2)) & \textit{for} \quad i \geqq 2, \end{cases}$$

$$\delta^* k_s^r = \mathscr{P}^2(j^{*-1} b_{p+s}^{r+1}),$$

$$\delta^* l_s' = W_{s+1} \mathscr{P}^{p-1}(j^{*-1} c_{s+1}^{p-1}), \quad \delta^* l_s = W_{s+1} \mathscr{P}^1(j^{*-1} b_{p+s-1}^1),$$

$$\delta^* b' = \Delta W_{p-2}(j^{*-1} b_{2p-2}^0).$$

The proof of Theorem 18.2 is delayed to § 20.

From the theorem, we have immediately the following two corollaries.

COROLLARY 18.3. *Let* $(p^2+3p+1)q-5 \leqq k \leqq (2p^2+p-2)q-6$. *Then, a* Z_p*-basis for* $H^{k+1}(\boldsymbol{K}_k)$ *is given by the following*:

(i) $Z_p\{a_{p^2+r}\}$ *for* $k = (p^2+r)q-1, \quad 3p+1 \leqq r \leqq p^2+p-3$
except $r = p^2-2p, (s-2)p+s-1 \ (5 \leqq s \leqq p), p^2+1$.

(ii) $Z_p\{a_{2p^2-2p}, c_1^{2p-3}\}$ *for* $k = (2p^2-2p)q-1, \ p \geqq 7$.

(iii) $Z_p\{a_{p^2+(s-2)p+s-1}, c_s^{p-2}\}$ *for* $k = (p^2+(s-2)p+s-1)q-1$,
$5 \leqq s \leqq p$.

(iv) $Z_p\{a_{2p^2+1}, k_1^{p-3}\}$ *for* $k = (2p^2+1)q-1$.

(v) $Z_p\{b_1^{p+r}\}$ *for* $k = (p^2+(r+1)p-1)q-2r-4, \ 3 \leqq r \leqq p$.

(vi) $Z_p\{b_s^r\}$ *for* $k = ((r+s)p+s-1)q-2r-2$,
$(r, s) \in I$ *except* $(r, s) = (1, 2p-2)$.

(vii) $Z_p\{c_1^{p+r}\}$ *for* $k = (p^2+(r+1)p+1)q-2r-5$,
$3 \leqq r \leqq p-2, \ r \neq p-3$.

(viii) $Z_p\{c_s^r\}$ *for* $k=((r+s)p+s)q-2r-3,\ (r,s)\in I_1\cup I_2,\ r\neq p-2$.

(ix) $Z_p\{e_i'(2)\}$ *for* $k=(2p^2+i)q-5,\ 1\leqq i\leqq p-3$.

(x) $Z_p\{e_i(2)\}$ *for* $k=(2p^2+i)q-4,\ 1\leqq i\leqq p-4$.

(xi) $Z_p\{b_{2p-2}^1, e_{p-3}(2)\}$ *for* $k=(2p^2+p-3)q-4$.

(xii) $Z_p\{k_s^r\}$ *for* $k=(p^2+(r+s+2)p+s+1)q-2r-5$,

$(r,s)\in J$ *except* $(r,s)=(p-3,1)$.

(xiii) 0 *for other* k.

COROLLARY 18.4. *For the same values of* k *as Corollary* 18.3, *the kernel of* $\Delta\colon H^{k+1}(\boldsymbol{K}_k)\to H^{k+2}(\boldsymbol{K}_k)$ *is generated by the element* a_{sp}, $p+4\leqq s\leqq 2p$.

§19. Calculations of ${}_p\pi_*(S)$

By (1.4), we remark the following

LEMMA 19.1. *If* $\Delta\colon H^{k+1}(\boldsymbol{K}_k)\to H^{k+2}(\boldsymbol{K}_k)$ *is monomorphic, then the homomorphism* $\phi\colon {}_p\pi_k(\boldsymbol{S})\to H^{k+1}(\boldsymbol{K}_k)$ *of* (18.2) *is isomorphic. If* $H^{k+1}(\boldsymbol{K}_k)=Z_p\{a\}$ *with* $\Delta a=0$ (*resp.* $Z_p\{a,b\}$ *with* $\Delta a=0, \Delta b\neq 0$), *then* ${}_p\pi_k(\boldsymbol{S})$ *is isomorphic to* Z_{p^t} (*resp.* $Z_{p^t}\oplus Z_p$) *for some* $t\geqq 2$ *and the epimorphism* ϕ *carries the factor* Z_{p^t} *to* $Z_p\{a\}$.

We consider the inverse-images of the elements in Corollary 18.3 by ϕ. First, by applying Theorems 3.3–3.4, we have

LEMMA 19.2. *For the elements* b_s^r *and* c_s^r, $(r,s)\in I_1\cup I_2$, *of Corollary* 18.3, *the following hold up to non-zero coefficients.*

(i) $\phi(\beta_s)=b_s^0$ *for* $(0,s)\in I_2$, *where* β_s *is the element due to L. Smith* [12] *and H. Toda* [18] (*cf.* [9; §5]).

(ii) $\phi(\beta_1^r\beta_s)=b_s^r$ *for* $(r,s)\in I_1$, $s\neq p$.

(iii) $\phi(\beta_1^{r-1}\beta_2\beta_{p-1})=b_p^r$ *for* $r\leqq p-1$.

(iv) $\phi(\alpha_1\beta_1^r\beta_s)=c_s^r$ *for* $(r,s)\in I_1\cup I_2$, $s\neq p$.

(v) $\phi(\alpha_1\beta_1^{r-1}\beta_2\beta_{p-1})=c_p^r$ *for* $r\leqq p-1$.

PROOF. By Corollaries 18.3 (vi), 18.4 and Lemma 19.1, ${}_p\pi_{(sp+s-1)q-2}(\boldsymbol{S})$ is isomorphic to Z_p and is generated by $\phi^{-1}(b_s^0)$ for $p+2\leqq s\leqq 2p-2$. By [12], this group contains the element β_s of order p, and hence (i) follows. By

Theorems 3.3–3.4, we have $\phi^{-1}(c_s^r)=\alpha_1\phi^{-1}(b_s^r)$ and $\phi^{-1}(b_s^r)=\beta_1\phi^{-1}(b_s^{r-1})$ $(r\geqq 1)$. For small r and s, (ii)–(iii) hold by Theorem 16.2. These facts show (ii)–(v). *q.e.d.*

Next, by Theorem 3.5 we have

LEMMA 19.3. *The following equalities hold up to non-zero coefficients*:

(i) $\phi(\beta_1^{p+r+1}) = b_1^{p+r}$ *for* $r \leqq 2p-1$.

(ii) $\phi(\beta_1^{r+1}\varepsilon') = c_1^{p+r}$ *for* $r \leqq 2p-2$.

PROOF. For small r, these are proved in Theorem 16.2. By Theorem 3.5, $\phi^{-1}(b_1^{p+r})=\beta_1\phi^{-1}(b_1^{p+r-1})$ and $\phi^{-1}(c_1^{p+r})=\beta_1\phi^{-1}(c_1^{p+r-1})$. These show the lemma. *q.e.d.*

Now let $1\leqq s\leqq p-3$. We proved the relation $\alpha_1\beta_1\beta_{p+s}=0$ [10; Cor 1]. So the secondary composition $\{\beta_1\beta_{p+s}, \alpha_1, \alpha_1\}$ is defined. Since ${}_p\pi_k(S)$, $k=(p^2+(s+2)p+s)q-4$, vanishes by Corollary 18.3 (xiii) and Lemma 19.1, the indeterminacy of the composition is $\beta_1\beta_{p+s}\pi_{2q-1}(S)=Z_p\{\beta_1\beta_{p+s}\alpha_2\}=0$. Hence, the composition consists of a single element. We then define

$$\kappa_s = \{\beta_1\beta_{p+s}, \alpha_1, \alpha_1\}, \qquad 1 \leqq s \leqq p-3, \tag{19.1}$$

and obtain the following lemma, by Theorem 3.5.

LEMMA 19.4. *The following equalities hold up to non-zero coefficients*:

(i) $\phi(\beta_1^r\beta_s) = b_s^r$ *for* $(r, s)\in I_3 \cup I_4$.

(ii) $\phi(\beta_1^r\kappa_s) = k_s^r$ *for* $(r, s)\in J$.

Set $t=(2p^2+1)q-4$. Then, the group ${}_p\pi_t(S)$ is isomorphic to Z_p by Corollaries 18.3 (x), 18.4 and Lemma 19.1. Let

$$\lambda_1 \in {}_p\pi_{(2p^2+1)q-4}(S) = Z_p \tag{19.2}$$

be a generator. By Corollary 18.3 (xiii) and Lemma 19.1, we have ${}_p\pi_{t+iq+1}(S)=0$ for $1\leqq i\leqq p-5$, and also $p({}_p\pi_{t+iq}(S))=0$ for $1\leqq i\leqq p-4$ by Corollaries 18.3 (x)–(xi), 18.4 and Lemma 19.1. Hence we can define inductively the element

$$\lambda_i = \{\lambda_{i-1}, p\iota, \alpha_1\} \qquad for \quad 2 \leqq i \leqq p-3, \tag{19.3}$$

where the secondary composition in the right side consists of a single element.

By using Theorem 3.6, we see $\lambda_i\neq 0$, and by Theorem 3.3, $\alpha_1\lambda_i\neq 0$ for $i\leqq p-4$. More precisely we have

LEMMA 19.5. *The following hold up to non-zero coefficients*:

(i) $\phi(\lambda_i) = e_i(2)$ *for* $1 \leqq i \leqq p-3$.

(ii) $\phi(\alpha_1\lambda_i) = e'_{i+1}(2)$ *for* $1 \leqq i \leqq p-4$.

By well known formulas [14–IV; Th. 4.4] of the secondary composition and by Theorem 6.2, we have

$$\{\alpha_1, \alpha_1, \beta_1^p\} = \{\beta_1^p, \alpha_1, \alpha_1\} = \varepsilon',$$

$$\{\alpha_1, \beta_1^p, \alpha_1\} = \{\beta_1^p, \alpha_1, \alpha_1\} + \{\alpha_1, \alpha_1, \beta_1^p\} = 2\varepsilon',$$

$$\{\beta_1^p, \alpha_1, \beta_1^p\} = 0,$$

and so

$$\{\beta_1^{2p}, \alpha_1, \alpha_1\} = \beta_1^p \varepsilon' = -(1/2)\{\beta_1^p, \alpha_1, \beta_1^p\}\alpha_1 = 0.$$

Hence the tertiary composition $\{\beta_1^{2p}, \alpha_1, \alpha_1, \alpha_1\}$ is defined. By easy calculations, this consists of a single element, and we put

$$\lambda' = \{\beta_1^{2p}, \alpha_1, \alpha_1, \alpha_1\}. \tag{19.4}$$

LEMMA 19.6. *The following hold up to non-zero coefficients*:

(i) $\phi(\lambda') = e'_i(2)$.

(ii) $\phi(\beta_1^{2p+1}) = b_1^{2p}$.

The lemma is an application of the following result.

THEOREM. *Let* $a \in H^{k+1}(\boldsymbol{K}_k)$ *and* $\gamma \in \pi_k(\boldsymbol{S})$ *such that* $\phi(\gamma) = a \neq 0$ *and* $\mathscr{P}^3 a = 0$. *Let* $b \in H^{k+3q}(\boldsymbol{K}_{k+1})$ *be an element such that* $\delta^* b = \mathscr{P}^3 j^{*-1} a$. *Assume that* $\mathscr{P}^2 a \neq 0$, $\gamma\alpha_1 = 0$, $\{\gamma, \alpha_1, \alpha_1\} \ni 0$ *and* $b \neq 0$ *in* $\boldsymbol{K}_{k+3q-1}$. *Then the tertiary composition* $\{\gamma, \alpha_1, \alpha_1, \alpha_1\}$ *is defined and contains an element* δ *such that* $\phi(\delta) = xb$ *for some* $x \not\equiv 0 \bmod p$. *Assume further that* $\mathscr{P}^{p-3} b = 0$, $\mathscr{P}^{p-4} b \neq 0$ *in* $\boldsymbol{K}_{k+3q-1}$ *and* $c \neq 0$ *in* $\boldsymbol{K}_{k+pq-2}$, *where* c *in* $\boldsymbol{K}_{k+3q}$ *is defined by* $\delta^* c = \mathscr{P}^{p-3} j^{*-1} b$. *Then,* $\phi(\beta_1\gamma) = yc$ *for some* $y \not\equiv 0 \bmod p$.

