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Impact of Delays on Oscillation in General Functional Equations

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1. Introduction

Mathematical models representing physical systems in which the rate of change of the system is related to the past history, make use of differential equations with time lag [1]. The oscillatory behavior of such equations need to be very carefully watched in regard to certain high speed mechanisms which might become unstable due to oscillations caused by delays; see Minorsky [8, p. 518].

The following equation

$$y''(t)-y(t-\pi)=0 \tag{1}$$

points out the difference in the oscillatory nature of (1) and the corresponding ordinary differential equation

$$y''(t)-y(t)=0. \tag{2}$$

Equation (1) has $\sin t$ and $\cos t$ as solutions while (2) is nonoscillatory. The difference between equations

$$y''(t)-y(t-\pi)=0$$

and

$$y''(t)+y(t-\pi)=0 \tag{3}$$

is also interesting. The oscillatory behavior of equation (3) and

$$y''(t)+y(t)=0 \tag{4}$$

is the same regardless of the delay term. In fact a great many oscillatory criteria pertaining to equations of the form

$$y^{(n)}(t)+a(t)y(t)=0 \tag{5}$$

remain valid for the corresponding retarded equation

$$y^{(n)}(t)+a(t)y(t-\tau(t))=0 \tag{6}$$

as long as the delay $\tau(t)$ is bounded. For information in this regard, the reader is referred to [2, 3, 4, 6, 7, 10–11].

Recently G. Ladas, G. Ladde and J. S. Papadakis [7] found the oscillatory criterion for an equation of the type

$$y''(t) - p(t)y(g(t)) = 0 \tag{7}$$

where $p(t) > 0$ on some positive half line. They showed that if

$$\limsup_{t\to\infty} \int_{g(t)}^{t} (g(t) - g(s))p(s)ds > 1 \tag{8}$$

then bounded solutions of equation (7) are oscillatory.

Our purpose, here, is to study a much more general equation

$$y^{(n)}(t) + (-1)^{n+1}p(t)y(g(t)) = f(t). \tag{9}$$

Section (2) of this paper is devoted to finding the oscillatory criteria for equation (9) and thus generalizing the existing results of [7]. Section (3) characterizes solutions of equation (9). There are obvious examples of equations of the type (9) which have bounded solutions satisfying conditions found in theorems (1) and (2). Therefore, the existence of bounded solutions of equation (9) on some positive half line will be assumed throughout this manuscript.

In what follows, the term "solution" applies only to continuously extendable solution (on $[T, \infty)$, $T > 0$) of equations it pertains to.

We call $h(t) \in C[T, \infty)$ oscillatory if $h(t)$ has arbitrarily large zeros. Otherwise $h(t)$ is called nonoscillatory. Among other assumptions:

(i) $f(t),\ p(t),\ g(t) \in C[T, \infty), R],\ p(t) > 0,\ g(t) > 0.$

(ii) $g(t) \le t,\ g(t) \to \infty$ as $t \to \infty$. $g'(t) > 0.$

2. On oscillation

THEOREM 1. *Suppose*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{(g(t) - g(s))^{n-1} p(s)}{(n-1)!} ds \ge 1. \tag{10}$$

Let there exist a function $r(t)$ such that

$$r^{(n)}(t) = f(t),\ r(t) \text{ remains bounded as } t \to \infty, \tag{11}$$

and suppose the function

(12) $$\phi(t) = \int_{g(t)}^{t} p(s)r(g(s))ds$$

is oscillatory.

Then bounded solutions of equation (9) *are oscillatory.*

PROOF. Let $y(t)$ be a bounded solution of equation (9). If $y(t)$ is non-oscillatory then $y(t)$ is eventually of the same sign. Without any loss of generality, let $t_0 > T$ be large enough so that for $t \geq t_0$, $y(t)$ and $y(g(t))$ are both positive. In a manner of Kartsatos [4] (also see [5]), let

(13) $$x(t) = y(t) - r(t).$$

From equation (9), (11) and (13) we have

(14) $$x^{(n)}(t) + (-1)^{n+1} p(t) y(g(t)) = 0.$$

From (11) and (13), $x(t)$ is bounded and since from (14)

$$(-1)^n x^{(n)}(t) > 0 \qquad \text{for} \quad t \geq t_0,$$

there exists a conveniently large $T_0 > t_0$ such that

(15) $$(-1)^i x^{(i)}(t) \geq 0, \qquad i = 1, 2, \ldots, n-1; \quad t \geq T_0.$$

