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On the Boltzmann Equation for Rough Spherical Molecules with Internal Energy

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§1. Introduction, notations and the results

Consider a system of the same kind of a large number of (say, N) rough elastic spherical molecules with the mass m, the radius a, and the moment of inertia $\boldsymbol{I}=ma^2\kappa$ $(0\leq\kappa\leq 2/3)$. Such a molecule possesses energy of rotation which is interconvertible with energy of translation. This model was treated by F. B. Pidduck [5] (see also [3]). The motion of a molecule can be specified by a pair (ξ, α), where $\xi\in R^3$ is the velocity of its center and $\alpha\in R^3$ is its angular velocity. The dynamics of a collision is given in the following way: Let (ξ, α) and (η, β) be the velocity pairs after collision of two molecules (ξ', α') and (η', β'). Then,

$$(1.1)\qquad \begin{cases} \xi' = \xi+\dfrac{\kappa V+(\eta-\xi,\ \ell)\ell}{\kappa+1}, \\[2ex] \eta' = \eta-\dfrac{\kappa V+(\eta-\xi,\ \ell)\ell}{\kappa+1}, \\[2ex] \alpha' = \alpha+\dfrac{\ell\times V}{a(\kappa+1)}, \\[2ex] \beta' = \beta+\dfrac{\ell\times V}{a(\kappa+1)}, \end{cases}$$

where ℓ denotes the unit vector in the direction of the line from the center of the molecule (ξ, α) to that of the molecule (η, β) at collision and V the relative velocity, after impact, of the points of the spheres which come into contact, that is,

$$(1.2)\qquad V=\eta-\xi+a\ \ell\times(\alpha+\beta).$$

(In (1.1) and (1.2), (,) and $\times$ denote the inner and the outer product respectively.) Note that this dynamics preserves the linear momentum and the total energy, i.e.,

$$(1.3)\qquad \begin{cases} \xi'+\eta' = \xi+\eta\,, \\ [\xi',\ \alpha']^2+[\eta',\ \beta']^2 = [\xi,\ \alpha]^2+[\eta,\ \beta]^2, \end{cases}$$

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where $[\xi, \alpha]=\sqrt{\frac{m}{2}|\xi|^2+\frac{I}{2}|\alpha|^2}$, and leaves the volume element invariant;

$$d\xi' d\alpha' d\eta' d\beta' = d\xi d\alpha d\eta d\beta. \tag{1.4}$$

Let $Nu(t, \xi, \alpha)d\xi d\alpha$ be the number of molecules with linear and angular velocities in the ranges $d\xi$, $d\alpha$ respectively at time t. Then the spatially homogeneous Boltzmann equation in the absence of exterior forces takes the form

$$\begin{aligned}\frac{\partial u(t, \xi, \alpha)}{\partial t} = \int_{S^2\times R^3\times R^3} & (\eta-\xi, \ell)^+ \\ & \cdot\{u(t, \xi', \alpha')u(t, \eta', \beta')-u(t, \xi, \alpha)u(t, \eta, \beta)\}d\ell d\eta d\beta,\end{aligned} \tag{1.5}$$

where $(\eta-\xi, \ell)^+=\max\{(\eta-\xi, \ell), 0\}$, $d\ell$ denotes the uniform probability measure on the unit surface S^2 in R^3 (cf. S. Chapman and T. G. Cowling [3]) and the multiplication $4Na^2$ of the right hand part of (1.5) is neglected.

For *non-rotating* models the initial value problem of the Boltzmann equation was solved by T. Carleman [2] for the molecules with elastic spheres, and by A. Ya. Povzner [6] and L. Arkeryd [1] for the molecules with differential collision cross-section satisfying certain boundedness conditions.

In this paper we prove the existence and the uniqueness of the solutions of the equation (1.5). The method is similar to those of the non-rotating case due to A. Ya. Povzner [6] and H. Tanaka [7]. The entropy argument such as in L. Arkeryd [1] would work if we consider only those solutions that are even in α; however, we do not employ this argument since it cannot be expected that there are many even solutions.

The kernel $(\eta-\xi, \ell)^+$ is not invariant under changing ℓ to $-\ell$, so it is convenient to introduce the following notations

$$\begin{cases} \xi'_- = \xi+\dfrac{\kappa V_-+(\eta-\xi, \ell)\ell}{\kappa+1}, \\ \eta'_- = \eta-\dfrac{\kappa V_-+(\eta-\xi, \ell)\ell}{\kappa+1}, \\ \alpha'_- = \alpha-\dfrac{\ell\times V_-}{a(\kappa+1)}, \\ \beta'_- = \beta-\dfrac{\ell\times V_-}{a(\kappa+1)}, \end{cases} \tag{1.6}$$

$$V_- = \eta-\xi-a\,\ell\times(\alpha+\beta), \tag{1.7}$$

and write $x=(\xi, \alpha)$, $y=(\eta, \beta)$, $x'=(\xi', \alpha')$, $y'=(\eta', \beta')$, $x'_-=(\xi'_-, \alpha'_-)$, $y'_-=(\eta'_-, \beta'_-)$, $[x]=[\xi, \alpha]$, $[y]=[\eta, \beta]$, and so on. Note that the relations similar to

