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Nonoscillation Generating Delay Terms in Even Order Differential Equations

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1. Introduction

The study of differential equations with time lag is growing increasingly significant due to technological dependence on physical systems with after effects. Mathematically, such systems [1] are governed by some sort of differential equation with an appropriate delay term which in itself may be a variable quantity. The oscillatory behavior of such equations becomes an interesting phenomenon especially when delay is chiefly responsible for causing oscillations. For example, following Teodorick [20], (also see Norkin [12, pp. 4-6]), the equation

$$x''(t)+\frac{r}{m}x'(t)+\frac{k}{m}x(t)+\frac{2p}{\pi am}x(t-\Delta)=0 \tag{1}$$

represents the working of an electric hammer of mass m. A study of this system shows that without the delay term Δ, there will be no vibrations.

Results concerning the oscillatory behavior of a wide variety of retarded equations can be found in [2, 3, 5, 9, 10, 13, 16, 19, 21]. However most of these results are such that the delay term does not play any role at all. But an obvious example such as

$$y''(t)-y(t-\pi)=0 \tag{2}$$

clearly indicates by its solutions $\sin t$ and $\cos t$, that its oscillatory behavior is different from that of the ordinary differential equation

$$y''(t)-y(t)=0 \tag{3}$$

which is nonoscillatory. This difference in the behavior of equations (2) and (3) is clearly due to the delay term π.

Recently Ladas and Lakshmikantham [10] showed that if $p(t)>0$, $p'(t)\leq 0$ and $\tau^2 p(t)\geq 2$, then the bounded solutions of the equation

$$y''(t)-p(t)y(t-\tau)=0 \tag{4}$$

are oscillatory. Taking $p(t)\equiv 1$ and $\tau=0$, equation (4) reduces to equation (3) which we know is nonoscillatory. Ladas, Ladde and Pappadakis in [9, Theorem

3.1] generalized these results to a more general equation

$$y''(t)-\sum_{i=1}^{m} p_i(t)y(g_i(t)) = 0 \tag{5}$$

where

$$g_i(t) \leq t, \qquad i = 1, 2,\ldots, m.$$

Our purpose, here, is to consider a still more general equation

$$y^{(2n)}(t)-\sum_{i=1}^{m} p_i(t)y(g_i(t)) = f(t), \quad (n \geq 1 \text{ an integer}) \tag{6}$$

and accomplish the following:

(a) to find conditions which will ensure that bounded nonoscillatiory solutions of equation (6) approach zero,

(b) to find conditions such that nonscillatory solutions of equation (6) do not approach zero.

In what follows, we call a function $y(t) \in C[t_0, \infty)$, $t_0 > 0$, oscillatory if it has arbitrarily large zeros. Otherwise call it nonoscillatory. All solutions referred to equations (2), (3), (4), (5), and (6) will be, henceforth, continuously extendable solutions on some positive half real axis.

It is assumed and showed by examples that such solutions exist. For more on this see Gustafson [6].

2. Main restlts

The following assumptions are to hold for the rest of this paper (see [9, p. 386]).

(i) $p_i, g_i \in C[[t_0, \infty), R]$, $p_i \geq 0$, $i = 1, 2,\ldots, m$, and for some index i_0, $1 \leq i_0 \leq m$, $p_{i_0}(t) > 0$ for $t \geq t_0$.

(ii) $g_i(t) \leq t$ and $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1, 2,\ldots, m$.

(iii) $g_i'(t) \geq 0$ for $i = 1, 2,\ldots, m$.

(iv) $f(t) \in C[t_0, \infty)$ and nonnegative.

Theorem 1. *Suppose*

$$\limsup_{t\to\infty} \sum_{i=1}^{m} \int_{g^*(t)}^{t} \left[\frac{(g_i(t)-g_i(s))^{2n-1}}{(2n-1)!}\right] p_i(s)ds > 1 \tag{A}$$

where $g^*(t) = \max_{1\leq i\leq m} [g_i(t)]$. *Then bounded nonoscillatory solutions of equation (6) are eventually negative.*

PROOF. Let $y(t)$ be a bounded nonoscillatory solution of equation (6). Then $y(t)$ eventually assumes a constant sign. Suppose, to the contrary, that $y(t)$ is eventually positive. Let T be large enough so that for $t \geq T \geq t_0$, $y(g_i(t)) > 0$ for $i = 1, 2, \ldots, m$. Due to condition (i) and $f(t) \geq 0$, it follows from equation (6) that $y^{(2n)}(t) > 0$. Due to the fact that $y(t)$ is bounded and positive, the conclusion $y^{(2n)}(t) > 0$ implies the following

$$(7) \qquad (-1)^i y^{(i)}(t) \geq 0, \qquad i = 1, 2, \ldots, 2n$$

for sufficiently large t. Without any loss of generality, we can assume that T is large enough so that for $t \geq T$, (7) holds. Let $p, q \geq T$. By generalized mean value theorem we have

$$(8) \qquad y(p) = y(q) + (p-q)y'(q) + \frac{(p-q)^2}{2!} y''(q) + \cdots + \frac{(p-q)^{2n-1}}{(2n-1)!} y^{(2n-1)}(q) + \frac{(p-q)^{2n}}{(2n)!} y^{(2n)}(\beta)$$

where $\beta \in (p, q)$.