From (13) and (15) $x(t) \geq 0$ eventually since if $x(t_2) < 0$ for some t_2 then $x'(t) \leq 0 => x(t) \to -\infty$ as $t \to \infty$, a contradication. Now by generalized mean value theorem we have

(16) $$x(a) = x(b) + (a-b)x'(b) + \frac{(a-b)^2}{2!} x''(b) + \cdots +$$
$$\frac{(a-b)^{n-1}}{(n-1)!} x^{(n-1)}(b) + \frac{(a-b)^n}{(n!)} x^{(n)}(c),$$

where $c \in (a, b)$.

Let $T_0 < s < t$, then $g(s) \leq g(t)$, since $g'(t) > 0$.
Taking $a \equiv g(s)$ and $b \equiv g(t)$ in (16) and invoking (15) we have

(17) $$x(g(s)) \geq \frac{(g(s) - g(t))^{n-1}}{(n-1)!} x^{(n-1)}(g(t)).$$

Multiplying (17) by $p(s)$ we have

(18) $$p(x(g(s)) \geq (-1)^{n-1} x^{(n-1)}(g(t)) \frac{(g(t) - g(s))^{n-1}}{(n-1)!} p(s)$$

which gives from (14)

$$(19)\quad (-1)^n x^{(n)}(t) = p(s)x(g(s)) \geq (-1)^{n-1} x^{(n-1)}(g(t))\left[\frac{(g(t)-g(s))^{n-1}}{(n-1)!}\right]p(s) + p(s)r(g(s)).$$

Integrating (19) between $g(t)$ and t we have

$$(20)\quad (-1)^n x^{(n-1)}(t) \geq \left[\int_{g(t)}^{t} \frac{(g(t)-g(s))^{n-1}}{(n-1)!} p(s)ds - 1\right] x^{n-1}(g(t))(-1)^{n-1} + \int_{g(t)}^{t} p(s)r(g(s))ds.$$

Choose t_1 large enough so that $t_1 > 0$ and

$$(21)\quad \phi(t_1) > 0.$$

Then from (20) and (21) we have

$$(22)\quad (-1)^n x^{(n-1)}(t_1) \geq (-1)^{n-1} x^{(n-1)}(g(t_1))\left[\int_{g(t_1)}^{t_1} \frac{(g(t)-g(s))^{n-1}}{(n-1)!} p(s) - 1\right] + \phi(t_1).$$

Since $(-1)^n x^{n-1}(t_1) < 0$, (22) yields a contradiction due to condition (10).

The proof is now complete.

The following example clarifies this theorem.

EXAMPLE 1. Consider the equation

$$(23)\quad y''(t) - 8e^{-\pi/2} y(t-\pi/2) = -2e^{-t}\cos t.$$

Let $r(t) = e^{-t}\sin t$, $r(t) \to 0$ as $t \to \infty$

$$r''(t) = -2e^{-t}\cos t.$$

Thus $r(t)$ satisfies Condition (11). Also

$$\begin{aligned}\int_{g(t)}^{t} r(g(s))p(s)ds &= \int_{t-\pi/2}^{t} e^{-\pi/2} \cdot 8e^{-s+\pi/2} \sin\left(s - \frac{\pi}{2}\right) ds \\ &= -8\int_{t-\pi/2}^{t} e^{-s}\cos(s)ds \\ &= 4[e^{-t}(\sin t - \cos t) - e^{-t+\pi/2}(\sin(t-\pi/2) - \cos(t-\pi/2))] \\ &= 4e^{-t}[\sin t - \cos t + e^{\pi/2}\cos t + e^{\pi/2}\sin t] \\ &= 4e^{-t}[\sin t(1+e^{\pi/2}) - \cos t(1-e^{\pi/2})].\end{aligned}$$

Since

$$\tan t = \frac{1-e^{\pi/2}}{1+e^{\pi/2}}$$

has arbitrarily large solutions due to periodicity of $\tan t$, we conclude that

$$\phi(t)=\int_{g(t)}^{t} p(s)r(g(s))ds$$

is oscillatory. Thus $r(t)$ satisfies condition (12). As for condition (10) we have

$$\liminf_{t\to\infty}\int_{t-\pi/2}^{t} 8e^{-\pi/2}(t-s)ds = 8e^{-\pi/2}\frac{\pi^2}{8} > 1.$$

Thus all the conditions of this theorem are satisfied. Hence all bounded solutions of equation (23) are oscillatory. In fact $y=5e^{-t}\sin t$ is a solution of equation (23).