(1.3) also hold;

$$(1.3)' \quad \begin{cases} \xi'_- + \eta'_- = \xi + \eta, \\ [x'_-]^2 + [y'_-]^2 = [x]^2 + [y]^2 . \end{cases}$$

Then, making use of (1.4) and (1.3)′ we can easily transfer the equation (1.5) to the equations for measures;

$$(1.8) \quad \frac{\partial u(t, \Gamma)}{\partial t} = \int_{S^2 \times R^6 \times R^6} (\eta - \xi, \ell)^+ \{\delta(x'_-, \Gamma) - \delta(x, \Gamma)\} d\ell u(t, dx) u(t, dy),$$

or equivalently,

$$(1.9) \quad \frac{\partial u(t, \Gamma)}{\partial t} = \int_{S^2 \times R^6 \times R^6})\eta - \xi, \ell)^+ \cdot \left\{ \frac{\delta(x'_-, \Gamma) + \delta(y'_-, \Gamma)}{2} - \frac{\delta(x, \Gamma) + \delta(y, \Gamma)}{2} \right\} d\ell u(t, dx) u(t, dy),$$

where $u(t, \Gamma) = \int_\Gamma u(t, x) dx$ for $\Gamma \in \mathscr{B}(R^6)$. For the solution u of the equation (1.8) (=(1.9)) with the given initial probability distribution f, we use the terminology "*u preserves the mass, the linear momentum, the total energy*", when

$$\int_{R^6} \psi(x) u(t, dx) = \int_{R^6} \psi(x) f(dx)$$

for $\psi(x) = 1, \xi, [x]^2$ respectively.

Now we can state the results.

THEOREM 1. *Assume that* $\sigma^2 = \int_{R^6} [x]^2 f(dx) < \infty$ *for the probability distribution* f *on* R^6. *Then there exists a solution* u *of the equation* (1.8) *with the initial distribution* f *such that* u *preserves the mass and the linear momentum.*

THEOREM 2. *Assume that* $\mu^{(p)} = \int_{R^6} [x]^p f(dx) < \infty$ *for the probability distribution* f *on* R^6 *for some* $p \geq 3$. *Then there exists a solution* u *of the equation* (1.8) *with the initial distribution* f *such that*

(1.10) $\mu^{(p)}(t) = \int_{R^6} [x]^p u(t, dx)$ *is a locally bounded function of* t,

(1.11) u *preserves the mass, the linear momentum, and the total energy.*

THEOREM 3. *If the assumption for* f *in* Theorem 2 *is valid with* $p=4$, *then the solution* u *of the equation* (1.8) *with the initial distribution* f *satisfying* (1.10) *and* (1.11) *is unique, and has a density whenever* f *does. Moreover, if we put*

$$
(1.12)\qquad \begin{cases} \tilde{f}(dx) = \dfrac{1+[x]^2}{1+\sigma^2} f(dx)\,, \\[2ex] \tilde{u}(t, dx) = \dfrac{1+[x]^2}{1+\sigma^2} u(t, dx)\,, \end{cases}
$$

then $\tilde{u}$ *is the unique solution of the following equation* (1.13) *with the initial distribution* $\tilde{f}$:

$$
(1.13)\qquad \frac{\partial \tilde{u}(t, \Gamma)}{\partial t}
$$

$$
= (1+\sigma^2) \int_{R^6 \times R^6} (1+[x]^2)\,\{\tilde{\Pi}(x, y, \Gamma) - \delta(x, \Gamma)\}\tilde{u}(t, dx)\tilde{u}(t, dy)\,,
$$

where

$$
(1.14)\qquad \begin{cases} \tilde{\Pi}(x, y, \Gamma) = \dfrac{1}{1+\dfrac{[x]^2+[y]^2}{2}} \displaystyle\int_\Gamma (1+[z]^2)\Pi(x, y, dz)\,, \\[3ex] \Pi(x, y, \Gamma) = \dfrac{\delta(x, \Gamma)+\delta(y, \Gamma)}{2} + \dfrac{1}{(1+[x]^2)(1+[y]^2)} \end{cases}
$$

$$
\cdot \int_{S^2} (\eta - \xi, \ell)^+ \left\{ \frac{\delta(x'_-, \Gamma)+\delta(y'_-, \Gamma)}{2} - \frac{\delta(x, \Gamma)+\delta(y, \Gamma)}{2} \right\} d\ell\,.
$$

In § 2 we prove the existence of solutions of the equation (1.8) with the replacement of $(\eta-\xi, \ell)^+$ by $(\eta-\xi, \ell)^+ \wedge N$, and prove the local boundedness of the corresponding moments $\mu^{(p)}(t)$ of this u. In § 3 we demonstrate the above theorems. Finally, in § 4 we mention some remarks on steady state solutions which are even in α.