Since $y^{(2n)}(\beta) > 0$, we have from (8)

$$(9) \qquad y(p) \geq y(q) + (p-q)y'(q) + \frac{(p-q)^2}{2!} y''(q) + \cdots + \frac{(p-q)^{2n-1}}{(2n-1)!} y^{(2n-1)}(q).$$

Let

$$p = g_i(s), \quad q = g_i(t).$$

Then from (9), we obtain

$$(10) \qquad y(g_i(s)) \geq y(g_i(t)) + (g_i(s) - g_i(t))y'(g_i(t)) + \frac{(g_i(s) - g_i(t))^2}{2!} y''(g_i(t)) + \cdots + \frac{(g_i(s) - g_i(t))^{2n-1}}{(2n-1)!} y^{(2n-1)}(g_i(t))$$

from which

$$(11) \qquad y^{(2n)}(s) = \sum_{i=1}^{m} p_i(s) y(g_i(s)) + f(s) \geq \sum_{i=1}^{m} p_i(s) y(g_i(t)) + \sum_{i=1}^{m} y'(g_i(t)) p_i(s)(g_i(s) - g_i(t)) + \cdots + \cdots + \frac{1}{(2n-1)!} \sum_{i=1}^{m} p_i(s)(g_i(s) - g_i(t))^{2n-1} y^{(2n-1)}(g_i(t)) + f(s).$$

Integrating (11) between $[g^*(t), t]$ we get

$$(12)\qquad y^{(2n-1)}(t)-y^{(2n-1)}(g^*(t)) \geq \sum_{i=1}^{m} y(g_i(t))\int_{g^*(t)}^{t} p_i(s)ds$$

$$+\sum_{i=1}^{m} y'(g_i(t))\int_{g^*(t)}^{t}(g_i(s)-g_i(t))p_i(s)ds$$

$$+\cdots+\frac{1}{(2n-1)!}\sum_{i=1}^{m} y^{(2n-1)}(g_i(t))\,.$$

$$\cdot\int_{g^*(t)}^{t} p_i(s)(g_i(s)-g_i(t))^{2n-1}ds+f(t)(t-g^*(t))\,.$$

(12) now yields

$$(13)\quad y^{(2n-1)}(t) \geq \sum_{i=1}^{m} y(g_i(t))\int_{g^*(t)}^{t} p_i(s)ds+\sum_{i=1}^{m} y'(g_i(t))\int_{g^*(t)}^{t}(g_i(s)-g_i(t))p_i(s)ds$$

$$+\frac{1}{2}\sum_{i=1}^{m} y''(g_i(t))\int_{g^*(t)}^{t}(g_i(s)-g_i(t))^2p_i(s)ds+\cdots$$

$$-\frac{y^{(2n-1)}(g^*(t))}{(2n-1)!}\left[\sum_{i=1}^{m}\int_{g^*(t)}^{t}(g_i(t)-g_i(s))^{2n-1}p_i(s)ds-(2n-1)!\right]$$

$$+f(t)(t-g^*(t))\,.$$

Due to condition (A) of this theorem, conclusion (7) and the fact that $g_i(s)-g_i(t)\leq 0$ (since $s\leq t$), each term to the right of (13) is nonnegative while the left side of (13) is nonpositive. Since $p_{i_0}>0$, at least for one i_0, the right hand side of (13) is positive. This contradiction proves the theorem.

THEOREM 2. *Suppose condition* (A) *of* Theorem 1 *is satisfied. In addition to this suppose*

(B) *there exists* $k>0$ *such that* $\liminf_{t\to\infty}\int_t^{t+k} p_j(t)dt\geq\varepsilon>0$: *for some index* j, $1\leq j\leq m$.

$$(\mathrm{C})\qquad \int^{\infty} f(t)dt<\infty\,.$$

Then bounded nonoscillatory solutions of equation (6) *approach zero as* $t\to\infty$.