Of course, this theorem is not valid for $p=3$. This is an analogy of Theorem 3.5 and proved by a modification of the proof of Theorem 3.5. So, we omit the proof.

Finally we consider the inverse-image of the element a_r. The following result is proved by J. F. Adams [1] (cf. [9; §4]).

(19.5) *There exist the elements* $\alpha_r \in {}_p\pi_{rq-1}(\boldsymbol{S})$, $r \geqq 1$, *of order* p *such that* α_r

$\in \{\alpha_{r-1}, p\iota, \alpha_1\}$ *and that* α_r *generates a summand if* $r \not\equiv 0 \bmod p$. *There exist the elements* $\alpha'_{rp} \in {}_p\pi_{rpq-1}(S)$, $r \geqq 1$, *of order* p^2 *such that* $p\alpha'_{rp} = \alpha_{rp}$, $\alpha'_{rp} \in \{\alpha'_{(r-1)p}, p^2\iota, \alpha'_p\}$ *and that* α'_{rp} *generates a summand if* $r \not\equiv 0 \bmod p$. *Also there exist the elements* $\alpha''_{rp^2} \in {}_p\pi_{rp^2q-1}(S)$, $r \geqq 1$, *of order* p^3 *such that* $p\alpha''_{rp^2} = \alpha'_{rp^2}$, $\alpha''_{rp^2} \in \{\alpha''_{(r-1)p^2}, p^3\iota, \alpha''_{p^2}\}$ *and that* α''_{rp^2} *generates a summand if* $r \not\equiv 0 \bmod p$.

Combining the above fact and a method similar to (4.11)–(4.12) of [14–IV] with Corollaries 18.3 (i)–(iv), 18.4 and Lemma 19.1, we get the following lemma.

LEMMA 19.7. *The following equalities hold up to non-zero coefficients*:

(i) $\phi(\alpha_r) = a_r$ *for* $p^2+3p+1 \leqq r \leqq 2p^2+p-3$, $r \not\equiv 0 \bmod p$.

(ii) $\phi(\alpha'_{sp}) = a_{sp}$ *for* $p+4 \leqq s \leqq 2p-1$.

(iii) $\phi(\alpha''_{2p^2}) = a_{2p^2}$.

The following lemma gives representations of the element λ' by secondary compositions.

LEMMA 19.8. *The following relations hold*:

$$\{\beta_1^p, \varepsilon', \alpha_1\} = \{\alpha_1, \varepsilon', \beta_1^p\} = \lambda',$$

$$\{\beta_1^p, \alpha_1, \varepsilon'\} = \{\varepsilon', \alpha_1, \beta_1^p\} = 2\lambda',$$

$$\{\varepsilon', \beta_1^p, \alpha_1\} = \{\alpha_1, \beta_1^p, \varepsilon'\} = 3\lambda'.$$

PROOF. By the formula [14–IV; Th. 4.4 i)], any $\{\alpha, \beta, \gamma\}$ above satisfies $\{\alpha, \beta, \gamma\} = \{\gamma, \beta, \alpha\}$. Using the formula [14–IV; Th. 4.4 ii)], we have

$$(*) \qquad \{\beta_1^p, \varepsilon', \alpha_1\} - \{\varepsilon', \beta_1^p, \alpha_1\} + \{\beta_1^p, \alpha_1, \varepsilon'\} = 0.$$

Applying the formula [14–IV; (4.4) ii)] for $\alpha = \delta = \beta_1^p$ and $\beta = \gamma = \varepsilon = \alpha_1$, we have

$$(**) \qquad \{\varepsilon', \beta_1^p, \alpha_1\} + \{\beta_1^p, \varepsilon', \alpha_1\} - 2\{\beta_1^p, \alpha_1, \varepsilon'\} = 0,$$

since $\{\beta_1^p, \alpha_1, \alpha_1\} = \{\alpha_1, \alpha_1, \beta_1^p\} = \varepsilon'$ and $\{\alpha_1, \beta_1^p, \alpha_1\} = 2\varepsilon'$. From $(*)$ and $(**)$, we see easily $\{\beta_1^p, \alpha_1, \varepsilon'\} = 2\{\beta_1^p, \varepsilon', \alpha_1\}$ and $\{\varepsilon', \beta_1^p, \alpha_1\} = 3\{\beta_1^p, \varepsilon', \alpha_1\}$. Thus, it suffices to show the equality $\lambda' = \{\beta_1^p, \varepsilon', \alpha_1\}$.

Let n be a sufficiently large integer. Since $\alpha_1\alpha_1 = 0$, we can construct a double mapping cone

$$P_n = S^n \cup_{\alpha_1} e^{n+q} \cup_{\alpha_1} e^{n+2q}.$$

(By using the notation (11.2), $P_n = C^n(\alpha_1, \alpha_1)$). Since $\{\alpha_1, \alpha_1, \alpha_1\} = 0$ and $\{\beta_1^{2p},$

$\alpha_1, \alpha_1\}=0$, there exist elements $\alpha\colon S^{n+2b+3q-1}\to P_{n+2b}$ and $\beta\colon P_{n+2b}\to S^n$, $b=(p^2-1)q-2$, such that $j\alpha=\alpha_1$ and $\beta i=\beta_1^{2p}$, where $i\colon S^n\to P_n$ is the inclusion and $j\colon P_{n+2b}\to S^{n+2b+2q}=P_{n+2b}/S^{n+2b}\cup e^{n+2b+q}$ is the projection. The element β is unique because of ${}_p\pi_{2b+q}(S)=0$ and ${}_p\pi_{2b+2q}(S)=0$. We put

$$Q_n = S^n \cup_{\varepsilon'} e^{n+a}, \qquad a = (p^2+1)q-2,$$

a mapping cone of a representative of $\varepsilon'\in{}_p\pi_{a-1}(S)$. Since $\beta_1^p\varepsilon'=0$, there exists an extension $\beta'\colon Q_{n+b}\to S^n$ of β_1^p. Since $\varepsilon'=\{\beta_1^p, \alpha_1, \alpha_1\}$, there exists $\gamma\colon P_{n+2b}\to Q_{n+b}$ such that $\gamma i=i'\beta_1^p$ and $j'\gamma=j$, where $i'\colon S^{n+b}\to Q_{n+b}$ is the inclusion and $j'\colon Q_{n+b}\to S^{n+a+b}$ is the projection. Then the element $\beta'\gamma$ satisfies $(\beta'\gamma)i=\beta_1^{2p}$, and so $\beta=\beta'\gamma$ by the uniqueness of β.

By the definition of the secondary and tertiary compositions and by (19.4), we obtain $\lambda'=\{\beta_1^{2p}, \alpha_1, \alpha_1, \alpha_1\}=\beta\alpha=\beta'(\gamma\alpha)=\{\beta_1^p, \varepsilon', \alpha_1\}$ as desired. *q.e.d.*

Summarizing Corollaries 18.3–18.4, Lemmas 19.1–19.8 and (19.1)–(19.5), we have obtained the following main result of this section.

THEOREM 19.9 ($p\geqq 5$). *Let $(p^2+3p+1)q-5\leqq k\leqq(2p^2+p-2)q-6$, $q=2(p-1)$. Then, the group ${}_p\pi_k(S)$ is the direct sum of the cyclic groups generated by the following elements of degree k.*

generator γ		$\deg\gamma\ (=k)$	order of γ
α_{p^2+r}	$(3p+1\leqq r\leqq p^2+p-3,$ $r\not\equiv 0 \bmod p)$	$(p^2+r)q-1$	p
α'_{p^2+sp}	$(4\leqq s\leqq p-1)$	$(p^2+sp)q-1$	p^2
α''_{2p^2}		$2p^2q-1$	p^3
β_1^r	$(p+4\leqq r\leqq 2p+1)$	$rpq-2r$	p
$\beta_1^r\beta_s$	$((r,s)\in I)$	$((r+s)p+s-1)q-2r-2$	p
$\beta_1^r\beta_2\beta_{p-1}$	$(2\leqq r\leqq p-2)$	$(p^2+(r+2)p-1)q-2r-4$	p
$\alpha_1\beta_1^r\beta_s$	$((r,s)\in I')$	$((r+s)p+s)q-2r-3$	p
$\alpha_1\beta_1^r\beta_2\beta_{p-1}$	$(2\leqq r\leqq p-2)$	$(p^2+(r+2)p)q-2r-5$	p
$\beta_1^r\varepsilon'$	$(4\leqq r\leqq p-1)$	$(p^2+rp+1)q-2r-3$	p
$\beta_1^r\kappa_s$	$((r,s)\in J)$	$(p^2+(r+s+2)p+s+1)q-2r-5$	p
λ'		$(2p^2+1)q-5$	p
λ_i	$(1\leqq i\leqq p-3)$	$(2p^2+i)q-4$	p
$\alpha_1\lambda_i$	$(1\leqq i\leqq p-4)$	$(2p^2+i+1)q-5$	p

Here the index sets I, I' and J are given by

$$I = \{(r,s)|0 \leqq r < p,\ 4 \leqq s \leqq 2p-2,\ s \neq p,\ p+3 \leqq r+s \leqq 2p-1\},$$

$$I' = \{(r,s)|0 \leqq r < p,\ 4 \leqq s < p,\ r+s \geqq p+3\} \cup \{(0,s)|p+2 \leqq s \leqq 2p-2\},$$

$$J = \{(r,s)|r \geqq 0,\ 1 \leqq s \leqq p-3,\ r+s \leqq p-2\}.$$

The following formulas give representations of the generators:

$$\kappa_s = \{\beta_1\beta_{p+s}, \alpha_1, \alpha_1\}, \qquad 1 \leqq s \leqq p-3,$$

$$\lambda' = \{\beta_1^p, \varepsilon', \alpha_1\} = (1/2)\{\beta_1^p, \alpha_1, \varepsilon'\} = (1/3)\{\varepsilon', \beta_1^p, \alpha_1\},$$

$$\lambda_i = \{\lambda_{i-1}, p\iota, \alpha_1\}, \qquad 2 \leqq i \leqq p-3.$$

REMARK. In the theorem, we do not fix the element λ_1 of (19.2). We shall prove in §22 the formula

$$\lambda_1 = \{\varepsilon_1, \beta_1^p, \alpha_1\},$$

which defines a fixed λ_1.

REMARK. Combining the theorem with Theorem A of [8–II], we have determined the group ${}_p\pi_k(S)$ for all $k \leqq N = (2p^2+p-2)q-6$. In Theorem 18.2, the coefficients x, u_1 and u_2 are not known to us, and we can not determine $H^*(\boldsymbol{K}_t)$, $t=(2p^2+p-3)q-3$, in degree $\geqq N+2$. By Theorem 3.3 and Lemmas 19.4 (i), 19.5 (i), $x=0$ implies $\alpha_1\beta_1\beta_{2p-2} \neq 0$ and $u_1=u_2=0$ implies $\alpha_1\lambda_{p-3} \neq 0$. From these facts, we obtain a partial result:

$${}_p\pi_{N+1}(S) \approx Z_p \oplus Z_p,\ Z_p \text{ or } 0, \quad \textit{generated by } \alpha_1\beta_1\beta_{2p-2} \textit{ and } \alpha_1\lambda_{p-3}.$$

§20. Proof of Theorem 18.2

We prepare some results on A^*-submodules of A^*, which are staightforward consequences of [7] and [14–I].

By Proposition 1.5 of [14–I], we have

LEMMA 20.1. *Let $R_k = (k+1)\mathscr{P}^1\Delta - k\Delta\mathscr{P}^1$. The relations in the submodule A^*R_k of A^* are generated by the relation $R_{k+1}\cdot R_k = 0$ for $1 \leqq k \leqq p-3$, and by $R_{p-1}\cdot R_{p-2} = 0$ and $\Delta\mathscr{P}^1\Delta\cdot R_{p-2} = 0$ for $k = p-2$. The relations in $A^*R_{p-1} + A^*\Delta\mathscr{P}^1\Delta$ are given by $\Delta\cdot R_{p-1} = 0$, $\Delta\cdot\Delta\mathscr{P}^1\Delta = 0$, $\Delta\mathscr{P}^1\cdot R_{p-1} - \mathscr{P}^1\cdot\Delta\mathscr{P}^1\Delta = 0$. The relations in the submodule $(A^*/A^*\Delta)(\Delta\mathscr{P}^1, \mathscr{P}^1)$ of $A^*/A^*\Delta \oplus A^*/A^*\Delta$ are given by $R_1(\Delta\mathscr{P}^1, \mathscr{P}^1) = 0$.*

By (1.1), (3.3) and (3.4) of [7], we have

LEMMA 20.2. *Let* $1 \leqq i < p$. *The* A^**-module structure of the submodule* $A^*\mathscr{P}^i$ *of* A^* *is given by the relation* $\mathscr{P}^{p-i} \cdot \mathscr{P}^i = 0$.