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§ 2. Preliminary lemmas

In this section, we consider the following equation.

$$
(2.1)\qquad \frac{\partial u(t, \Gamma)}{\partial t} = \int_{S^2 \times R^6 \times R^6} q_N(x, y, \ell)
$$

$$
\cdot \left\{ \frac{\delta(x'_-, \Gamma)+\delta(y'_-, \Gamma)}{2} - \frac{\delta(x, \Gamma)+\delta(y, \Gamma)}{2} \right\} d\ell\, u(t, dx)u(t, dy),
$$

$$
q_N(x, y, \ell) = (\eta - \xi, \ell)^+ \wedge N.
$$

If the initial distribution is f, the equation is easily written in the integral form:

$$(2.2)\qquad u(t, \Gamma) = e^{-Nt}f(\Gamma)+N\int_0^t e^{-N(t-s)} \int_{R^6\times R^6} \Pi_N(x, y, \Gamma)u(s, dx)u(s, dy)ds,$$

using the probability measure $\Pi_N(x, y, \cdot)$ defined by

$$(2.3)\qquad \Pi_N(x, y, \Gamma) = \frac{\delta(x, \Gamma)+\delta(y, \Gamma)}{2}+\frac{1}{N}\int_{S^2} q_N(x, y, \ell) \cdot\left\{\frac{\delta(x'_-, \Gamma)+\delta(y'_-, \Gamma)}{2}-\frac{\delta(x, \Gamma)+\delta(y, \Gamma)}{2}\right\}d\ell .$$

LEMMA 1. *For any probability distribution f on R^6, there exists a unique probability solution u of the equation* (2.2) *with initial distribution f.*

PROOF. Define $u^n(t, \Gamma)$, $n=0, 1, 2,\dots$, $t\geq 0$, $\Gamma\in\mathscr{B}(R^6)$ successively as follows:

$$(2.4)\qquad \begin{cases} u^0(t, \Gamma) = e^{-Nt}f(\Gamma), \\ u^{n+1}(t, \Gamma) = e^{-Nt}f(\Gamma)+N\int_0^t e^{-N(t-s)} \end{cases}$$

$$\cdot\int_{R^6\times R^6} \Pi_N(x, y, \Gamma)u^n(s, dx)u^n(s, dy)ds, \qquad n = 0, 1, 2,\dots.$$

Then we can easily prove that $u^n(t, \cdot)$ increases, as $n\to\infty$, to a solution $u(t, \cdot)$ of the equation (2.2). The proof that $u(t, \cdot)$ is a unique probability solution of the equation (2.2) is also routine.

Next we estimate the moment of the solutions of the equation (2.2). For this purpose we first consider the moment of the approximate solution $u^n(t, \cdot)$. We put

$$\mu^{(p)} = \int_{R^6}[x]^p f(dx), \qquad \mu_n^{(p)}(t) = \int_{R^6}[x]^p u^n(t, dx),$$

and assume that $\mu^{(p)}<\infty$ for some $p\geq 3$. Since

$$\int_{R^6}[z]^2\Pi_N(x, y, dz)=\frac{[x]^2+[y]^2}{2}$$

by (1.3)′, we have

$$\int_{R^6}[z]^2\int_{R^6\times R^6} \Pi_N(x, y, dz)u^n(s, dx)u^n(s, dy) \leq \mu_n^{(2)}(s).$$

Using (2.4), we have

$$\mu_{n+1}^{(2)}(t) \le e^{-Nt}\sigma^2 + N\int_0^t e^{-N(t-s)}\mu_n^{(2)}(s)ds, \qquad n = 0, 1, 2,\dots,$$

and $\mu_0^{(2)}(t)=e^{-Nt}\sigma^2$. Therefore $\mu_n^{(2)}(t)\le\sigma^2$ for all n. Next we deal with $\mu_n^{(p)}(t)$ for general $p\ge 3$. By (2.4) we have

$$(2.5)\qquad \mu_{n+1}^{(p)}(t)=e^{-Nt}\mu^{(p)}+N\int_0^t e^{-N(t-s)}\int_{R^6\times R^6}\Big[\frac{[x]^p+[y]^p}{2}$$
$$+\frac{1}{N}\int_{S^2} q_N(x, y, \ell)\Big\{\frac{[x'_-]^p+[y'_-]^p}{2}-\frac{[x]^p+[y]^p}{2}\Big\}d\ell\Big]$$
$$\cdot u^n(s, dx)u^n(s, dy)ds\,.$$

LEMMA 2. *There exists a constant* C_p *such that*

$$[x'_-]^p+[y'_-]^p-([x]^p+[y]^p) \le C_p([x]^{p-2}[y]^2+[x]^2[y]^{p-2})$$

for all $x=(\xi, \alpha)$, $y=(\eta, \beta)\in R^6$.

PROOF. The following elementary inequality holds

$$0\le (a^2+b^2)^{\frac{p}{2}}-(a^p+b^p)\le C_p(a^{p-2}b^2+a^2b^{p-2})$$

for $p\ge 2$, a, $b\ge 0$. Thus we have

$$\begin{aligned}
&[x'_-]^p+[y'_-]^p-([x]^p+[y]^p)\\
&\le [x'_-]^p+[y'_-]^p-([x]^2+[y]^2)^{\frac{p}{2}}+C_p([x]^{p-2}[y]^2+[x]^2[y]^{p-2})\\
&\le ([x'_-]^2+[y'_-]^2)^{\frac{p}{2}}-([x]^2+[y]^2)^{\frac{p}{2}}+C_p([x]^{p-2}[y]^2+[x]^2[y]^{p-2})\\
&= C_p([x]^{p-2}[y]^2+[x]^2[y]^{p-2});
\end{aligned}$$

in deriving the last equality we have used (1.3)′.