PROOF. Let $y(t)$ be a bounded nonoscillatory solution of equation (6). By Theorem 1, $y(t)$ and $y(g(t))$ are negative to the right of some conveniently large t_0. From condition (B)

$$\int_{t_0}^{\infty} p_j(t)dt = \infty . \tag{14}$$

Integrating equation (6) between t_0 and t, we get

$$y^{(2n-1)}(t) - y^{(2n-1)}(t_0) - \sum_{1}^{m} \int_{t_0}^{t} p_j(s)y(g_j(s))ds = \int_{t_0}^{t} f(s)ds , \tag{15}$$

or

$$y^{(2n-1)}(t) - y^{(2n-1)}(t_0) - \int_{t_0}^{t} p_j(s)y(g_j(s))ds \leq \int_{t_0}^{t} f(s)ds . \tag{16}$$

Equation (16) suggests that

$$-\int_{t_0}^{\infty} p_j(s)y(g_j(s))ds < \infty . \tag{17}$$

In fact if

$$-\int_{t_0}^{\infty} p_j(s)y(g_j(s))ds = \infty , \tag{18}$$

then due to $\int_{t_0}^{\infty} f(t)dt < \infty$, it follows from (15) that $y^{(2n-1)} \to -\infty$ as $t \to \infty$. But this will force $y(t)$ to be unbounded, a contradiction. Hence (17) holds. Now we will show that

$$\lim_{t\to\infty} y'(t) = 0 . \tag{19}$$

From (16) and (17), it follows that

$$y^{(2n-1)}(t) \to 0 \qquad \text{as} \quad t \to \infty . \tag{20}$$

Now we shall make use of Kolmogorov's general theorem that, if $|y| \leq M_0$ and $|y^{(2n-1)}| \leq M_{2n-2}$ on $(0, \infty)$, then

$$|y^{(i)}| \leq C_{n,i} \cdot M_0^{1-\frac{i}{2n-1}} M_{2n-1}^{\frac{i}{2n-1}} , \tag{21}$$

where $C_{n,i}$ is a numerical constant depending on n and i and $0 < i < 2n-1$ (see [14, p. 22]). Thus if $y^{(2n-1)}(t) \to 0$ as $t \to \infty$ and $y(t)$ is bounded, then $y^{(i)}(t) \to 0$ $(0 < i < 2n-1)$ as $t \to \infty$. Hence (19) holds.

Thus we have shown that

$$\lim_{t\to\infty} y'(t) = 0 .$$

Now from (14) and (17) we have

(22) $$\liminf_{t\to\infty}(-y(g(t)) = 0\,.$$

The subscript j in (22) is dropped for convenience. For convenience, let $-y(t)=z(t)>0$. If $\lim_{t\to\infty} z(t)\neq 0$, then let

(23) $$\limsup_{t\to\infty} z(t) > r > 0\,.$$

In view of (11), there exists a sequence $\{\beta_v\}$, $v\geq 0$ with the following properties (see, Singh [16], Hammett [7]):

(24) $\lim_{v\to\infty}\beta_v=\infty$, $\beta_v\geq t_1$ for all v, t_1 is the same as above.

(25) For each v, $z(g(\beta_v))>r$.

(26) For each $v\geq 1$, there exist numbers β'_v such that $\beta_{v-1}<\beta'_v<\beta_v$ and $z(g(\beta'_v))<r/2$.

Let α_v be the largest number less than β_v such that $z(g(\alpha_v))=r/2$ and δ_v be the smallest number greater than β_v such that

(27) $$z(g(\delta_v))=\frac{r}{2} \qquad \text{for} \quad v\geq 1\,.$$

Now in the interval $[\alpha_v, \beta_v]$, there exists a ζ_v such that by mean value theorem

(28) $$g'(\zeta_v)z'(g(\zeta_v))=\frac{z(g(\beta_v))-z(g(\alpha_v))}{\beta_v-\alpha_v}$$

or

(29) $$|g'(\zeta_v)z'(g(\zeta_v))|>\frac{r-r/2}{\delta_v-\alpha_v}=\frac{r}{2(\delta_v-\alpha_v)}\,.$$

But $z'(g(\zeta_v))\to 0$ as $g(\zeta_v)\to\infty$. Also $g'(\zeta_v)$ is bounded. Therefore from (29), it follows that

(30) $$\lim_{v\to\infty}(\delta_v-\alpha_v) = \infty\,.$$

Also because of the way α_v and δ_v were chosen

(31) $$z(g(t)) \geq r/2 > 0 \quad \text{on} \quad [\alpha_v, \delta_v]\,.$$

Now from (17), it follows

$$\infty > \int_{t_1}^{\infty} p_j(t)z(g_j(t))dt$$

$$\geq \sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\delta_\nu} p_j(s) z(g_j(s)) ds$$

$$> \frac{r}{2} \sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\delta_\nu} p_j(s) ds = \infty ,$$

due to condition (B) of this theorem. This contradiction shows that $r=0$ and the proof is complete.