LEMMA 20.3. *Let* $1 \leqq i < p$. *The submodule* $A^*\mathscr{P}^i + A^*\mathscr{P}^i\Delta$ *of* A^* *is given by the relations* $\mathscr{P}^{p-i} \cdot \mathscr{P}^i = 0$, $R_i \cdot \mathscr{P}^i - \mathscr{P}^1 \cdot \mathscr{P}^i \Delta = 0$ *and* $i\Delta\mathscr{P}^1\Delta \cdot \mathscr{P}^i - \mathscr{P}^1\Delta \cdot \mathscr{P}^i\Delta = 0$.

The following is a restatement of (4.3), (4.5) and (4.8) of [7].

LEMMA 20.4. *Let* $W_k = (k+1)\mathscr{P}^p\mathscr{P}^1\Delta - k\mathscr{P}^{p+1}\Delta + (k-1)\Delta\mathscr{P}^{p+1}$. *Then, there exist elements* $\overline{W}_k$ *and* A_k *such that* $\mathscr{P}^1 W_k = \overline{W}_k \mathscr{P}^1$ *and* $W_k W_{k-1} = A_k \mathscr{P}^1$. *The submodule* $A^*\mathscr{P}^1 + A^*W_k$, $2 \leqq k < p-1$, *of* A^* *is determined by the relations* $\mathscr{P}^{p-1} \cdot \mathscr{P}^1 = 0$, $\overline{W}_k \cdot \mathscr{P}^1 - \mathscr{P}^1 \cdot W_k = 0$, $A_{k+1} \cdot \mathscr{P}^1 - W_{k+1} \cdot W_k = 0$, *and in addition* $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta \cdot W_{p-2} = 0$ *for* $k = p-2$. *Also the structure of* $A^*\mathscr{P}^1 + A^*W_{p-1}$, *in degree* $< (p^2+p+2)q+3$, *is determined by* $\mathscr{P}^{p-1} \cdot \mathscr{P}^1 = 0$, $\overline{W}_{p-1} \cdot \mathscr{P}^1 - \mathscr{P}^1 \cdot W_{p-1} = 0$, $\Delta\mathscr{P}^1\Delta \cdot W_{p-1} = 0$ *and* $A_p \cdot \mathscr{P}^1 - W_p \cdot W_{p-1} = 0$.

The following two lemmas are easily obtained from Lemmas 20.2 and 20.4.

LEMMA 20.5. *Let* W_k, $\overline{W}_k$ *and* A_k *be as above. Then, the* A^**-module structure of* $A^*\mathscr{P}^{i+1} + A^*W_k\mathscr{P}^i$, $1 \leqq i < p-1$, $2 \leqq k \leqq p-1$, *is given by* $\mathscr{P}^{p-i-1} \cdot \mathscr{P}^{i+1} = 0$, $(i+1)\overline{W}_k \cdot \mathscr{P}^{i+1} - \mathscr{P}^1 \cdot W_k\mathscr{P}^i = 0$, $(i+1)A_{k+1} \cdot \mathscr{P}^{i+1} - W_{k+1} \cdot W_k\mathscr{P}^i = 0$, $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta \cdot W_{p-2}\mathscr{P}^i = 0$ *for* $k = p-2$, $\Delta\mathscr{P}^1\Delta \cdot W_{p-1}\mathscr{P}^i = 0$ *for* $k = p-1$ *and some relations of degree* $\geqq (p^2+p+i+2)q+3$ *for* $k = p-1$.

LEMMA 20.6. *The relations in* $A^*W_k\mathscr{P}^{p-1}$, $2 \leqq k \leqq p-1$, *are given by* $\mathscr{P}^1 \cdot W_k\mathscr{P}^{p-1} = 0$, $W_{k+1} \cdot W_k\mathscr{P}^{p-1} = 0$, *and in addition* $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta \cdot W_{p-2}\mathscr{P}^{p-1} = 0$ *for* $k = p-2$, *and* $\Delta\mathscr{P}^1\Delta \cdot W_{p-1}\mathscr{P}^{p-1} = 0$, $\alpha_j \cdot W_{p-1}\mathscr{P}^{p-1} = 0$ $(\deg \alpha_j \geqq (p^2+1)q+2)$ *for* $k = p-1$.

Now we shall prove Theorem 18.2.

PROOF OF THEOREM 18.2. We prove inductively the theorem by repeating Proposition 1.2. Let $(p^2+3p)q-4 \leqq k \leqq (2p^2+p-3)q-5$ and assume that the theorem holds for any $H^*(\boldsymbol{K}_l)$, $l \leqq k$. We consider the 13 cases of k which are (i)–(xiii) of Corollary 18.3 with (xi) replaced by

(xi)′ $\qquad k = (p^2+3p)q-4$, *i.e.*, $\quad H^{k+1}(\boldsymbol{K}_k) = Z_p\{b^1_{p+1}\}$,

and we prove the theorem for $H^*(\boldsymbol{K}_{k+1})$ in each case of k.

CASE (xi)′. The induction starts from this case. By Lemma 18.1, the relations in the submodule $A^*b^1_{p+1}$ of $H^*(\boldsymbol{K}_k)$ are generated by two relations

$\mathscr{P}^2 b^1_{p+1}=0$ and $W_3\mathscr{P}^1 b^1_{p+1}=0$. Applying Proposition 1.2 and Lemma 20.5, we obtain the new generators k^0_1 and l_2 of $H^*(\boldsymbol{K}_{k+1})$ and the new relations

$$\mathscr{P}^{p-2}k^0_1 = i^*w_1, \quad -2\overline{W}_3 k^0_1+\mathscr{P}^1 l_2 = i^*w_2,$$

$$-2A_4 k^0_1 + W_4 l_2 = i^*w_3, \quad (\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta l_2 = i^*w_4 \quad \text{if} \quad p=5(*)),$$

in $H^*(\boldsymbol{K}_{k+1})$, where k^0_1 and l_2 satisfy $\delta^* k^0_1=\mathscr{P}^2 j^{*-1}b^1_{p+1}$ and $\delta^* l_2=W_3\mathscr{P}^1 j^{*-1}b^1_{p+1}$ as desired in (18.4).

The last relation is of degree$\geqq(2p^2+p-1)q$ and we can omit it. In the following, we shall put a mark $(*)$ on the heel of the new relaitons of degree $\geqq(2p^2+p-1)q$.

Since $i^*w_1\in i^*H^{k+pq}(\boldsymbol{K}_k)=0$, $i^*w_2\in i^*H^{k+(p+3)q+1}(\boldsymbol{K}_k)=0$ and $i^*w_3\in i^*H^{k+(2p+3)q+2}(\boldsymbol{K}_k)=Z_p\{\mathscr{P}^{p+1}l'_2\}$, we obtain the relations $\mathscr{P}^{p-2}k^0_1=0$, $-2\overline{W}_3 k^0_1+\mathscr{P}^1 l_2=0$ and $-2A_3 k^0_1-y_2\mathscr{P}^{p+1}l'_2+W_4 l_2=0$, $y_2\in Z_p$, as desired in Table B10 (k–1), (l–2) and (l–3).

Thus, the theorem holds for $H^*(\boldsymbol{K}_{k+1})$.

Case (vi). For $(r,s)=(1,p+s')\in I_3$, $s'<p-2$, we see in the same way as the case (xi)′ above that the new generators of $H^*(\boldsymbol{K}_{k+1})$ are $k^0_{s'}$ and $l_{s'+1}$ and that the new relations in $H^*(\boldsymbol{K}_{k+1})$ are

$$\mathscr{P}^{p-2}k^0_{s'} = i^*w_1 = 0, \quad -2\overline{W}_{s'+2}k^0_{s'}+\mathscr{P}^1 l_{s'+1} = i^*w_2 = 0,$$

$$-2A_{s'+3}k^0_{s'}+W_{s'+3}l_{s'+1} = i^*w_3\in Z_p\{\mathscr{P}^{p+1}l'_{s'+1}\},$$

and in addition

$$\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta l_{p-3} = i^*w_4(*) \qquad \text{for} \quad s'=p-4^{*)},$$

$$\Delta\mathscr{P}^1\Delta l_{p-2} = i^*w_5(*) \qquad \text{for} \quad s'=p-3.$$

Therefore the relations (k–1), (l–2) and (l–3) are obtained, and the theorem holds for $H^*(\boldsymbol{K}_{k+1})$.

Let $(r,s)\in I_1\cup I_4$. Then the relations in $A^*b^r_s$ are given by $\mathscr{P}^1 b^r_s=0$ for $(r,s)\in I_1$ and $\mathscr{P}^2 b^r_s=0$ for $(r,s)\in I_4$. So, by Proposition 1.2 and Lemma 20.2, we obtain the new generator c^r_s (resp. k^{r-1}_{s-p}) and the new relation $\mathscr{P}^{p-1}c^r_s=i^*w$ (resp. $\mathscr{P}^{p-2}k^{r-1}_{s-p}=i^*w$) for $(r,s)\in I_1$(resp. I_4). Since $i^*w\in i^*H^{k+pq}(\boldsymbol{K}_k)=Z_p\{l'_{s-2}\}$ for $r=p-1$, $s\leqq p$, and $=0$ otherwise for $(r,s)\in I_1\cup I_4$, the new relations (c–2), (c–4) and (k–1) are obtained. Also $\mathscr{P}^{p-1}c^{p-1}_s=yl'_{s-2}$ for some $y\in Z_p$, $s<p$. We have proved $\phi(\beta^r_1\beta_s)=b^r_s$ in Lemma 19.2 (ii). We notice that this equality has been proved only using the results of $H^*(\boldsymbol{K}_l)$ for $l\leqq k$, i.e., the induction hypothesis. Hence it follows from Theorem 3.4 that $\mathscr{P}^{p-1}c^{p-1}_s=0$ implies

---

*) $(*)$ indicates the new relations in $H^*(\boldsymbol{K}_{k+1})$ of degree $\geqq(2p^2+p-1)q$, and we can omit them.

$\beta_1^{r+1}\beta_s=0$. Since $\beta_1^p\beta_s=0$ [18; Th. 5.8], we see $\mathscr{P}^{p-1}c_s^{p-1}\neq 0$ and $y\neq 0$. Replacing l'_{s-2} by $(1/y)l'_{s-2}$, we obtain the relation (c–3). Thus the theorem holds for $k+1$.

Finally let $(r, s)=(0, p+s')\in I_2$. For $s'<p-2$, $A^*b_{p+s'}^0$ has the relations $\mathscr{P}^1 b_{p+s'}^0=0$ and $W_{s'}b_{p+s'}^0=0$. So the new generators $c_{p+s'}^0$ and $b_{p+s'+1}^0$ are obtained, and the new relations are

$$\mathscr{P}^{p-1}c_{p+s'}^0 = i^*w_1, \quad \mathscr{P}^1 b_{p+s'+1}^0 - \overline{W}_{s'+1}c_{p+s'}^0 = i^*w_2,$$

$$W_{s'+1}b_{p+s'+1}^0 - A_{s'+1}c_{p+s'}^0 = i^*w_3,$$

by Lemma 20.4. We have $i^*w_1\in i^*H^{k+pq}(\boldsymbol{K}_k)=0$, $i^*w_2\in i^*H^{k+(p+2)q+1}(\boldsymbol{K}_k)$ $=Z_p\{\Delta\mathscr{P}^1\Delta l_{s'}, \mathscr{P}^1\Delta l'_{s'}, \mathscr{P}^{p+1}b_{s'+2}^{p-1}\}(+Z_p\{\mathscr{P}^p b_1^{2p-2}\}$ if $s'=p-3)$ and $i^*w_3\in$ $i^*H^{k+(2p+2)q+2}(\boldsymbol{K}_k)=A^{(2p+2)q}a_t+A^{(2p+1)q+1}b_{s'+2}^{p-1}+Z_p\{\Delta\mathscr{P}^p\mathscr{P}^1\Delta l'_{s'}\}(+Z_p\{d_7,$ $\mathscr{P}^{2p}\Delta b_1^{2p-2}\}$ if $s'=p-3)$, $t=p^2+(s'+1)p+s'-1$. Hence we obtain the relations (b–4), (b–8) and (b–9) by replacing $b_{p+s'+1}^0$ by $b_{p+s'+1}^0+x\Delta l'_{s'}+y\mathscr{P}^p b_{s'+2}^{p-1}$ for some $x, y\in Z_p$. For $s'=p-2$, $A^*b_{2p-2}^0$ has the relations $\mathscr{P}^1 b_{2p-2}^0=0$, $W_{p-2}b_{2p-2}^0=0$ if $z=0$ and $\mathscr{P}^1 b_{2p-2}^0=0$, $\Delta W_{p-2}b_{2p-2}^0=0$ if $z\neq 0$, by (b–4), (b–9), and (d–2). So we obtain the new generators c_{2p-2}^0, b_{2p-1}^0 (if $z=0$), b' (if $z\neq 0$) and the new relations $\mathscr{P}^{p-1}c_{2p-2}^0=0$, $\mathscr{P}^1 b_{2p-1}^0-\overline{W}_{p-1}c_{2p-2}^0=i^*w_1(*)$, $W_{p-1}b_{2p-1}^0$ $-A_{p-1}c_{2p-2}^0=i^*w_2(*)$, $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta b_{2p-1}^0=i^*w_3(*)$, $\Delta b'=i^*w_4\in A^*\{b_p^{p-1},$ $a_{2p^2-3}\}$, and some relations of higher degree involving b'.