Applying this lemma to (2.5), we have

$$(2.6)\qquad \mu_{n+1}^{(p)}(t)\le e^{-Nt}\mu^{(p)}+N\int_0^t e^{-N(t-s)}\{\mu_n^{(p)}(s)+C_p\sigma^2\mu_n^{(p-2)}(s)\}ds\,.$$

LEMMA 3. $\mu_n^{(p)}(t)$ *is a locally bounded function of* t, *that is, for any* $T\in(0, \infty)$ *there exists a constant* K *depending only on* σ, $\mu^{(p)}$, T, N *and* p *such that*

$$(2.7)\qquad \mu_n^{(p)}(t)\le K, \qquad 0\le t\le T.$$

PROOF. Let us prove this by induction on p. In the case $1\le p-2\le 2$, we have $\mu_n^{(p-2)}(s)\le\sigma^{p-2}$ by Hölder's inequality. Combining this with (2.6), we have

$$\mu_{n+1}^{(p)}(t) \leq e^{-Nt}\mu^{(p)} + N\int_0^t e^{-N(t-s)}\{\mu_n^{(p)}(s) + C_p\sigma^p\}ds,$$

and hence by induction in n we obtain

$$\mu_n^{(p)}(t) \leq \mu^{(p)} + C_p\sigma^p Nt.$$

Especially $\mu_n^{(4)}(t) \leq \mu^{(4)} + C_4\sigma^4 Nt$, and so in the case $2 < p-2 \leq 4$, using Hölder's inequality for $\mu_n^{(p-2)}(s)$ in (2.6), we have

$$\mu_{n+1}^{(p)}(t) \leq e^{-Nt}\mu^{(p)} + N\int_0^t e^{-N(t-s)}$$
$$\cdot[\mu_n^{(p)}(s) + C_p\sigma^2\{(\mu^{(4)})^{\frac{p-2}{4}} + (C_4\sigma^4)^{\frac{p-2}{4}}(1+NS)\}]ds.$$

From this we have $\mu_n^{(p)}(t) \leq \text{const}(1+N^2t^2)$, $n=0, 1, 2,\dots$ This method is successively applicable to the case $2k < p-2 \leq 2k+2$, $k \geq 1$; the result is $\mu_n^{(p)}(t) \leq \text{const}(1+N^kt^k)$, $n=0, 1, 2,\dots$. Thus the lemma is proved.

LEMMA 4. *Let u be the solution of the equation* (2.1) *with initial probability distribution f satisfying $\mu^{(p)} < \infty$ for some $p \geq 3$, and put $\mu^{(p)}(t) = \int_{R^6}[x]^p u(t, dx)$. Then for any $T \in (0, \infty)$ there exists a constant K depending only on σ, $\mu^{(p)}$, p and T such that*

$$\mu^{(p)}(t) \leq K, \qquad 0 \leq t \leq T.$$

PROOF. Integrating the equation (2.1), we have

$$u(t, \Gamma) = f(\Gamma) + \int_0^t \int_{S^2\times R^6\times R^6} q_N(x, y, \ell)$$
$$\cdot\left\{\frac{\delta(x'_-, \Gamma) + \delta(y'_-, \Gamma)}{2} - \frac{\delta(x, \Gamma) + \delta(y, \Gamma)}{2}\right\}$$
$$\cdot d\ell u(s, dx)u(s, dy)ds.$$

Since $\mu^{(p)}(t)$ satisfies the same estimate as in (2.7), using Lemma 2 we have

$$\mu^{(p)}(t) = \mu^{(p)} + \int_0^t \int_{S^2\times R^6\times R^6} q_N(x, y, \ell)$$
$$\cdot\left\{\frac{[x'_-]^p + [y'_-]^p}{2} - \frac{[x]^p + [y]^p}{2}\right\}d\ell u(s, dx)u(s, dy)ds$$

$$\leq \mu^{(p)} + \int_0^t \int_{S^2\times R^6\times R^6} q_N(x, y, \ell) \tag{2.8}$$

$$\cdot C_p \frac{[x]^{p-2}[y]^2+[x]^2[y]^{p-2}}{2} d\ell u(s, dx)u(s, dy)ds$$

$$\leq \mu^{(p)} + C_p\sqrt{\frac{2}{m}}\int_0^t \int_{R^6\times R^6}([x]+[y])$$

$$\cdot[x]^{p-2}[y]^2 u(s, dx)u(s, dy)ds\,.$$

For the case $p=3$, this inequality (2.8) implies that

$$\mu^{(3)}(t) \leq \left(\mu^{(3)} + C_3\sqrt{\frac{2}{m}}\sigma^4 t\right)\exp\left(C_3\sqrt{\frac{2}{m}}\sigma t\right).$$

In the case $p>3$, using Hölder's inequality we have

$$\mu^{(p-k)}(s) \leq (\mu^{(p)}(s))^{\frac{p-k}{p}} \leq 1+\mu^{(p)}(s), \qquad k=1, 2,$$

and then by (2.8)

$$\mu^{(p)}(t) \leq \mu^{(p)} + C_p\sqrt{\frac{2}{m}}\int_0^t\left\{\sigma^2 + \left(\mu^{(3)} + C_3\sqrt{\frac{2}{m}}\sigma^4 s\right)\right.$$

$$\left.\cdot\exp\left(C_3\sqrt{\frac{2}{m}}\sigma s\right)\right\}(1+\mu^{(p)}(s))ds\,.$$

Now an application of Gronwall's inequality completes the proof of the lemma.