EXAMPLE 1. Consider the equation

$$y^{(IV)}(t) - e^{t/2-\pi} y(t-\pi) = e^{-t/2} - e^{-t}, \qquad t > \pi; \tag{32}$$

which has $y = -e^{-t}$ as nonoscillatory negative solution that goes to zero as $t \to \infty$.

Now from condition (A), we have

$$\limsup_{t\to\infty} \frac{1}{6} \int_{g^*(t)}^{t} (g(t)-g(s))^3 p(s) ds \tag{33}$$

$$= \limsup_{t\to\infty} \frac{1}{6} \int_{t-\pi}^{t} (t-s)^3 e^{s/2-\pi} ds > 1 .$$

From condition (B), it follows

$$\liminf_{t\to\infty} \int_{t}^{t+k} e^{s/2-\pi} ds = 2e^{-\pi} \cdot \liminf_{t\to\infty} e^{t/2}(e^k - 1) = \infty, \qquad \text{since} \quad k>0. \tag{34}$$

Also

$$f(t) = e^{-t}(e^{t/2} - 1) \geq 0, \quad \text{which is integrable on} \quad (\pi, \infty)$$

and

$$\int^{\infty} e^{-t}(e^{t/2} - 1) dt < \infty .$$

Thus all the conditions of Theorems 1 and 2 are satisfied.

THEOREM 3. *Suppose condition* (A) *of* Theorem 1 *holds. In addition to this suppose*

$$\lim_{t\to\infty} \int^{t} p(s) ds = \infty \tag{D}$$

and

$$\liminf_{t\to\infty} \frac{\int^{t} f(s) ds}{\int^{t} p(s) ds} \geq \beta > 0 . \tag{E}$$

Then no nonoscillatory solution of equation

$$y^{(2n)}(t)-p(t)y(g_1(t))=f(t) \tag{35}$$

approaches zero.

PROOF. Let $y(t)$ be a nonoscillatory solution of equation (6) which approaches zero. Then $y(t)$ is bounded on $[t_1, \infty)$. By Theorem 1, $y(t)$ is eventually negative. Without any loss we can assume that $y(t)<0$ for $t\geq t_1$. From equation (6) we have

$$y^{(2n-1)}(t)-y^{(2n-1)}(t_1)=\sum_1^m \int_{t_1}^t p_k(s)y_k(g(s))ds+\int_{t_1}^t f(s)ds. \tag{36}$$

Since $y(t)\to 0$ as $t\to\infty$ and is negative, there exists $t_2>t_1$ such that

$$y(g_k(t)) > -\beta/2 \qquad \text{for} \quad t\geq t_2. \tag{37}$$

Replacing t_2 for t_1 in (37), we get

$$y^{(2n-1)}(t)-y^{(2n-1)}(t_2)\geq -\frac{\beta}{2}\int_{t_2}^t p(s)\,ds+\int_{t_2}^t f(s)ds,$$

which yields

$$\frac{y^{(2n-1)}(t)}{\int_{t_2}^t p(s)ds}-\frac{y^{(2n-1)}(t_2)}{\int_{t_2}^t p(s)ds}\geq -\frac{\beta}{2}+\frac{\int_{t_2}^t f(s)ds}{\int_{t_2}^t p(s)ds}. \tag{38}$$

Now as $t\to\infty$, the right hand side of (38) is bounded away from zero due to condition (E). Since

$$\lim_{t\to\infty}\int_{t_2}^t p(s)ds=\infty,$$

it follows from (38) that $y^{(2n-1)}(t)\to\infty$ which in turn forces $y(t)$ to be positive, a contradiction since $y(t)<0$ for $t\geq t_2$. This contradiction proves the theorem.

EXAMPLE 2. Consider the equation

$$y^{(IV)}(t)-y(t-\pi)=r+2\sin t, \qquad r>1. \tag{39}$$

The solution is $y(t)=-r+2\sin t$, a negative nonoscillatory solution which does not approach zero as $t\to\infty$. From condition (E), we have

$$\liminf_{t\to\infty}\frac{\int_{t_1}^t f(s)ds}{\int_{t_1}^t p(s)ds}=\lim_{t\to\infty}\frac{r(t-t_1)+2(\cos t_1-\cos t)}{t-t_1}=r>0 \tag{40}$$

Similarly for condition (A), we have

$$\text{(41)} \qquad \limsup_{t\to\infty} \frac{1}{6}\int_{g^*(t)}^{t} (g(t)-g(s))^3 p(s)\,ds$$

$$\geq \limsup_{t\to\infty} \frac{1}{6}\int_{t-\pi}^{t} (t-s)^3\,ds = \frac{\pi^4}{24} > 1\,.$$

Thus all the conditions of this theorem are satisfied to justify the solution $y(t) = -r + 2\sin t$ of equation (39).

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