3. On characterization

In this section solutions of equation (9) are characterized. Our next theorem generalizes theorem (2.1) of [7]. Let first $n=2k$.

THEOREM 2. *Suppose*

$$\int^{\infty} g(t)p(t)dt = \infty,\ g'(t) > 0 \tag{24}$$

and the function $r(t)$ of Theorem 1 is such that

$$r^{(i)}(t) \to 0 \qquad as \quad t\to\infty,\ i = 0, 1, 2,\dots, 2k-1 \tag{25}$$

where

$$r^{(i)}(t) \equiv \frac{d^i}{dt^i}(r(t)), \qquad r^{(0)}(t) = r(t)\,.$$

Let $y(t)$ be a nonoscillatory solution of equation (9). Then either

$$|y^{(i)}(t)| \to \infty \quad or \quad |y^{(i)}(t)| \to 0, \quad as\ t\to\infty,\ i = 0, 1, 2, 3,\dots, 2\,k-1.$$

PROOF. Without any loss of generality, suppose that $y(t)$ and $y(g(t))$ are both positive for $t\geq t_2$, where t_2 is conveniently large. Let $x(t)\equiv y(t)-r(t)$. Since

$$r^{(2k)}(t) = f(t),$$

we get from equation (9)

$$x^{(2k)}(t)-p(t)y(g(t)) = 0 \tag{25a}$$

which gives $x^{(2k)}(t)>0$. Thus $x^{(2k-1)}(t)$ is increasing for $t\geq t_2$. By L'Hospital's rule

$$\lim_{t\to\infty}\frac{(2k-1)!x(g(t))}{g^{2k-1}(t)}=\lim_{t\to\infty}x^{(2k-1)}(g(t))=\lim_{t\to\infty}y^{(2k-1)}(g(t)) \tag{26}$$

since (25) holds. Thus $\lim\limits_{t\to\infty}y^{(2k-1)}(g(t))=y^{(2k-1)}(\infty)$ esists since

$$\lim_{t\to\infty}x^{(2k-1)}(g(t)) = x^{(2k-1)}(\infty) \text{ exists.}$$

Now suppose $x^{(2k-1)}(t)>0$. Since $x^{(2k)}(t)>0$, $t\geq t_2$ and $r(t)\to 0$, there exists a $t_3>t_2$ such that

$$y^{(2k-1)}(g(t)) > 0,\ t\geq t_3. \tag{27}$$

Thus

$$\lim_{t\to\infty}\frac{y(g(t))(2k-1)!}{g^{2k-1}(t)}= y^{(2k-1)}(\infty)>0. \tag{28}$$

We will show that $y^{(2k-1)}(\infty)=\infty$. Suppose to the contrary $y^{(2k-1)}(\infty)<\infty$. In a manner of Theorem (2.1) of [7] we have from (25a) for a conveniently large t_3

$$\begin{aligned}x^{(2k-1)}(t) &= x^{(2k-1)}(t_3)+\int_{t_3}^{t}p(s)y(g(s))ds\\ &= x^{(2k-1)}(t_3)+\frac{1}{(2k-1)!}\int_{t_3}^{t}g^{2k-1}(s)p(s)\\ &\qquad\cdot\frac{y(g(s))(2k-1)!}{g^{2k-1}(s)}ds\\ &\geq x^{(2k-1)}(t_3)+\frac{1}{2(2k-1)!}y^{(2k-1)}(\infty)\int_{t_3}^{t}g^{2k-1}(s)p(s)ds\end{aligned}$$

from (28) and choice of t_3. Thus

$$x^{(2k-1)}(t)\geq x^{(2k-1)}(t_3)+\frac{y^{(2k-1)}(\infty)}{2(2k-1)!}\int_{t_3}^{t}g(s)p(s)ds\to\infty \text{ as } t\to\infty,$$