§3. Proof of the theorems

Let $u_N(t, \cdot)$ be the unique solution of the equation (2.2) constructed in Lemma 1. If the initial distribution f has the finite second moment σ^2, then the following (3.1)~(3.3) hold.

(3.1) $u_N(t, \cdot)$ is a probability measure on R^6.

(3.2) $\int_{R^6}[x]^2 u_N(t, dx)=\sigma^2$, $\int_{R^6}\xi\; u_N(t, dx)=\int_{R^6}\xi\; f(dx)$.

(3.3) There exists a constant K, independent of N, such that

$$\left|\int_{R^6}\phi(x)u_N(t, dx)-\int_{R^6}\phi(x)u_N(s, dx)\right| \leq K\cdot\|\phi\|\cdot|t-s|$$

for any bounded continuous function ϕ on R^6.

Thus the proof of Theorem 1 is now evident by the relative compactness of $\{u_N(t, \cdot)\}_{N=1,2,\dots}$ and the equicontinuity (3.3), that is, there exists a subsequence $\{N_k\}_{k=1,2,\dots}$ such that $\{u_{N_k}(t, \cdot)\}$ converges to a probability measure $u(t, \cdot)$ as

$k\to\infty$ for all rational $t\geq 0$, and hence for all $t\geq 0$ by (3.3). Moreover $u(t,\cdot)$ satisfies (1.8).

This construction of the solution $u(t,\cdot)$ of (1.8) is also useful for the proof of Theorem 2, because the local boundedness of the moment of $u(t,\cdot)$ is already proved in Lemma 4.

PROOF OF THEOREM 3. The existence part is contained in Theorem 2, so we prove the uniqueness and the existence of density solutions. Let $\Pi(x, y, \Gamma)$ and $\tilde{\Pi}(x, y, \Gamma)$ be as in (1.14). Then by

$$\int_{R^6}(1+[z]^2)\Pi(x, y, dz) = 1+\frac{[x]^2+[y]^2}{2},$$

$\tilde{\Pi}(x, y, \Gamma)$ is also a probability measure in $\Gamma\in\mathscr{B}(R^6)$ for each $x, y\in R^6$. On the other hand, for a solution u of the equation (1.8) we have

$$\begin{aligned}
&\int_\Gamma(1+[z]^2)\int_{S^2\times R^6\times R^6}(\eta-\xi, \ell)^+ \\
&\qquad\cdot\left\{\frac{\delta(x_\ell', dz)+\delta(y_\ell', dz)}{2}-\frac{\delta(x, dz)+\delta(y, dz)}{2}\right\}d\ell u(t, dx)u(t, dy) \\
&=\int_\Gamma(1+[z]^2)\int_{R^6\times R^6}(1+[x]^2)(1+[y]^2) \\
&\qquad\cdot\left\{\Pi(x, y, dz)-\frac{\delta(x, dz)+\delta(y, dz)}{2}\right\}u(t, dx)u(t, dy) \\
&=\int_{R^6\times R^6}(1+[x]^2)(1+[y]^2)\left\{\left(1+\frac{[x]^2+[y]^2}{2}\right)\tilde{\Pi}(x, y, \Gamma)\right. \\
&\qquad\left.-\frac{(1+[x]^2)\delta(x, \Gamma)+(1+[y]^2)\delta(y, \Gamma)}{2}\right\}u(t, dx)u(t, dy) \\
&=\int_{R^6\times R^6}(1+[x]^2)\{\tilde{\Pi}(x, y, \Gamma)-\delta(x, \Gamma)\}(1+[x]^2)u(t, dx)(1+[y]^2)u(t, dy) \\
&=(1+\sigma^2)^2\int_{R^6\times R^6}(1+[x]^2)\{\tilde{\Pi}(x, y, \Gamma)-\delta(x, \Gamma)\}\tilde{u}(t, dx)\tilde{u}(t, dy),
\end{aligned}$$

where $\tilde{u}(t, dx)$ is defined in (1.12). Hence $\tilde{u}$ is a solution of (1.13). $\tilde{u}(t,\cdot)$ is still a probability measure on R^6, so we obtain

$$\begin{aligned}
\frac{\partial\tilde{u}(t, \Gamma)}{\partial t} &= (1+\sigma^2)\int_{R^6\times R^6}(1+[x]^2)\tilde{\Pi}(x, y, \Gamma)\tilde{u}(t, dx)\tilde{u}(t, dy) \\
&\quad -(1+\sigma^2)\int_\Gamma(1+[x]^2)\tilde{u}(t, dx).
\end{aligned}$$

Therefore we can write

$$(3.4)\qquad \tilde{u}(t,\Gamma) = \int_\Gamma e^{-\tilde{q}(x)t}\tilde{f}(dx) + \int_0^t \int_{R^6\times R^6} \tilde{u}(s,dx)\tilde{u}(s,dy)\tilde{q}(x)\int_\Gamma e^{-\tilde{q}(z)(t-s)}\tilde{\Pi}(x,y,dz)ds,$$

where $\tilde{q}(x)=(1+\sigma^2)(1+[x]^2)$.