CASE (viii). The discussions are divided into three parts: $(r, s)\in I_1$, $r<p-1$; $(r, s)\in I_1$, $r=p-1$; $(r, s)\in I_2$. For the first part $(r, s)\in I_1$, $r<p-1$, $A^*c_s^r$ has the relation $\mathscr{P}^{p-1}c_s^r=0$ and we obtain easily the new generator b_s^{r+1} and the new relation $\mathscr{P}^1 b_s^{r+1}=0$.

Next, consider the second part $(p-1, s)\in I_1$. Let $s<p$. If $y_{s-2}\neq 0$ in (l–3), then $\alpha W_s l_{s-2}=0$ in $\boldsymbol{K}_k$ if and only if $\alpha\in A^*\{\mathscr{P}^1, W_{s+1}\}(+A^*\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta$ for $s=p-2$, $+A^*\Delta\mathscr{P}^1\Delta$ for $s=p-1)$, by (l–2): $\mathscr{P}^1 l_{s-2}=0$ and Lemma 20.4. So $A^*l'_{s-2}$ has the relations $\mathscr{P}^1 l'_{s-2}=0$, $W_s l'_{s-2}=0$, and in addition $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta\mathscr{P}^{p+1}l'_{s-2}=0$ for $s=p-2$ if $y_{p-4}\neq 0$, $p\geqq 7$, $\Delta\mathscr{P}^1\Delta\mathscr{P}^{p+1}l'_{s-2}=0$ for $s=p-1$ if $y_{p-3}\neq 0$, by the congruence $W_{t+1}\mathscr{P}^{p+1}\equiv\mathscr{P}^{p+1}W_t \bmod A^*\mathscr{P}^1$. The only relation involving c_s^{p-1} is $\mathscr{P}^{p-1}c_s^{p-1}=l'_{s-2}$, and hence $A^*c_s^{p-1}$ has the relations $W_s\mathscr{P}^{p-1}c_s^{p-1}=0$, and in addition $\Delta\mathscr{P}^{3p}\Delta\mathscr{P}^1\Delta c_{p-2}^{p-1}=0$ if $y_{p-4}\neq 0$ $(p\geqq 7)$, $\Delta\mathscr{P}^{2p}\Delta\mathscr{P}^1 c_{p-1}^{p-1}=0(*)$ if $y_{p-3}\neq 0$. So the new generators are l'_{s-1} (and d_7 for $s=p-2$). From Lemma 20.6 and the same discussion on the relations of d_7 for $p=5$ [8–II; p. 145], the new relations are $\mathscr{P}^1 l'_{s-1}=i^*w_1$, $W_{s+1}l'_{s-1}$ $=i^*w_2$, $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta l'_{p-3}=i^*w_3(*)$, $\Delta\mathscr{P}^1\Delta l'_{p-2}=i^*w_4(*)$, $\Delta d_7=i^*w_5$, $\mathscr{P}^1 d_7$ $=i^*w_6$ and $\mathscr{P}^p d_7-\Delta\mathscr{P}^{2p+2}\Delta l'_{p-3}=i^*w_7(*)$, $w_i\in H^*(\boldsymbol{K}_k)$. These imply ($l$–1) and ($d$–2) by easy calculations. For $s=p$, $A^*c_p^{p-1}$ has the relation $\mathscr{P}^{p-1}c_p^{p-1}=0$ if $y=0$ in (c–4), and no relations in our degree if $y\neq 0$. So, if $y=0$, the new

generator is b_p^p and the new relation is $\mathscr{P}^1 b_p^p = i^* w(*)$.

Finally, consider the third part $(r, s) = (0, p+s') \in I_2$. $A^* c_{p+s'}^0$ has the relation $\mathscr{P}^{p-1} c_{p+s'}^0 = 0$, which gives the new generator $b_{p+s'}^1$. The new relation is $\mathscr{P}^1 b_{p+s'}^1 = i^* w$ by Lemma 20.2, and we can put $\mathscr{P}^1 b_{p+s'}^1 = x l_{s'}$ from the results on $H^*(\boldsymbol{K}_k)$. Now, $\phi(\alpha_1 \beta_{p+s'}) = c_{p+s'}^0$ by Lemma 19.2 (iv). (Notice that this equality is proved only using the results for $H^*(\boldsymbol{K}_l)$, $l \leqq k$). So, it follows from Theorem 3.3 that $\mathscr{P}^1 b_{p+s'}^1 = 0$ implies $\alpha_1 \beta_1 \beta_{p+s'} \neq 0$. But we have $\alpha_1 \beta_1 \beta_{p+s'} = 0$ [10; Cor. 1] and so $x \neq 0$ for $s' < p-2$. Replacing $l_{s'}$ by $(1/x) l_{s'}$, we obtain $(b\text{–}5)$ and $(b\text{–}6)$ as desired.

Cases (v) and (vii). By $(c\text{–}1)$, $A^* c_1^{p+r}$ has the relation $\mathscr{P}^{p-2} c_1^{p+r} = 0$, which gives the new generator b_1^{p+r+1} of $H^*(\boldsymbol{K}_{k+1})$. The new relation is $\mathscr{P}^2 b_1^{p+r+1} = 0$ for $r < p-2$, $= x d_5$ for $r = p-2$, by Lemma 20.2 and the results on $H^*(\boldsymbol{K}_k)$. By Lemma 19.3 (ii) $\phi(\beta_1^{p-1} \varepsilon') = c_1^{2p-2}$, and we have $\beta_1^p \varepsilon' = 0$, which implies $\mathscr{P}^2 b_1^{2p-2} \neq 0$ by Theorem 3.3. Hence we obtain the new relations $(b\text{–}1)$ and $(b\text{–}2)$.

Next, $A^* b_1^{p+r}$ has the relation $\mathscr{P}^2 b_1^{p+r} = 0$ for $r < p-1$, $\mathscr{P}^3 b_1^{2p-1} = 0$ and $\mathscr{P}^3 \Delta b_1^{2p-1} = 0$ for $r = p-1$, and no relations in our degree for $r = p$. These relations give c_1^{p+r} for $r < p-1$, and $e_1'(2)$ and $e_1(2)$ for $r = p-1$, whose relations are given by $(c\text{–}1)$, $(e\text{–}1)$, $(e\text{–}2)$ and $(e\text{–}4)$, by using Lemmas 20.2–20.3.

Cases (ix) and (x). $A^* e_1'(2)$ has relation $\mathscr{P}^{p-3} e_1'(2) = 0$ for $i = 1$, and no relations for $i > 1$. So the new generator is b_1^{2p}, whose relation is $\mathscr{P}^{p-3} b_1^{2p} = i^* w(*)$. $A^* e_i(2)$ has the relations $\mathscr{P}^1 e_i(2) = 0$ and $\mathscr{P}^1 \Delta e_i(2) = 0$, which give the new generators $e_{i+1}'(2)$ and $e_{i+1}(2)$. The new relations are $\mathscr{P}^{p-1} e_{i+1}'(2) = i^* w_1(*)$, $\mathscr{P}^1 e_{i+1}(2) - R_1 e_{i+1}'(2) = i^* w_2$ and $\mathscr{P}^1 \Delta e_{i+1}(2) - \Delta \mathscr{P}^1 \Delta e_{i+1}'(2) = i^* w_3$, by Lemma 20.3. The elements $i^* w_2$ and $i^* w_3$ belong to $i^* H^{k+2q+1}(\boldsymbol{K}_k)$ and $i^* H^{k+2q+2}(\boldsymbol{K}_k)$, any elements of which are written as the right sides of the equalities in $(e\text{–}2)$, $(e\text{–}3)$, $(e\text{–}4)$ and $(e\text{–}5)$.

Case (xii). The proof of this case is easy. The new generator and relation of $H^*(\boldsymbol{K}_{k+1})$ are b_{s+p}^{r+2} and $(b\text{–}7)$, respectively.

Cases (i)–(iv). For the case (i), the proof is easy but tedious, by using Lemma 20.1 and by straightforward calculations, and we omit the details. For the cases (ii)–(iv), the theorem for $k+1$ is proved by combining the discussions in the cases (i), (vii), (viii) and (xii).

Case (xiii). This case is clear, since $i^*: H^*(\boldsymbol{K}_k) \to H^*(\boldsymbol{K}_{k+1})$ is isomorphic by (1.5).

From the above discussions, the proof is complete. *q.e.d.*

§21. Adams spectral sequence

In his thesis [4], J. P. May calculated extensively the cohomology $H^{*,*}(A^*)$ $=\mathrm{Ext}_{A^*}^{*,*}(Z_p, Z_p)$ of A^*. Recently, O. Nakamura [5] has extended (and corrected) May's calculations and determined $H^{s,t}(A^*)$ for $t-s\leqq(3p^2+3p+4)q-2$.

From now on, we shall use Nakamura's notations [5] for the elements in $E^0H^{*,*}(A^*)$, the associated graded algebra of $H^{*,*}(A^*)$, but for the simplicity we shall denote the elements b_{01} $(=b_1^0$ in May's notation [4]), b_{11} $(=b_1^1$ in [4]), b_{02} $(=b_2^0$ in [4]), $g_{1,l}$ $(=g_1^l$ in [4], $g_{1,0}=h_0)$, $k_{1,l}$ $(=k_1^l$ in [4]) and a_1 of Nakamura [5; Th. 4.4] by b, b_1, b_2, g_l $(g_0=h_0)$, k_l and a, respectively. Also, any element in $H^{*,*}(A^*)$ will be denoted by the same symbol for the element in $E^0H^{*,*}(A^*)$ corresponding to it.

Since the Adams spectral sequence (converging to ${}_p\pi_*(S)$) has $H^{*,*}(A^*)$ as its E_2 term, we can obtain the differentials of the Adams spectral sequence $\{E_r^{s,t}\}$ in the range $t-s\leqq(2p^2+p-2)q-5$ from our results on ${}_p\pi_*(S)$ ([8–II; Th. A] and Theorem 19.9).

THEOREM 21.1. *All non-trivial differentials of the* mod p, $p\geqq 5$, *Adams spectral sequence, in the range* $t-s\leqq(2p^2+p-2)q-5$, *are exhausted by the following equalities up to non-zero coefficients* ($i, j, k\geqq 0$ *and* $s=0, 1$):

I. (i) $d_2(a_0^ih_1b^k) = a_0^{i+1}b^{k+1}$, $d_2(a_0^ih_2b^kb_1^s) = a_0^{i+1}b^kb_1^{s+1}$.

(ii) $d_2(g_{2,l}b^kb_1^s) = g_{l+1}b^{k+1}b_1^s$, $d_2(g_{3,l}b^k) = g_{2,l+1}b^kb_2$.

(iii) $d_2(g_lb^ka^ju) = g_lb^{k+1}a^{j+1}$,

$d_2(a_0^ib^ka^ju) = a_0^ib^{k+1}a^{j+1}$ $\quad$ ($j\not\equiv -2 \bmod p$ *if* $k=0$).

(iv) $d_2(a_0^ib^kc) = a_0^{i+1}b^kh_1b_2$.

(v) $d_2(g_{2,l}b^ka_2) = b^ke_{l+2}$, $d_2(g_lb^ka^jua_2) = b^ka^{j+1}e_{l+1}$.