a contradiction. Hence

$$\lim_{n\to\infty}x^{(2k-1)}(t) = \lim_{n\to\infty}y^{(2k-1)}(t) = \infty \tag{29}$$

which in turn implies

$$y^{(i)}(t) \to \infty,\ i = 0, 1, 2, .., 2k-1. \tag{30}$$

Suppose now

$$x^{(2k-1)}(t) < 0,\ t \geq t_4 \geq t_3. \tag{31}$$

From (31), we must have

$$x^{(2k-2)}(t) \geq 0 \tag{32}$$

eventually. In fact if $x^{(2k-2)}(t)<0$ eventually, then $x(t) \to -\infty$ and hence $y(t) \to -\infty$ as $t \to \infty$. Hence (32) holds. Thus there exists $t_5 > t_4$ such that

$$x^{(2k-2)}(t) \geq 0,\ t \geq t_5. \tag{33}$$

It also follows from nonnegative nature of $y(t)$ and the fact $r(t) \to 0$ as $t \to \infty$ that

$$\lim_{t\to\infty} x^{(2k-1)}(t) = 0 = \lim_{t\to\infty} y^{(2k-1)}(t). \tag{34}$$

Now from (25a), in a manner of theorem (2.1) of [7] we have

$$x^{(2k-1)}(t) = x^{(2k-1)}(t_5) + \int_{t_5}^{t} p(s)y(g(s))\,ds \tag{35}$$

which gives

$$x^{(2k-1)}(t_5) = -\int_{t_5}^{\infty} p(t)y(g(t))dt. \tag{36}$$

Again from (35) we have on using (36)

$$\begin{aligned} x^{(2k-2)}(t) &= x^{(2k-2)}(t_5) + (t-t_5)x^{(2k-1)}(t_5) + \int_{t_5}^{t} (t-s)p(s)y(g(s))ds \\ &= x^{(2k-2)}(t_5) + \int_{t_5}^{t} (t_5-s)p(s)y(g(s))ds \\ &\quad -(t-t_5)\int_{t}^{\infty} p(s)y(g(s))ds \\ &\leq x^{(2k-2)}(t_5) + t_5(x^{(2k-1)}(t) - x^{(2k-1)}(t_5)) \\ &\quad -\int_{t_5}^{t} s\,p(s)y(g(s))ds. \end{aligned} \tag{37}$$

Thus

$$x^{(2k-2)}(t) \leq x^{(2k-2)}(t_5) - t_5 x^{(2k-1)}(t_5) - \int_{t_5}^{t} g(s)p(s)y(g(s))ds. \tag{38}$$

Now

$$\lim_{t\to\infty} y(g(s)) = 0 \tag{39}$$

because otherwise the right hand side of (38) tends to $-\infty$ as $t\to\infty$ forcing $x^{(2k-2)}(t)$ to be eventually negative. This contradiction to (33) shows that (39) holds.

From (34) and (39), for arbitrarily small $\varepsilon_1 > 0$, $\varepsilon_2 > 0$

$$y(t) \leq \varepsilon_2,$$

$$|y^{(2k-1)}(t)| \leq \varepsilon_1 \tag{40}$$

for $t \geq t_6 > t_5$, where t_6 is conveniently large.

We now invoke Landau's inequalities between derivatives of bounded functions. See Schoenberg [9]. There exists a constant $k>0$ such that

$$|y^{(i)}(t)| \leq k\left(\varepsilon_2^{1-\frac{i}{2k-1}} \varepsilon_1^{\frac{i}{2k-1}}\right). \tag{41}$$

Conclusion follows from (40) and (41).

A similar argument holds when $y(t)<0$ in the beginning of the proof.

The proof is now complete.

Corollaly 1. *Suppose the conditions of Theorem 1 and Theorem 2 hold. Let $y(t)$ be a nonoscillatory solution of equation (9). Then $|y(t)|\to\infty$ as $t\to\infty$.*

When $n=2k+1$, equation (9) takes the form

$$y^{(2k+1)}(t) + p(t)y(g(t)) = f(t). \tag{42}$$

We shall prove stronger result for (42).