On the other hand we can easily construct the minimum solution $\tilde{u}_0(t,\cdot)$ of the equation (3.4) by iteration. If we take

$$\tilde{v}(t,\Gamma) = \tilde{u}(t,\Gamma)-\tilde{u}_0(t,\Gamma), \qquad \Gamma\in\mathscr{B}(R^6),$$

then $\tilde{v}(t,\Gamma)\geq 0$ and

$$\begin{cases} \dfrac{d\tilde{v}(t,R^6)}{dt} = \tilde{\mu}(t)\tilde{v}(t,R^6),\ \tilde{\mu}(t) = \displaystyle\int_{R^6}\tilde{q}(x)\tilde{u}_0(t,dx), \\ \tilde{v}(0,R^6) = 0. \end{cases}$$

However $\tilde{\mu}(t)\leq\int_{R^6}\tilde{q}(x)\tilde{u}(t,dx)\leq \text{const}\,(1+\mu^{(4)}(t))$ is locally bounded in t, so $\tilde{v}(t,\cdot)$ vanishes identically. Therefore the uniqueness of the solutions of (1.8) is proved. Finally assume that f and hence $\tilde{f}$ have densities with respect to the Lebesgue measure in R^6. Then the usual method of iteration for solving the equation (3.4) shows that the solution $\tilde{u}_0(t,\cdot)=\tilde{u}(t,\cdot)$ has a density for each $t\geq 0$.

§4. Remarks on the steady state

Let $u(t,\xi,\alpha)$ be a (density) solution of the equation (1.5), and put

$$H(t) = \int_{R^3\times R^3} u(t,\xi,\alpha)\log u(t,\xi,\alpha)d\xi d\alpha.$$

Then we have (at least formally)

$$\begin{aligned} \frac{d}{dt}H(t) &= \int_{R^3\times R^3} \frac{\partial u(t,\xi,\alpha)}{\partial t}\{\log u(t,\xi,\alpha)+1\}d\xi d\alpha \\ &= -\frac{1}{4}\int\{(\eta-\xi,\ell)^-\log u(t,\xi',\alpha')u(t,\eta',\beta') \\ &\qquad -(\eta-\xi,\ell)^+\log u(t,\xi,\alpha)u(t,\eta,\beta)\} \\ &\qquad \cdot\{u(t,\xi',\alpha')u(t,\eta',\beta')-u(t,\xi,\alpha)u(t,\eta,\beta)\}d\ell d\xi d\alpha d\eta d\beta \\ &\quad +\int(\eta-\xi,\ell)^+\{u(t,\xi',\alpha')u(t,\eta',\beta')-u(t,\xi,\alpha)u(t,\eta,\beta)\}d\ell d\xi d\alpha d\eta d\beta. \end{aligned}$$

In general, the non-increasing property of $H(t)$ does not follow from this formula. But if we assume that $u(t, \xi, \alpha)$ is even in α, then by considering the collision of two molecules $(\xi', -\alpha')$ and $(\eta', -\beta')$ with the unit vector $-\ell$ which produces $(\xi, -\alpha)$ and $(\eta, -\beta)$, we obtain the following inequality (see also [3, 1st ed.]).

$$\begin{aligned}\frac{d}{dt} H(t) &= -\frac{1}{4}\int (\eta-\xi, \ell)^+ \log \frac{u(t, \xi', \alpha')u(t, \eta', \beta')}{u(t, \xi, \alpha)u(t, \eta, \beta)} \\ &\quad \cdot \{u(t, \xi', \alpha')u(t, \eta', \beta') - u(t, \xi, \alpha)u(t, \eta, \beta)\} d\ell d\xi d\alpha d\eta d\beta \\ &\leq 0.\end{aligned}$$

We next consider a probability density function $f = f(\xi, \alpha)$ in the steady state:

$$\int_{S^2 \times R^3 \times R^3} (\eta-\xi, \ell)^+ \{f(\xi', \alpha')f(\eta', \beta') - f(\xi, \alpha)f(\eta, \beta)\} d\ell d\eta d\beta = 0.$$

We only consider those functions $f(\xi, \alpha)$ which are even in α. Assuming that $[\xi, \alpha]^2 f(\xi, \alpha) \log f(\xi, \alpha) \in L^1(R^6)$, we have

$$\begin{aligned}&-\frac{1}{4}\int (\eta-\xi, \ell)^+ \log \frac{f(\xi', \alpha')f(\eta', \beta')}{f(\xi, \alpha)f(\eta, \beta)} \\ &\qquad \cdot \{f(\xi', \alpha')f(\eta', \beta') - f(\xi, \alpha)f(\eta, \beta)\} d\ell d\xi d\alpha d\eta d\beta = 0,\end{aligned}$$

and hence we have

(4.1) $f(\xi', \alpha')f(\eta', \beta') = f(\xi, \alpha)f(\eta, \beta)$ for almost all $\xi, \alpha, \eta, \beta \in R^3$, $\ell \in S^2$.

The purpose of this section is to show that an even f satisfying the relation (4.1) is Maxwellian (Theorem 4). This fact is found in [3, 1st ed.]; the proof we give here is a probabilistic one based upon the function $\mathfrak{e}$ of [4].