(vi) $d_2(a_0^ib^{k+1}h_1a_2) = a_0^{i+1}b^kf$, $d_2(a_0^ib^{k+1}a^jua_2) = a_0^ib^ka^{j+1}f$.

(vii) $d_2(a_0^ib^ka^jw) = a_0^ib^ka^jub_2$ $\quad$ ($j\not\equiv -3 \bmod p$ *if* $k=0$),

$d_2(g_lb^ka^jw) = g_lb^ka^jub_2$.

(viii) $d_2(b^kh_1g_{3,0}) = b^kh_0k_1b_1$, $d_2(b^kj_l) = b^kh_0k_{l+1}b_1$.

(ix) $d_2(a_0^ia^{p-3}a_2u) = a_0^il$.

II. (i) $d_3(a_0^{p^2-1-p+i}h_2) = a_0^ia^{p-1}b$, $d_3(a_0^{p^2-1+i}a^{p-3}a_2u) = a_0^ia^{2p-1}b$.

III. (i) $d_{p+1}(b^k k_0 b_2) = b^{k+1} e_1, \quad d_p(b^k k_l b_2) = b^{k+1} h_0 k_{l-1} a_2 \quad (l \geqq 1)$.

(ii) $d_{2p-1}(b^k b_1) = h_0 b^{p+k}, \quad d_{2p-1}(b^k b_1^2) = h_0 b^{p+k} b_1$,

$d_{2p-1}(b^k k_l b_1) = h_0 b^{p+k} k_l$.

(iii) $d_{2p-1}(b^k h_1 b_2) = b^{p+k} k_0, \quad d_{2p}(b^k k_{2,l}) = b^{p+k} k_{l+1}$.

REMARK. Since the spectral sequence is multiplicative, I. (i) is essentially $d_2(h_1) = a_0 b$ and $d_2(h_2) = a_0 b_1$. The first is the main result of [14–II]. In general, $d_2(h_i) = a_0 b_1^{i-1}$, $i \geqq 1$, hold. This is equivalent to the triviality of the mod p Hopf invariant (A. Liulevicius [3], N. Shimada and T. Yamanoshita [11], cf. H. H. Gershenson [2; Appendix]). J. P. May [4; Th. II. 7.5.] proved the first of I. (ii), I. (iii) and the first of II. (i). Also, he pointed out the possibility of the second of I. (ii) and the first of I. (vii), and conjectured I. (ix) and the second of II. (i) [4; p. II–7.6]. III. (ii) is essentially $d_{2p-1}(b_1) = h_0 b^p$, which is the main result of H. Toda [16]. The first of III. (iii) is equivalent to the relation $\beta_1^p \beta_2 = 0$ in ${}_p\pi_*(S)$, which is proved by H. Toda [17]. Also, the second of III. (iii) is equivalent to $\beta_1^p \beta_{l+2} = 0$, proved by H. Toda [18]. III. (i) is equivalent to $\alpha_1 \beta_1 \beta_{p+s} = 0$ $(1 \leqq s \leqq p-3)$ of [10; Cor. 1].

REMARK. O. Nakamura [6] has determined all differentials in the mod 3 Adams spectral sequence in the range $t-s \leqq 104$. In particular, the differentials corresponding to our results [8–II; Th. B] on ${}_3\pi_*(S)$ are seen in Theorems 2.1–2.4 of [6].

We also obtain the following list of the elements surviving to E_∞.

THEOREM 21.2. *In the* mod p *Adams spectral sequence,* $p \geqq 5$, *the elements surviving to* E_∞ *term (and corresponding generators of* ${}_p\pi_*(S)$*) are listed, in total degree* $\leqq (2p^2+p-2)q-6$, *by the following*:

$1, a_0, a_0^2, \ldots (\iota)$;

$g_l a^j (\alpha_{jp+l+1})$ *for* $0 \leqq l \leqq p-2, j \geqq 0$;

$a_0^{p-2} h_1, a_0^{p-1} h_1 (\alpha'_p)$; $a_0^{p-1} a^j u, a_0^p a^j u (\alpha'_{(j+2)p})$ *for* $j \not\equiv -2 \bmod p, j \geqq 0$;

$a_0^{p^2-2} h_2, a_0^{p^2-1} h_2, a_0^{p^2} h_2 (\alpha''_{p^2})$;

$a_0^{p^2+p-2} a^{p-3} a_2 u, a_0^{p^2+p-1} a^{p-3} a_2 u, a_0^{p^2+p} a^{p-3} a_2 u (\alpha''_{2p^2})$;

$b^k (\beta_1^k)$ *for* $1 \leqq k \leqq 2p+1$; $h_0 b^k (\alpha_1 \beta_1^k)$ *for* $1 \leqq k < p$;

$h_0^s b^k k_l (\alpha_1^s \beta_1^k \beta_{l+2})$ *for* $s = 0, 1, 0 \leqq k < p, 0 \leqq l \leqq p-3$;

$a_0^{p-1} c (\beta_{p+1})$; $e_1 (\alpha_1 \beta_{p+1})$; $b^k f (\beta_1^{k+1} \beta_{p+1})$ *for* $0 \leqq k \leqq p-3$;

$b^k k_l a_2(\beta_1^k \beta_{p+l+2})$ *for* $0 \leqq k < p, 0 \leqq l \leqq p-4, k+l \leqq p-4$;

$h_0 k_l a_2(\alpha_1 \beta_{p+l+2})$ *for* $0 \leqq l \leqq p-4$;

$b^k g_{2,p-3} b_1(\beta_1^k \beta_2 \beta_{p-1})$ *for* $0 \leqq k \leqq p-2$;

$a_0^{p-3} bh_2$, $a_0^{p-2} bh_2(\varphi)$; $a_0^{p-3} b^k h_2(\alpha_1 \beta_1^{k-2} \beta_2 \beta_{p-1})$ *for* $2 \leqq k \leqq p$;

$h_0 b^k b_1(\beta_1^k \varepsilon')$ *for* $0 \leqq k < p$;

$g_l h_2(\varepsilon_{l+1})$ *for* $0 \leqq l \leqq p-2$; $g_l b_1(\alpha_1 \varepsilon_l)$ *for* $1 \leqq l \leqq p-3$;

$h_0 b^k k_l b_2(\beta_1^k \kappa_{l+1})$ *for* $k \geqq 0, 0 \leqq l \leqq p-4, k+l \leqq p-3$;

$h_0 b_1^2(\lambda')$; $g_l b_1 h_2(\lambda_{l+1})$ *for* $0 \leqq l \leqq p-4$; $g_l b_1^2(\alpha_1 \lambda_l)$ *for* $1 \leqq l \leqq p-4$.

Since the Adams spectral sequence is multiplicative, there are many differentials of higher degree which can be mechanically determined from Theorem 21.2. The following proposition follows from Nakamura's results [5; Th. 4.4].

PROPOSITION 21.3. *In the range* $(2p^2+p-2)q-5 \leqq t-s \leqq (2p^2+p)q-4$ *except for* $(s, t-s)=(2p, 2p^2+p-1)q-3)$, $(p, (2p^2+p-1)q-2)$, *the following elements give a* Z_p*-basis for* $E_\infty^{s,t}$:

$$h_0 bk_{p-4} a_2 \in (2p+1, N-2q-5), \quad b^{p-1} g_{2,p-3} b_1 \in (3p-1, N-2q-4),$$

$$k_{p-3} a_2 \in (2p-1, N-2q-2), \quad g_{p-3} a^{2p} \in (2p^2+p-2, N-2q-1),$$

$$h_0 k_{p-3} b_2 \in (p+2, N-q-5), \quad a_0^{p-3} b^{p+1} h_2 \in (3p, N-q-5),$$

$$b^{p-2} f \in (3p, N-q-2), \quad g_{p-2} a^{2p} \in (2p^2+p-1, N-q-1),$$

where $N=(2p^2+p)q$ *and* $\alpha \in (a, b)$ *means* $\operatorname{bideg} \alpha=(a, a+b)$.

For the above two exceptions of $(s, t-s)$, E_∞ *term is trivial if* $d_p(x) \neq 0$, *and is generated by the following elements if* $d_p(x)=0$:

$$h_0 k_{p-3} a_2 \in (2p, N-q-3), \quad x \in (p, N-q-2).$$

The element $h_2 b_2 \in (3, N-3)$ *survives to* E_∞.

We determine the first unsolved differential $d_p(x)$.

PROPOSITION 21.4. $d_p(x)=h_0 k_{p-3} a_2$ *up to a non-zero coefficient.*

PROOF. Let $n=(2p^2+p-2)q-2$. The group ${}_p\pi_n(S)$ is Z_p by Proposition 21.3, and hence it is generated by the element β_{2p-1} of L. Smith [12] and H. Toda [18]. So, $k_{p-3} a_2$ converges to β_{2p-1}.

Consider the element $h_0bk_{p-3}a_2 \in (2p+2, n+(p+1)q-3)$. It converges to the element $\alpha_1\beta_1\beta_{2p-1}$, which is zero [10; Cor. 1]. So, $h_0bk_{p-3}a_2$ is killed by some $\alpha \in (s, n+(p+1)q-2)$ with $s \leqq 2p$. Hence, α is a linear combination of the elements bx and $g_{p-2}h_2b_2$ by [5; Th. 4.4]. Since g_{p-2} and h_2b_2 are permanent cycles, $d_r(g_{p-2}h_2b_2)=0$ for $r \geqq 2$. Thus, we obtain a differential $d_p(bx) = h_0bk_{p-3}a_2$, which implies the proposition. *q.e.d.*

COROLLARY 21.5. $\alpha_1\beta_{2p-1}=0$.

REMARK. The same result as Proposition 21.4 for $p=3$ is recently proved by O. Nakamura [6; Prop. 3.1].

From the above discussion, we have known all differentials in the range $t-s \leqq (2p^2+p)q-4$. So, the group ${}_p\pi_{t-s}(S)$ is determined up to extension in the cited range. Recalling (19.5), we obtain immediately the following

THEOREM 21.6 ($p \geqq 5$). *Let* $N=(2p^2+p)q$, $q=2(p-1)$. *Then the group* ${}_p\pi_k(S)$, $N-2q-5 \leqq k \leqq N-4$, *is given as follows*:

${}_p\pi_{N-2q-5}(S) = Z_p$, *generated by* $\alpha_1\beta_1\beta_{2p-2}$;

${}_p\pi_{N-2q-4}(S) = Z_p$, *generated by* $\beta_1^{p-1}\beta_2\beta_{p-1}$;

${}_p\pi_{N-2q-2}(S) = Z_p$, *generated by* β_{2p-1};

${}_p\pi_{N-2q-1}(S) = Z_p$, *generated by* α_{2p^2+p-2};

${}_p\pi_{N-q-5}(S) = Z_{p^2}$ *or* $Z_p \oplus Z_p$, *in which* $\alpha_1\beta_1^{p-1}\beta_2\beta_{p-1} \neq 0$;

${}_p\pi_{N-q-3}(S) = 0$, *in particular* $\alpha_1\beta_{2p-1} = 0$;

${}_p\pi_{N-q-2}(S) = Z_p$, *generated by* $\beta_1^{p-1}\beta_{p+1}$;

${}_p\pi_{N-q-1}(S) = Z_p$, *generated by* α_{2p^2+p-1};

${}_p\pi_k(S) = 0$ *otherwise for* $N-2q-5 \leqq k \leqq N-4$.

REMARK. In the next section, we shall show that the group ${}_p\pi_{N-q-5}(S)$ is cyclic.