Theorem 3. *Suppose*

$$\int^{\infty} g(t)p(t)dt = \infty, \; g'(t) > 0$$

and there exists a function $r(t)$ such that

$$r^{(2k+1)}(t) = f(t), \; r^{(i)}(t) \to 0 \quad as \quad t\to\infty, \; i = 0, 1, \ldots, 2k.$$

Let $y(t)$ be a nonoscillatory solution of (42). Then

$$|y^{(i)}(t)| \to 0 \qquad as \quad t \to \infty,\ i = 0, 1, 2,\dots, 2k.$$

PROOF. We proceed as in Theorem 2. Let $y(g(t))>0$ for $t \geq t_5$. As before we have

$$x^{(2k+1)}(t) + p(t)y(g(t)) = 0 \tag{43}$$

where $x(t) = y(t) - r(t)$. Thus $x^{(2k+1)}(t) \leq 0$, $t \geq t_5$. Integration yields

$$x^{(2k)}(t) - x^{(2k)}(t_5) + \int_{t_5}^{t} p(s)g(s) \cdot \frac{y(g(s))}{g(s)}\, ds = 0. \tag{44}$$

Since $x^{(2k)}(t) \geq 0$ for large t, (44) yields

$$\lim_{s\to\infty} \frac{y(g(s))}{g(s)} = \lim_{s\to\infty} y'(g(s)) = 0 = \lim_{s\to\infty} x'(g(s)), \tag{45}$$

which in a manner of Theorem 2 implies (by Schoenberg's inequalities)

$$|y^{(i)}(t)| \to 0 \quad \text{as} \quad t \to \infty,\ i = 1, 2,\dots, 2k, \tag{46}$$

since we see that (44) and (45) also yield $\lim_{t\to\infty} x^{(2k)}(t) = \lim_{t\to\infty} y^{(2k)}(t) = 0$. Let $t_6 \geq t_5$ be large enough so that $x^{(2k)}(t) > 0$ for $t \geq t_6$. Multiplying (43) by $g(t)$ and integrating we get

$$x^{(2k)}(t)g(t) - \int_{t_6}^{t} x^{(2k)}(s)g'(s)ds + \int_{t_6}^{t} p(s)g(s)y(g(s))ds = x^{(2k)}(t_6)g(t_6). \tag{47}$$

Now $x^{(2k)}(t)$ is decreasing. Therefore (47) yields

$$x^{(2k)}(t)g(t) - x^{(2k)}(t_6)g(t_6) - \int_{t_6}^{t} x^{(2k)}(g(s))g'(s)ds + \int_{t_6}^{t} p(s)g(s)y(g(s))ds < 0$$

or

$$-x^{(2k)}(t_6)g(t_6) - x^{(2k-1)}(g(t)) + x^{(2k-1)}(g(t_6)) + \int_{t_6}^{t} p(s)g(s)y(g(s))ds < 0. \tag{48}$$

From (46), (48) and condition on $r(t)$, we must have

$$\lim_{t\to\infty} \int_{t_6}^{t} p(s)y(g(s))g(s)ds < \infty$$

$$\Rightarrow y(g(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

COROLLARY 1. *In addition to the hypothesis of Theorem* 3 *suppose* (10), (12) *hold* with $n = 2k+1$. *Then every solution of* (42) *is oscillatory.*

3. Remarks and examples

REMARK 1. The next example shows that it is not possible to weaken condition (12) of Theorem 1 if all other conditions of this theorem are satisfied.

EXAMPLE 2. The equation

$$y^{(iv)}(t)-2e^{t/2-\pi}y(t-\pi) = -e^{-t}+2e^{-t/2} \tag{49}$$

has

$$y(t) = -e^{-t}$$

as a solution. Taking

$$r(t) = -e^{-t}+32e^{-t/2}$$

we find that all conditions of Theorem 1 except condition (12) are satisfied. For condition (12)

$$\begin{aligned}\phi(t) &= e^{-\pi}\int_{t-\pi}^{t} 2e^{-t/2}[-e^{-s+\pi}+32e^{-s/2+\pi/2}]ds \\ &= 2\int_{t-\pi}^{t} e^{-s/2}ds+64e^{-\pi/2}\int_{t-\pi}^{t} ds \\ &= 4(e^{-t}-e^{-(t-\pi)})+64\pi e^{-\pi/2} > 0\end{aligned}$$

eventually.

REMARK 2. This example also satisfies the conditions and hence conclusion of Theorem 2.

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