THEOREM 4. *Let $f = f(\xi, \alpha)$ be a probability density function on R^6, even in α, and satisfy the relation* (4.1). *Then there exist a constant vector $b \in R^3$ and a positive constant c such that*

$$f(\xi, \alpha) = \text{const} \exp\{(b, \xi) - c[\xi, \alpha]^2\}.$$

The proof is divided into some steps. Before stating these lemmas let us introduce the change of α-scale:

$$\begin{pmatrix} \xi \\ a\sqrt{\kappa}\,\alpha \end{pmatrix} \longrightarrow \begin{pmatrix} \xi \\ \alpha \end{pmatrix}.$$

Then the dynamics of a collision is transformed to the following form:

$$(4.2)\quad \begin{cases} \xi' = \xi + \dfrac{\kappa V + (\eta-\xi,\ \ell)\ell}{\kappa+1}, & \alpha' = \alpha + \dfrac{\sqrt{\kappa}\ \ell \times V}{\kappa+1}, \\ \eta' = \eta - \dfrac{\kappa V + (\eta-\xi,\ \ell)\ell}{\kappa+1}, & \beta' = \beta + \dfrac{\sqrt{\kappa}\ \ell \times V}{\kappa+1}, \end{cases} .$$

$$(4.3)\quad V = \eta - \xi + \frac{1}{\sqrt{\kappa}} \ell \times (\alpha+\beta),$$

and the total energy of a molecule (ξ, α) is changed to $\frac{m}{2}(|\xi|^2 + |\alpha|^2)$. For each fixed $\ell = (\ell_1, \ell_2, \ell_3) \in S^2$, the relation (4.2) is written in short by

$$(4.4)\quad \begin{bmatrix} \xi' \\ \alpha' \\ \eta' \\ \beta' \end{bmatrix} = U \begin{bmatrix} \xi \\ \alpha \\ \eta \\ \beta \end{bmatrix} .$$

The invariance of the total energy then implies that U is an orthogonal matrix. By elementary calculations, we obtain the following

LEMMA 5.

$$(4.5)\quad U = \begin{bmatrix} I-M & N & M & N \\ -N & M & N & -(I-M) \\ M & -N & I-M & -N \\ -N & -(I-M) & N & M \end{bmatrix},$$

where

$$(4.6)\quad \begin{cases} M = \dfrac{\kappa}{\kappa+1} I + \dfrac{1}{\kappa+1} \begin{bmatrix} \ell_1^2 & \ell_1\ell_2 & \ell_1\ell_3 \\ \ell_1\ell_2 & \ell_2^2 & \ell_2\ell_3 \\ \ell_1\ell_3 & \ell_2\ell_3 & \ell_3^2 \end{bmatrix}, \\ N = \dfrac{\sqrt{\kappa}}{\kappa+1} \begin{bmatrix} 0 & -\ell_3 & \ell_2 \\ \ell_3 & 0 & -\ell_1 \\ -\ell_2 & \ell_1 & 0 \end{bmatrix}. \end{cases}$$

Moreover,

$$
(4.7)\qquad \begin{cases} M^2 = -\dfrac{\kappa}{\kappa+1}I + \dfrac{2\kappa+1}{\kappa+1}M, \\[2ex] N^2 = -\dfrac{\kappa}{\kappa+1}I + \dfrac{\kappa}{\kappa+1}M, \\[2ex] MN = NM = \dfrac{\kappa}{\kappa+1}N. \end{cases}
$$

The following lemma is a key part in the proof of Theorem 4.

LEMMA 6. *If a probability density function f in R^6 satisfies the assumption of* Theorem 4, *then there exist a constant vector $b \in R^3$ and symmetric 3×3-matrices Q_1 and Q_2 such that*

$$
(4.8)\qquad f(\xi, \alpha) = \text{const}\exp\{(b, \xi)-(Q_1\xi, \xi)-(Q_2\alpha, \alpha)\}.
$$

PROOF. The function $\tilde{f}(\xi, \alpha)=f(\xi, \alpha)\exp\{-|\xi|^2-|\alpha|^2\}$ satisfies the relation (4.1) and $(1+|\xi|^2+|\alpha|^2)\tilde{f}\in L^1(R^6)$. To prove the representation (4.8) for f, it is sufficient to prove it for $\tilde{f}$ (so normalized that $\int_{R^6}\tilde{f}(\xi, \alpha)d\xi d\alpha=1$), and therefore we may assume from the first that f itself has finite second moment. In general, suppose we are given an R^d-valued random variable X with finite second moment. As in [4], taking a suitable probability space $(\Omega, \mathscr{F}, P)$ we put

$$
\mathfrak{e}[X] = \inf E\{|\tilde{X}-\tilde{Y}|^2\},
$$

where the infimum is taken over all pairs of R^d-valued random variables $\tilde{X}$ and $\tilde{Y}$ defined on $(\Omega, \mathscr{F}, P)$ such that the distribution of $\tilde{X}$ is the same as that of X and $\tilde{Y}$ is a Gaussian random variable with the same mean vector and covariance matrix as those of X. H. Murata and H. Tanaka [4] proved the following statement.

(4.9) *If X_1 and X_2 are independent random variables with finite second moments, then we have*

$$
\mathfrak{e}[X_1+X_2] < \mathfrak{e}[X_1]+\mathfrak{e}[X_2]
$$

unless both X_1 and X_2 are Gaussian.