The first unknown differential after Proposition 21.4 is $d_r(d)$, $2 \leqq r \leqq p$, where $d \in (2, (2p^2+p)q-2)$, and we obtain the following partial result:

$$ {}_p\pi_{(2p^2+p)q-3}(S) \neq 0, $$

since h_2b_2 is a permanent cycle. Also, this group contains the element γ_2 due to E. Thomas and R. Zahler [13] [19]. Concerning [13; Cor. E], it follows from our results on ${}_p\pi_*(S)$ that any (non-zero) element in ${}_p\pi_{(2p^2+p)q-3}(S)$ is indecomposable. Especially, we have

PROPOSITION 21.7. *The element γ_2 is indecomposable.*

§22. Some relations concerning the elements λ_i

We consider the ring $\mathscr{A}_*(M)=\Sigma_k\mathscr{A}_k(M)$ and the homomorphism π_*i^*: $\mathscr{A}_k(M)\to{}_p\pi_{k-1}(S)$. Let $M_n=S^n\cup{}_pe^{n+1}$, $n\geqq 1$, and i: $S^n\to M_n$ and π: $M_n\to S^{n+1}$ be the inclusion and the projection, respectively. Denote the stable track group Dir lim $[M_{n+k}, M_n]$ by $\mathscr{A}_k(M)$ and the direct sum $\Sigma\mathscr{A}_k(M)$ by $\mathscr{A}_*(M)$, which is an algebra over Z_p and has a differential D of degree $+1$ such that $D(\xi\eta)=D(\xi)\eta+(-1)^{\deg\xi}\xi D(\eta)$ and $\operatorname{Ker} D$ is a commutative subalgebra, cf. [9; §1]. Also, the homomorphism π_*i^*: $\mathscr{A}_k(M)\to\pi_{k-1}(S)$ is defined by the composition: $\pi_*i^*(\xi)=\pi\xi i$. The image of π_*i^* is the subgroup $\pi_{k-1}(S)*Z_p$ of ${}_p\pi_{k-1}(S)$, and for any $\xi\in\mathscr{A}_k(M)$ there is $\eta\in\mathscr{A}_k(M)\cap\operatorname{Ker} D$ such that $\pi_*i^*\xi=\pi_*i^*\eta$ [9; §3].

In [9], we determined the algebra $\mathscr{A}_*(M)$, in degree $<(p^2+3p+1)q-6$. There exists the indecomposable element $\varepsilon\in\mathscr{A}_{s+q-1}(M)$, $s=p^2q$, such that

$$\pi_*i^*\varepsilon=\varepsilon_1,\quad D(\varepsilon)=0,\quad \varepsilon\in\{\alpha,\beta_{(1)},\delta\alpha(\delta\beta_{(1)})^{p-1}\}, \tag{22.1}$$

where $\delta=i\pi\in\mathscr{A}_{-1}(M)$ with $D(\delta)=1$, $\alpha=(\pi_*i^*)^{-1}\alpha_1\in\mathscr{A}_q(M)$ with $D(\alpha)=0$ and $\beta_{(1)}\in(\pi_*i^*)^{-1}\beta_1\subset\mathscr{A}_{pq-1}(M)$ with $D(\beta_{(1)})=0$ ([9; Prop. 5.2]). Since ε is of odd degree, we have

$$\varepsilon^2=0 \tag{22.2}$$

from the commutativity of $\operatorname{Ker} D$. For the element $\varepsilon\delta\varepsilon$, we have the following

LEMMA 22.1. *There exists uniquely an element $\lambda\in\mathscr{A}_{2s+q-3}(M)$, $s=p^2q$, such that $\varepsilon\delta\varepsilon=-\alpha\lambda=-\lambda\alpha$, $D(\lambda)=0$ and $\lambda=\{\beta_{(1)},\delta\alpha(\delta\beta_{(1)})^{p-1},\delta\varepsilon\}$* (mod *zero*). *The homomorphism π_*i^*: $\mathscr{A}_{2s+q-3}(M)\to{}_p\pi_{2s+q-4}(S)$ is isomorphic.*

PROOF. The secondary composition $\lambda=\{\beta_{(1)},\delta\alpha(\delta\beta_{(1)})^{p-1},\delta\varepsilon\}$ is well defined by (ii), (v), (ix) of [9; Th. 0.1] and (5.8) of [9]. The indeterminacy of λ is $\beta_{(1)}\mathscr{A}_b(M)+\mathscr{A}_{s-1}(M)\delta\varepsilon$, $b=(2p^2-p+1)q-2$. Since ${}_p\pi_{b+1}(S)=Z_p$, generated by α_{2p^2-p+1}, ${}_p\pi_b(S)=0$ and ${}_p\pi_{b-1}(S)=0$ by Theorem 19.9, it follows from the discussions in [9; §4] that $\mathscr{A}_b(M)=Z_p$, generated by $\alpha^{2p^2-p}\delta\alpha\delta$. Also we have $\mathscr{A}_{s-1}(M)=Z_p\{\alpha^{p^2}\delta,\alpha^{p^2-1}\delta\alpha,(\delta\beta_{(1)})^{p-2}\delta\beta_{(2)}\delta\}$ by [9; Th. 0.1]. By (i)–(ii) of [9; Th. 0.1], $\beta_{(1)}\mathscr{A}_b(M)=0$, and by (i), (vi) and (ix) of [9; Th. 0.1], $\mathscr{A}_{s-1}(M)\varepsilon\delta=0$. So, the composition λ consists of a single element, and we denote it by the same symbol λ. Then, the relation $\varepsilon\delta\varepsilon=-\alpha\lambda$ follows from (22.1) and the formula [14–IV; (4.4) i)] (cf. [9; p. 645]). Since ${}_p\pi_{2s+q-3}(S)={}_p\pi_{2s+q-2}(S)=0$, the last assertion follows from [9; Prop. 2.3]. The last assertion implies

$D(\lambda)=0$, and hence $\lambda\alpha=\alpha\lambda$. *q.e.d.*

We can set uniquely the elements λ_j of Theorem 19.9 by the following

THEOREM 22.2. *The generator λ_j of ${}_p\pi_{(2p^2+j)q-4}(S)$ can be chosen such as $\lambda_1=\pi_*i^*\lambda=\{\varepsilon_1, \beta_1^p, \alpha_1\}$ for $j=1$ and $\lambda_j=\pi_*i^*\lambda\alpha^{j-1}$ for $j\geqq 2$. These elements satisfy $\lambda_{p-2}=\{\lambda_{p-3}, p\iota, \alpha_1\}=x\beta_1^{p-1}\beta_2\beta_{p-1}$ and $\lambda_j=0$ for $j\geqq p-1$, where the coefficient x ($\not\equiv 0 \bmod p$) is the same one as in the relation $\beta_1\varphi=-x\alpha_1\beta_2\beta_{p-1}$ in* (14.2).

PROOF. By (14.2), (7.5) and (6.2), we have

$$-x\alpha_1\beta_1^{p-1}\beta_2\beta_{p-1} = \beta_1^p\varphi = \varphi\beta_1^p$$
$$= \{\varepsilon_{p-2}, \alpha_1, \alpha_1\}\beta_1^p = -\varepsilon_{p-2}\varepsilon'.$$

By [9; Prop. 5.2], there is an element $\bar{\varepsilon}\in\mathscr{A}_{(p^2+1)q-2}(M)$ such that $\pi_*i^*\bar{\varepsilon}=\varepsilon'$ and $D(\bar{\varepsilon})=0$. So, from [9; Cor. 6.6]

$$\varepsilon\alpha^{p-3}\delta\bar{\varepsilon} = \varepsilon(\delta\varepsilon\alpha^{p-3}\delta-\delta\varepsilon\alpha^{p-4}\delta\alpha)$$
$$= -\lambda\alpha^{p-2}\delta+\lambda\alpha^{p-3}\delta\alpha,$$

and hence

$$(*)\qquad x\alpha_1\beta_1^{p-1}\beta_2\beta_{p-1} = \varepsilon_{p-2}\varepsilon' = \pi_*i^*\varepsilon\alpha^{p-3}\delta\bar{\varepsilon} = \alpha_1\pi_*i^*\lambda\alpha^{p-3}.$$

This implies $\lambda\neq 0$ by Theorem 21.6. So we have $\pi_*i^*\lambda\neq 0$ by Lemma 22.1, and we can take $\lambda_1=\pi_*i^*\lambda$, which is equal to $\pi_*i^*\{\beta_{(1)}, \delta\alpha(\delta\beta_{(1)})^{p-1}, \delta\varepsilon\}=-\{\pi\beta_{(1)}, i\alpha_1\beta_1^{p-1}, \varepsilon_1\}=-\{\alpha_1, \beta_1^p, \varepsilon_1\}=\{\varepsilon_1, \beta_1^p, \alpha_1\}$. Then, the element $\lambda_j=\pi_*i^*\lambda\alpha^{j-1}$ satisfies $\lambda_j=\{\lambda_{j-1}, p\iota, \alpha_1\}$ for $j\geqq 2$, as defined in (19.3). Since $\alpha_{1*}\colon {}_p\pi_{(2p^2+p-2)q-4}(S)\to{}_p\pi_{(2p^2+p-1)q-5}(S)$ is monomorphic by Theorem 21.6, $(*)$ implies $\lambda_{p-2}=\pi_*i^*\lambda\alpha^{p-3}=x\beta_1^{p-1}\beta_2\beta_{p-1}$. The following result can be proved in a similar manner to [9; §§5–6], from Theorem 21.6:

$$\mathscr{A}_{(2p^2+p-2)q-3}(M) = Z_p\{\xi = \{\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta\beta_{(p-1)}, \delta\beta_{(2p-1)}\delta\}.$$

Then, $\lambda\alpha^{p-3}\equiv x\xi \bmod \delta\beta_{(2p-1)}\delta$ and so $\lambda\alpha^{p-2}\equiv 0 \bmod \delta\beta_{(2p-1)}\delta\alpha$, $\lambda\alpha^j=0$ for $j\geqq p-1$ by (5.8) of [9]. Hence $\lambda_j=0$ for $j\geqq p-1$. *q.e.d.*

We consider the element $\alpha_1\lambda_{p-3}$. This lies in the group ${}_p\pi_{(2p^2+p-2)q-5}(S)=Z_p$, generated by $\alpha_1\beta_1\beta_{2p-2}$. Hence we have a relation

$$(22.3)\qquad \alpha_1\lambda_{p-3} = y\alpha_1\beta_1\beta_{2p-2} \qquad \textit{for some } y\in Z_p.$$

We can prove $y\neq 0$. But the proof needs the results on $\mathrm{Ext}_{A^*}(Z_p, Z_p)$ higher than Theorems 21.1–21.2. We notice only the facts ${}_p\pi_{(2p^2+2p-2)q-6}(S)=0$

and $\beta_1\pi_{(2p^2+p-1)q-5}(S)\neq 0$ which imply $y\neq 0$, and we omit the details.

The next result determines the group ${}_p\pi_{(2p^2+p-1)q-5}(S)$. The relation (22.3) is used in the proof, but the result is independent of the claim $y\neq 0$.

THEOREM 22.3. *Consider the element*

$$\mu\in\{\lambda_{p-3}-y\beta_1\beta_{2p-2},\alpha_1,\alpha_1\}.$$

Then, μ is of order p^2 and generates the group ${}_p\pi_{(2p^2+p-1)q-5}(S)$. There is a relation

$$p\mu = -\alpha_1\lambda_{p-2} = -x\alpha_1\beta_1^{p-1}\beta_2\beta_{p-1}. \tag{22.4}$$

PROOF. By Theorems 21.6 and 22.2, it suffices to show $p\mu=-\alpha_1\lambda_{p-2}$. We have

$$\begin{aligned} p\mu &= \{\lambda_{p-3}-y\beta_1\beta_{2p-2},\alpha_1,\alpha_1\}(p\iota)\\ &= -(\lambda_{p-3}-y\beta_1\beta_{2p-2})\{\alpha_1,\alpha_1,p\iota\}\\ &= -(1/2)(\lambda_{p-3}-y\beta_1\beta_{2p-2})\alpha_2\\ &= -(1/2)\alpha_2\lambda_{p-3} = -(1/2)\pi_*i^*\lambda\alpha^{p-4}\delta\alpha^2\\ &= -\pi_*i^*\lambda\alpha^{p-3}\delta\alpha = -\alpha_1\lambda_{p-2}, \end{aligned}$$

by [14–IV; Th. 4.14 ii)] and [9; (4.4) (i)]. *q.e.d.*

THEOREM 22.4. *In $\mathscr{A}_*(M)$, the following relations hold:*

(i) $\lambda\alpha^{p-4}\delta\alpha-2\lambda\alpha^{p-3}\delta-\delta\lambda\alpha^{p-3} = y\alpha\delta\beta_{(1)}\delta\beta_{(2p-2)}$,

(ii) $\lambda\alpha^{p-3} = x(\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta\beta_{(p-1)}$,

(iii) $\lambda\alpha^{p-3}\delta\alpha\delta = \delta\lambda\alpha^{p-3}\delta\alpha$,

(iv) $\lambda\alpha^{p-2} = 0$,

where the coefficients y in (i) *and x in* (ii) *are the same ones as in* (22.3) *and* (22.4), *respectively.*

PROOF. As is seen in the proof of Theorem 22.2, we can write $\lambda\alpha^{p-3}=x\xi+x'\delta\beta_{(2p-1)}\delta$, $\xi=(\beta_{(1)}\delta)^{p-1}\beta_{(2)}\delta\beta_{(p-1)}$, for some x'. From $D\lambda=D\alpha=D\xi=0$ and $D(\delta\beta_{(2p-1)}\delta)=\beta_{(2p-1)}\delta+\delta\beta_{(2p-1)}\neq 0$, it follows that $x'=0$, and (ii) is proved. Since $\xi\alpha=0$, (iv) follows from (ii). Put $\eta=\lambda\alpha^{p-3}\delta\alpha$. Then, $D\eta=0$ and $\pi\eta i$ is divisible by p by Theorem 22.3. So, $\delta\eta\delta=0$ and $\eta\delta-\delta\eta=D(\delta\eta\delta)=0$. Thus (iii) is proved.