We now think of R^{12} as a probability space with probability measure $f(\xi, \alpha)f(\eta, \beta)$ $d\xi d\alpha d\eta d\beta$. Then both the random variables

$$
X = \begin{bmatrix} \xi \\ \alpha \\ \eta \\ \beta \end{bmatrix}, \quad X' = \begin{bmatrix} \xi' \\ \alpha' \\ \eta' \\ \beta' \end{bmatrix}
$$

on this probability space have the same probability density $f(\xi, \alpha)f(\eta, \beta)$ by the relation (4.1). We put

$$X_1 = \begin{pmatrix} \xi \\ \alpha \end{pmatrix}, X_2 = \begin{pmatrix} \eta \\ \beta \end{pmatrix}, X_1' = \begin{pmatrix} \xi' \\ \alpha' \end{pmatrix}, X_2' = \begin{pmatrix} \eta' \\ \beta' \end{pmatrix},$$

$$A = \begin{pmatrix} I-M & N \\ -N & M \end{pmatrix}, B = \begin{pmatrix} M & N \\ N & -(I-M) \end{pmatrix},$$

$$C = \begin{pmatrix} M & -N \\ -N & -(I-M) \end{pmatrix}, D = \begin{pmatrix} I-M & -N \\ N & M \end{pmatrix}.$$

Since X_1 and X_2 are independent, an application of (4.9) for $X_1' = AX_1 + BX_2$ and $X_2' = CX_1 + DX_2$ yields

$$\mathfrak{e}[X_1'] \leq \mathfrak{e}[AX_1] + \mathfrak{e}[BX_2], \quad \mathfrak{e}[X_2'] \leq \mathfrak{e}[CX_1] + \mathfrak{e}[DX_2]. \tag{4.10}$$

Taking pairs $(\tilde{X}_1, \tilde{Y}_1)$ and $(\tilde{X}_2, \tilde{Y}_2)$ which attain the infimums in the definition of $\mathfrak{e}[X_1]$ and $\mathfrak{e}[X_2]$ respectively, we have by (4.10)

$$\begin{aligned} \mathfrak{e}[X_1'] + \mathfrak{e}[X_2'] &\leq E\{|A(\tilde{X}_1 - \tilde{Y}_1)|^2 + |B(\tilde{X}_2 - \tilde{Y}_2)|^2 \\ &\quad + |C(\tilde{X}_1 - \tilde{Y}_1)|^2 + |D(\tilde{X}_2 - \tilde{Y}_2)|^2\} \\ &= \mathfrak{e}[X_1] + \mathfrak{e}[X_2]; \end{aligned} \tag{4.11}$$

here the last equality is derived from the orthogonality of U. On the other hand since X_1 and X_2 (resp., X_1' and X_2') are independent, we have from the definition of $\mathfrak{e}$,

$$\mathfrak{e}[X'] = \mathfrak{e}[X_1'] + \mathfrak{e}[X_2'], \quad \mathfrak{e}[X] = \mathfrak{e}[X_1] + \mathfrak{e}[X_2], \tag{4.12}$$

and also $\mathfrak{e}[X'] = \mathfrak{e}[X]$ since X and X' are identically distributed. Therefore the inequality sign in (4.11) must be the equality one. This can happen only when both X_1 and X_2 are Gaussian random variables by (4.9). Finally noting that the density f is an even function in α, we obtain the representation (4.8) of f.

To finish the proof of Theorem 4, it is sufficient to prove the following

LEMMA 7. *Let Q_1 and Q_2 be the symmetric matrices obtained in* Lemma 6. *Then there exists a positive constant c such that $Q_1 = Q_2 = cI$, that is, $(Q_1\xi, \xi) + (Q_2\alpha, \alpha) = c(|\xi|^2 + |\alpha|^2)$.*

PROOF. The relation (4.1) and Lemma 6 imply the following invariance:

$$(4.13)\qquad (Q_1\xi', \xi')+(Q_2\alpha', \alpha')+(Q_1\eta', \eta')+(Q_2\beta', \beta') = (Q_1\xi, \xi)+(Q_2\alpha, \alpha)+(Q_1\eta, \eta)+(Q_2\beta, \beta).$$

Inserting the relation (4.4) and using Lemma 5, we have five equations for matrices Q_1, Q_2, M and N:

$$(4.14)\qquad 2(NQ_1M-MQ_2N)+Q_2N-NQ_1=0,$$

$$(4.15)\qquad MQ_1(I-M)+NQ_2N=0,$$

$$(4.16)\qquad MQ_2(I-M)+NQ_1N=0,$$

$$(4.17)\qquad MQ_1M-NQ_2N-(Q_1M+MQ_1)/2=0,$$

$$(4.18)\qquad MQ_2M-NQ_1N-(Q_2M+MQ_2)/2=0.$$

Adding (4.17) to (4.15) (and also (4.18) to (4.16)), we have

$$(4.19)\qquad Q_1M=MQ_1, \quad Q_2M=MQ_2.$$

The relations (4.7) and (4.19) therefore imply $NQ_1=Q_2N$ by (4.14); here we have used $\kappa\neq 1$. By the way, it is easy to prove that any matrix Q which commutes with M for almost all $\ell\in S^2$ is a constant multiple of the identity matrix. Therefore $Q_1=c_1I$ and $Q_2=c_2I$. Finally, $c_1=c_2$ follows from the relation $NQ_1=Q_2N$.

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