By (22.3) and Theorem 21.6, we can write

$$\lambda\alpha^{p-4}\delta\alpha = a\lambda\alpha^{p-3}\delta + b\delta\lambda\alpha^{p-3} + y\alpha\delta\beta_{(1)}\delta\beta_{(2p-2)}.$$

Then, $-\lambda\alpha^{p-3} = D(\lambda\alpha^{p-4}\delta\alpha) = (b-a)\lambda\alpha^{p-3}$ and $2\lambda\alpha^{p-3}\delta\alpha = \lambda\alpha^{p-4}\delta\alpha^2 = a\lambda\alpha^{p-3}\delta\alpha$. Hence $a=2$ and $b=1$, as desired in (i). *q.e.d.*

PROPOSITION 22.5. *The following relations hold:*

(22.5) $\alpha_r\varepsilon_s = r\alpha_1\varepsilon_{r+s-1},\quad \alpha'_{rp}\varepsilon_s = 0,\quad \alpha''_{rp^2}\varepsilon_s = 0 \quad$ *for* $r \geqq 1$.

(22.6) $\alpha_r\lambda_s = r\alpha_1\lambda_{r+s-1},\quad \alpha'_{rp}\lambda_s = 0,\quad \alpha''_{rp^2}\lambda_s = 0 \quad$ *for* $r \geqq 1$.

(22.7) $\varepsilon_r\varepsilon_s = -\lambda_{r+s} \quad$ *for* $r, s \geqq 1$.

(22.8) $\varepsilon'\varepsilon_s = \alpha_1\lambda_s \quad$ *for* $s \geqq 1,\ \varepsilon'\varepsilon' = 0$.

Here, we interpret $\varepsilon_s=0$ *for* $s\geqq p$, $\lambda_{p-2}=x\beta_1^{p-1}\beta_2\beta_{p-1}$ *and* $\lambda_s=0$ *for* $s\geqq p-1$.

REMARK. Concerning the first of (22.5), we have obtained in §7 the following relations similar to (22.3)–(22.4):

(22.9) $$\alpha_1\varepsilon_{p-2} = 0,\quad \alpha_1\varepsilon_{p-1} = p\varphi.$$

PROOF OF PROPOSITION 22.5. By (12.5), [9; Cor. 6.6], (22.2) and Lemma 22.1, we have

$$\alpha_r\varepsilon_s = \pi_* i^*\varepsilon\alpha^{s-1}\delta\alpha^r = r\pi_* i^*\varepsilon\alpha^{r+s-1}\delta\alpha = r\alpha_1\varepsilon_{r+s-1},$$

$$\alpha_r\lambda_s = \pi_* i^*\lambda\alpha^{s-1}\delta\alpha^r = r\pi_* i^*\lambda\alpha^{r+s-1}\delta\alpha = r\alpha_1\lambda_{r+s-1},$$

$$\varepsilon_r\varepsilon_s = \pi_* i^*\varepsilon\alpha^{r-1}\delta\varepsilon\alpha^{s-1} = -\pi_* i^*\lambda\alpha^{r+s-1} = -\lambda_{r+s},$$

$$\varepsilon'\varepsilon_s = \pi_* i^*\varepsilon\alpha^{s-1}\delta\bar{\varepsilon} = -\pi_* i^*\varepsilon\delta\varepsilon\alpha^{s-2}\delta\alpha = \lambda_s\alpha_1 = \alpha_1\lambda_s,$$

where $\bar{\varepsilon}$ is the generator of $\mathscr{A}_*(M)$ introduced in [9; Prop. 5.2]. By the (graded) commutativity of ${}_p\pi_*(S)$, $\varepsilon'\varepsilon'=0$.

The equalities involving α'_{rp} and α''_{rp^2} are easy consequences of the following lemma. *q.e.d.*

LEMMA 22.6. *Let γ be an element of ${}_p\pi_k(S)$ of order p. Assume that there exists $\xi \in \mathscr{A}_{k+1}(M)$ such that $\pi_* i^*\xi = \gamma$ and $\alpha^t\xi = 0$ for some $t \geqq 1$. Then, $\alpha_r\gamma = 0$ for $r > t$, $\alpha'_{rp}\gamma = 0$ for $rp > t$ and $\alpha''_{rp^2}\gamma = 0$ for $rp^2 > t$.*

PROOF. In the algebra $\mathscr{A}_*(M)$, there is a relation $\alpha^r\delta = ((t+1)\alpha^{r-t}\delta - t\alpha^{r-t-1}\delta\alpha)\alpha^t$ [9; (4.4)]. Hence, $\alpha_r\gamma = \pi\alpha^r\delta\xi i = 0$ by $\alpha^t\xi = 0$.

Next we consider the algebra $\mathscr{A}_*(M_{p^2})$ studied in [9]. We use the same notations i_2, π_2, $\delta_2=i_2\pi_2$ and λ as in [9; §§3, 4, 7]. There is an element α' $\in\mathscr{A}_{pq}(\boldsymbol{M}_{p^2})$ which defines the elements α'_{rp} by the rule $\alpha'_{rp}=\pi_2\alpha'^r i_2$. Then, there is a relation $\alpha'^r\delta_2\lambda=-r\lambda\alpha\delta\alpha^{rp-1}$ by [9; Prop. 4.2 and (C) on p. 650]. Hence, $\alpha'_{rp}\gamma=\pi_2\alpha'^r\delta_2\lambda\xi i=0$. The last relation $\alpha''_{rp^2}\gamma=0$ follows from a similar discussion in $\mathscr{A}_*(\boldsymbol{M}_{p^3})$. *q. e. d.*

§23. The ring structure

It is well known [1] that the elements α_r, α'_{rp} and α''_{rp^2} of (19.5) lie in the image of the J-homomorphism. By [15–I] α_r is constructed on S^3, i.e., α_r belongs to the image of $S^\infty\colon \pi_{3+k}(S^3)\to\pi_k(S)$, $k=\deg\alpha_r$. From similar discussions using the results on the unstable groups ${}_p\pi_{pq+4}(S^5)$ [15–I] and ${}_p\pi_{p^2q+6}(S^7)$ [15–III], we see that α'_{rp} and α''_{rp^2} are constructed on S^5 and S^7 respectively. Then, the equalities $J(\xi)\alpha_s=J(\xi\alpha_s)$ for $J(\xi)=\alpha_r$, α'_{rp}, α''_{rp^2}, $J(\xi)\alpha'_{sp}=J(\xi\alpha'_{sp})$ for $J(\xi)=\alpha'_{rp}$, α''_{rp^2}, and $J(\xi)\alpha''_{sp^2}=J(\xi\alpha''_{rp^2})$ for $J(\xi)=\alpha''_{rp^2}$ hold. Since J is mod p trivial on even degrees, we obtain

(23.1) $$\alpha_r\alpha_s=0,\quad \alpha_r\alpha'_{sp}=0,\quad \alpha_r\alpha''_{sp^2}=0,$$
$$\alpha'_{rp}\alpha'_{sp}=0,\quad \alpha'_{rp}\alpha''_{sp^2}=0,\quad \alpha''_{rp^2}\alpha''_{sp^2}=0\qquad for\quad r,s\geqq 1.$$

PROPOSITION 23.1. *The following relations hold*:

(23.2) $$\alpha_r\beta_s=0\ (r\geqq 2),\quad \alpha_1\beta_{2p-1}=0,\quad \alpha'_{rp}\beta_s=0\ (r\geqq 1),$$
$$\alpha''_{rp^2}\beta_s=0\ (r\geqq 1).$$

(23.3) $$\alpha_r\kappa_s=0,\quad \alpha'_{rp}\kappa_s=0,\quad \alpha''_{rp^2}\kappa_s=0\qquad for\quad r\geqq 1.$$

(23.4) $$\alpha_r\varepsilon'=0,\quad \alpha'_{rp}\varepsilon'=0,\quad \alpha''_{rp^2}\varepsilon'=0,$$
$$\alpha_r\lambda'=0,\quad \alpha'_{rp}\lambda'=0,\quad \alpha''_{rp^2}\lambda'=0\qquad for\quad r\geqq 1.$$

(23.5) $$\alpha_r\varphi=0,\quad \alpha'_{rp}\varphi=0,\quad \alpha''_{rp^2}\varphi=0,\quad \alpha_r\mu=0\qquad for\quad r\geqq 1.$$

PROOF. There exist elements $\beta_{(s)}\in\mathscr{A}_*(\boldsymbol{M})$ such that $\pi\beta_{(s)}i=\beta_s$ and $\alpha\beta_{(s)}=0$ [18], cf. [9; §5]. Then (23.2) follows from Lemma 22.6 for $t=1$ and Corollary 21.5. Let $\kappa_{(s)}$ be an element of $\mathscr{A}_*(\boldsymbol{M})$ such that $\pi\kappa_{(s)}i=\kappa_s$. By dimensional reason, any $\kappa_{(s)}$ satisfies $\alpha\delta\kappa_{(s)}=0$ and $\alpha\kappa_{(s)}=0$, which imply (23.3). By dimensional reason and by easy calculations, $\alpha_r\varepsilon'=0$ for $r\leqq p$, $\alpha'_p\varepsilon'=0$ and $\alpha_r\lambda'=0$ for $r<p$. The element $\bar\varepsilon$ such that $\pi\bar\varepsilon i=\varepsilon'$ satisfies $\bar\varepsilon\alpha^p=0$, and we can take an element $\bar\lambda\in\mathscr{A}_*(\boldsymbol{M})$ so that $\pi\bar\lambda i=\lambda'$ and $\bar\lambda\alpha^{p-1}=0$. Then, (23.4) fol lows from Lemma 22.6.

Let $\bar\varphi=\varphi\wedge 1_{\boldsymbol{M}}\in\mathscr{A}_*(\boldsymbol{M})$ [9; (6.8)]. This element satisfies $\alpha\bar\varphi=0$ [9; (6.14)]. Hence $\alpha_r\varphi=-\pi\alpha^r\bar\varphi i=0$. By [9; §7], there exists an element $\varphi'\in\mathscr{A}_*(\boldsymbol{M}_{p^2})$ such that $\pi_2\varphi' i_2=\varphi$, $\alpha'\delta_2\varphi'=0$ and $\alpha'\varphi'=0$. Then, $\alpha'^r\delta_2\varphi'=0$ for any $r\geqq 1$ by [9; Prop. 4.2], and so $\alpha'_{rp}\varphi=\pi_2\alpha'^r\delta_2\varphi' i_2=0$. Also $\alpha''_{rp^2}\varphi=0$ is proved similarly to the proof of $\alpha'_{rp}\gamma=0$ in Lemma 22.6. Finally, let $\bar{\mu}=\mu\wedge 1_M\in\mathscr{A}_*(M)$. Then $\alpha\bar{\mu}\in\mathscr{A}_{(2p^2+p)q-5}(M)=0$, and hence $\alpha_r\mu=0$. *q.e.d.*

The following relations are proved by H. Toda [18; (5.7)].

(23.6) $$\beta_r\beta_s=(rs/(r+s-1))\beta_1\beta_{r+s-1} \quad \textit{for} \quad r+s\not\equiv 1 \bmod p,$$

$$\beta_r\beta_{sp-r+1}=(r(r-1)/2)\beta_2\beta_{sp-1}.$$

By these relations and (19.1), we have immediately

(23.7) $$\beta_r\kappa_s=(rs/(r+s-1))\beta_1\kappa_{r+s-1} \quad \textit{for} \quad r+s\leqq p-2.$$

PROPOSITION 23.2. *The following relations hold*:

(23.8) $$\beta_r\varepsilon'=0 \quad \textit{for} \quad 2\leqq r\leqq p-1, \quad \beta_r\varepsilon_s=0 \quad \textit{for} \quad 1\leqq r, s\leqq p-1.$$

(23.9) $$\beta_1\varphi=-x\alpha_1\beta_2\beta_{p-1}, \quad \beta_{p-1}\varphi=-3x\alpha_1\beta_1\beta_{2p-2},$$

$$\beta_r\varphi=0 \quad \textit{for} \quad 2\leqq r\leqq p-2.$$

PROOF. The first of (23.9) is the second of (14.2). For the same x, the relation $\bar{\varphi}\beta_{(1)}=-x\alpha\delta\beta_{(2)}\delta\beta_{(p-1)}$ in $\mathscr{A}_*(M)$ holds [9; (6.20)]. Then we have

$$\begin{aligned}\bar{\varphi}\beta_{(p-1)}&\in\bar{\varphi}\{\beta_{(1)},\alpha,\beta_{(p-2)}\}\subset-\{\bar{\varphi}\beta_{(1)},\alpha,\beta_{(p-2)}\}\\&=x\{\alpha\delta\beta_{(2)}\delta\beta_{(p-1)},\alpha,\beta_{(p-2)}\}\\&\supset-x\alpha\delta\beta_{(2)}\delta\{\beta_{(p-1)},\alpha,\beta_{(p-2)}\}\ni-x\alpha\delta\beta_{(2)}\delta\beta_{(2p-3)}\end{aligned}$$

and so $\pi_*i^*\bar{\varphi}\beta_{(p-1)}=-x\pi_*i^*\alpha\delta\beta_{(2)}\delta\beta_{(2p-3)}$. Hence $\beta_{p-1}\varphi=-x\alpha_1\beta_2\beta_{2p-3}=-3x\alpha_1\beta_1\beta_{2p-2}$.

Since ε_{p-1} is a non-zero multiple of β_p, $\beta_r\varepsilon_{p-1}=0$ follows for any $r\geqq 1$ from [18; Th. 5.3]. We can put $\beta_{p-1}\varepsilon_s=x_s\lambda_{s-2}$, $x_s\in Z_p$, for $3\leqq s\leqq p-2$. Then $0=\beta_{p-1}\varepsilon_{p-1}=-x_s\{\lambda_{s-2},p\iota,\alpha_{p-1-s}\}=-x_s\lambda_{p-3}$ and so $x_s=0$. Hence $\beta_{p-1}\varepsilon_s=0$.

The other compositions are clearly trivial, since the groups ${}_p\pi_k(S)$ containing them are zero. *q.e.d.*

By the relations (22.3)–(22.9) and (23.1)–(23.9), we can see that any element of ${}_p\pi_k(S)$, $k<(2p^2+p)q-3$, is a linear combination of the following monomials:

$$\alpha_r,\quad \alpha'_{rp},\quad \alpha''_{rp^2} \qquad \textit{for} \quad r\not\equiv 0 \bmod p,$$

$$\alpha_1^a\beta_1^r\xi \qquad \textit{for} \quad a=0,1,\ r\geqq 0,\ \xi=\iota,\ \beta_s\ (s\geqq 2),\ \beta_2\beta_{p-1}$$

$$\textit{except for} \quad \alpha_1\beta_1^{p-1}\beta_2\beta_{p-1},$$

$\beta_1^r\xi$ *for* $r \geqq 0,\ \xi = \varepsilon',\ \kappa_s,$

$\alpha_1^a\xi$ *for* $a = 0,\ 1,\ \xi = \varepsilon_j,\ \lambda_1,\ \varepsilon_1\varepsilon_i\ (= -\lambda_{i+1})\ (i \leqq p-4)$

except for $\alpha_1\varepsilon_{p-2},\ \alpha_1\varepsilon_{p-1}$ *and* $\alpha_1\varepsilon_1\varepsilon_{p-4}$,

$\varphi,\ \mu,\ \beta_{2p-1}$.

These elements form an additive basis for ${}_p\pi_k(S)$ by Theorems A and C, if we omit the elements $\alpha_1\beta_1^r\ (r \geqq p)$, $\beta_1^r\varepsilon'\ (r \geqq p)$, $\alpha_1^a\beta_1^r\beta_s\ (r \geqq p, s \geqq 2)$ and $\alpha_1\beta_1^r\beta_s$ $(r \geqq 1, p+1 \leqq s \leqq 2p-3)$, which are zero by the relations ([10], [16], [17], [18]):

(23.10) $$\alpha_1\beta_1^p = 0,\quad \beta_1^p\varepsilon' = 0,\quad \beta_1^p\beta_s = 0 \qquad (s \geqq 2),$$
$$\alpha_1\beta_1\beta_s = 0 \qquad (p+1 \leqq s \leqq 2p-3).$$

Thus, we have obtained the following

THEOREM 23.3. *Within the limits of degree less than* $(2p^2+p)q-3$, $q = 2(p-1)$, $p \geqq 5$, *the ring* ${}_p\pi_*(S)$ *is generated by the following elements* (*r satisfies* $r \geqq 1$ *and* $r \not\equiv 0 \bmod p$):

$$\alpha_r\ (r \leqq 2p^2+p-1),\quad \alpha'_{rp}\ (r \leqq 2p-1),\quad \alpha''_{p^2},\quad \alpha''_{2p^2},$$
$$\beta_r\ (r \leqq 2p-1),\quad \varepsilon',\quad \varepsilon_i\ (1 \leqq i \leqq p-1),\quad \varphi,$$
$$\kappa_s\ (1 \leqq s \leqq p-3),\quad \lambda',\quad \lambda_1,\quad \mu,$$

and the ring structure is given by the relations (22.3)–(22.9), (23.1)–(23.10).

We can also obtain several null compositions of degree higher than Theorem 23.3.

PROPOSITION 23.4. *The following relations hold*:

(i) $\beta_s\lambda_i = 0$ *for* $1 \leqq s \leqq p-1$, $\varepsilon_i\varphi = 0$, $\varphi\lambda_i = 0$.

(ii) $\varepsilon'\varphi = 0,\quad \varepsilon'\kappa_s = 0,\quad \varepsilon'\mu = 0,\quad \varphi\kappa_s = 0,\quad \varphi\mu = 0,$

$\kappa_s\kappa_t = 0,\quad \kappa_s\mu = 0.$

(iii) $\varepsilon'\lambda' = 0$ *if* $p \geqq 7$, $\varphi\lambda' = 0$.

(iv) $\varphi^2 = 0,\quad \lambda'^2 = 0,\quad \mu^2 = 0.$

PROOF. (i) By (22.7) and (23.8), $\beta_s\lambda_i = -\beta_s\varepsilon_1\varepsilon_{i-1} = 0$ for $i \geqq 2$. By Theorems 19.9, 21.6 and 22.3, we have $\{\beta_s, \varepsilon_1, \beta_1\}\alpha_1 = 0$, and hence $\beta_s\lambda_1 = -\{\beta_s, \varepsilon_1, \beta_1\}\alpha_1\beta_1^{p-1} = 0$. For $i \geqq 2$, $\varepsilon_i\varphi = -\pi_*i^*\varepsilon\alpha^{i-1}\bar{\varphi} = 0$ and $\lambda_i\varphi = -\pi_*i^*\lambda\alpha^{i-1}\bar{\varphi} = 0$ by

[9; Prop. 6.8].

Since ${}_p\pi_{N-5}(S)={}_p\pi_{N-4}(S)=0$, $N=(2p^2+p)q$, we have $\{\beta_1^p, \alpha_1, \varphi\}=0$ and $\{\varepsilon_1, \varepsilon_{p-2}, \alpha_1\}=0$. Hence $\varphi\lambda_1=-\varepsilon_1\{\beta_1^p, \alpha_1, \varphi\}=0$ and $\varepsilon_1\varphi=-\{\varepsilon_1, \varepsilon_{p-2}, \alpha_1\}\alpha_1=0$.

(ii) Let ξ and η be two of $\varepsilon', \varphi, \kappa$'s and μ. Then, $\xi\alpha_1=0$ by Proposition 23.1 and $\{\xi, \alpha_1, \alpha_1\}\subset{}_p\pi_k(S)=0$, $k=\deg\xi+2q-1$, if $\xi\neq\mu$. Also, $\eta\in\{\alpha_1, \alpha_1, \eta'\}$ for some η', and hence $\xi\eta\in\pm\{\xi, \alpha_1, \alpha_1\}\eta'=0$ if $\xi\neq\mu$.

(iii) $\varepsilon'\lambda'=\{\varepsilon', \alpha_1, \varepsilon'\}\beta_1^p\subset{}_p\pi_{(2p^2+3)q-6}(S)\beta_1^p=0$ ($p\geqq 7$), and also $\varphi\lambda'=-\lambda'\varphi=-(1/3)\varepsilon'\{\beta_1^p, \alpha_1, \varphi\}=0$.

(iv) This follows immediately from the commutativity of the ring $\pi_*(S)$. *q.e.d.*

At the end of this section, we give some relations for $p=5$.

Proposition 23.5. *If $p=5$, then $\{\lambda', \alpha_1, \alpha_1\}=\beta_1^{11}$ and $\lambda'\varepsilon'=\beta_1^{16}$ up to non-zero coefficients, and hence $\beta_1^{21}=0$.*

Remark. This is a slight improvement of the result $\beta_1^{26}=0$ of H. Toda [17].

Proof of Proposition 23.5. Applying Theorem 3.5 for $\gamma=\lambda'$ and using Lemma 19.6, we have the first relation. Then, $\lambda'\varepsilon'=x\beta_1^{16}$, $x\not\equiv 0 \bmod 5$, and $\beta_1^{21}=0$ by $\beta_1^5\varepsilon'=0$. *q.e.d.*

References

[1] J. F. Adams, *On the groups J (X)-IV*, Topology **5** (1966), 21–71.

[2] H. H. Gershenson, *Relationships between the Adams spectral sequence and Toda's calculations of the stable homotopy groups of spheres*, Math. Zeit. **81** (1963), 223–259.

[3] A. Liulevicius, *The factorization of cyclic reduced powers by secondary cohomology operations*, Mem. Amer. Math. Soc. **42**, Providence, 1962.

[4] J. P. May, *The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra*, Dissertation, Princeton University, Princeton, 1964.

[5] O. Nakamura, *On the cohomology of the* mod p *Steenrod algebra*, Bull. Sci. Engrg. Div. Univ. Ryukyus (Math. Nat. Sci.) **18** (1975), 9–58, Naha, Okinawa.

[6] ———, *Some differentials in the* mod 3 *Adams spectral sequence*, Bull. Sci. Engrg. Div. Univ. Ryukyus (Math. Nat. Sci.) **19** (1975), 1–26, Naha, Okinawa.

[7] S. Oka, *Some exact sequences of modules over the Steenrod algebra*, Hiroshima Math. J. **1** (1971), 109–121.

[8] ———, *The stable homotopy groups of spheres I, II*, Hiroshima Math. J. **1** (1971), 305–337; **2** (1972), 99–161.

[9] ———, *On the stable homotopy ring of Moore spaces*, Hiroshima Math. J. **4** (1974), 629–678.

[10] ——— and H. Toda, *Non-triviality of an element in the stable homotopy groups of spheres*, Hiroshima Math. J. **5** (1975), 115–125.

[11] N. Shimada and T. Yamanoshita, *On triviality of the* mod p *Hopf invariant*, Jap. J. Math. **31** (1961), 1–24.

[12] L. Smith, *On realizing complex bordism modules. Applications to the homotopy of spheres*, Amer. J. Math. **92** (1970), 793–856.

[13] P. E. Thomas and R. Zahler, *Generalized higher order cohomology operations and stable homotopy groups of spheres*, to appear in Adv. Math.

[14] H. Toda, *p-Primary components of homotopy groups, I. Exact sequences in Steenrod algebra; II.* mod p *Hopf invariant; III. Stable groups of the sphere; IV. Compositions and toric constructions*, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **31** (1958), 129–142; 143–160; 191–210; **32** (1959), 288–332.

[15] ———, *On iterated suspensions I, II, III*, J. Math. Kyoto Univ. **5** (1965), 87–142; 209–250; **8** (1968), 101–130.

[16] ———, *An important relation in homotopy groups of spheres*, Proc. Japan Acad. **43** (1967), 839–842.

[17] ———, *Extended p-th powers of complexes and applications to homotopy theory*, Proc. Japan Acad. **44** (1968), 198–203.

[18] ———, *Algebra of stable homotopy of Z_p-spaces and applications*, J. Math. Kyoto Univ. **11** (1971), 197–251.

[19] R. Zahler, *Existence of the stable homotopy family* $\{\gamma_t\}$, Bull. Amer. Math. Soc. **79** (1973), 787–789.

Department of Mathematics,
Faculty of Science,
Hiroshima University