HIROSHIMA MATH. J.
6 (1976), 183-189

Remarks on the Oscillatory Behavior of Solutions of Functional Differential Equations with Deviating Argument

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(Received September 12, 1975)

1. Introduction

We consider the following nth order functional differential equations with deviating argument

$$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_2(t)(r_1(t)y'(t))')'\cdots)')')' + (-1)^n y(g(t))F([y(g(t))]^2, t) = 0, \tag{1}$$

$$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_2(t)(r_1(t)y'(t))')'\cdots)')')' + (-1)^{n+1} y(g(t))F([y(g(t))]^2, t) = 0, \tag{2}$$

$$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_2(t)(r_1(t)y'(t))')')\cdots')')' + y(g(t))F([y(g(t))]^2, t) = 0. \tag{3}$$

The conditions we always assume for r_i, g, F are as follows:

(a) $g(t)$ is continuous on $[\tau, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(4) (b) each $r_i(t)$ is continuous and positive on $[\tau, \infty)$, and

$$\int_\tau^\infty \frac{dt}{r_i(t)} = \infty, \qquad i = 1,\dots, n-1\ ;$$

(c) $F(z, t)$ is nonnegative on $(0, \infty) \times [\tau, \infty)$. $yF(y^2, t)$ is continuous on $(-\infty, \infty) \times [\tau, \infty)$ and is nondecreasing in y for each $t \geqq \tau$.

We restrict our discussion to those solutions $y(t)$ of the above differential equations which exist on some ray $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)|: t_0 \leqq t < \infty\} > 0$$

for every $t_0 \in [T_y, \infty)$. Such a solution is said to be oscillatory (or to oscillate) if it has arbitrarily large zeros. Otherwise the solution is said to be nonoscillatory.

The oscillatory behavior of solutions of the above equations and/or related equations has recently been studied by Kartsatos [1], [2] and by Lovelady [4], [5]. We mention in particular the work of Lovelady [4] who presents oscillation criteria for bounded solutions of linear ordinary differential equations of the forms (1) and (2). Our main concern in this note is to extend Lovelady's results [4] to the nonlinear functional differential equations with deviating argument (1) and (2). We are also interested in providing criteria for all solutions of the equation (3) to be oscillatory. Our results seem to be new even in the case of ordinary differential equations, that is, in the case when $g(t)\equiv t$.

2. Main results

We begin by stating the main results of this paper.

THEOREM 1. *Assume that*

$$\int^{\infty} R_{n-1}(t)F(c^2, t)dt = \infty \qquad \text{for all} \quad c > 0, \tag{5}$$

where $R_{n-1}(t)$ *is defined by*

$$R_0(t) = 1,\ R_i(t) = \int_{\tau}^{t} \frac{R_{i-1}(s)}{r_i(s)} ds, \qquad i=1,\dots, n-1.$$

Then, all bounded solutions of (1) *are oscillatory.*

THEOREM 2. *Assume that* (5) *holds. Then, every bounded solution of* (2) *either oscillates or tends to zero monotonically as* $t\to\infty$.

THEOREM 3. *Assume that*

$$\int^{\infty} F(c^2, t)dt = \infty \qquad \text{for all} \quad c > 0. \tag{6}$$

Then, for n even, every solution of (3) *is oscillatory, while, for n odd, every solution of* (3) *either oscillates or tends monotonically to zero as* $t\to\infty$.

Each of these theorems follows from an appropriate combination of the following four lemmas which give information on the behavior of possible nonoscillatory solutions of the differential equations under consideration.

LEMMA 1. *Suppose* (5) *holds. If there exists a bounded nonoscillatory solution of* (1) *or* (2), *then it tends to zero as* $t\to\infty$.

LEMMA 2. *Suppose* (6) *holds. If there exists a nonoscillatory solution of* (3), *then it tends to zero as* $t\to\infty$.

The above lemmas are virtually particular cases of the theorems obtained by the present authors in [3]. The verification is left to the reader.

LEMMA 3. *Suppose* (5) *holds. Let* $y(t)$ *be a bounded nonoscillatory solution of* (1) *or* (2), *and define*

$$Y_0(t) = y(t), \quad Y_i(t) = r_i(t)Y'_{i-1}(t), \qquad i = 1,\dots, n-1. \tag{7}$$

(i) *If* $y(t)$ *is a solution of* (1), *then*

$$(-1)^{i+1}Y_0(t)Y_i(t) \geqq 0 \text{ eventually, and } \lim_{t\to\infty} Y_i(t) = 0,\ i=1,\dots, n-1.$$

(ii) *If* $y(t)$ *is a solution of* (2), *then*

$$(-1)^{i}Y_0(t)Y_i(t) \geqq 0 \text{ eventually, and } \lim_{t\to\infty} Y_i(t) = 0,\ i = 1,\dots, n-1.$$

PROOF. It suffices to prove that the statement (i) is true. Let $y(t)$ be a bounded positive solution of (1). By (4)(a) there exists t_1 such that $y(g(t))>0$ for $t\geqq t_1$. From (1) we have

$$(-1)^n Y'_{n-1}(t) = -y(g(t))F([y(g(t))]^2, t) \leqq 0 \qquad \text{for} \quad t \geqq t_1,$$

so that $(-1)^n Y_{n-1}(t)$ decreases to a limit $M_{n-1} \geqq -\infty$ as $t\to\infty$. Suppose $M_{n-1} > 0$. Then,

$$(-1)^n Y_{n-1}(t) = (-1)^n r_{n-1}(t)Y'_{n-2}(t) \geqq M_{n-1} \qquad \text{for} \quad t \geqq t_1. \tag{8}$$

Dividing (8) by $r_{n-1}(t)$ and integrating give

$$(-1)^n Y_{n-2}(t) - (-1)^n Y_{n-2}(t_1) \geqq M_{n-1}\int_{t_1}^{t} \frac{ds}{r_{n-1}(s)},$$

from which, using (4)(b), we obtain

$$\lim_{t\to\infty}(-1)^n Y_{n-2}(t) = \infty. \tag{9}$$

It is a matter of easy computation to deduce from (9) with the use of (4)(b) that

$$\lim_{t\to\infty}(-1)^n Y_{n-3}(t) = \cdots = \lim_{t\to\infty}(-1)^n Y_0(t) = \infty.$$

Consequently, we have $\lim_{t\to\infty}(-1)^n y(t) = \infty$. However, this contradicts the boundedness of $y(t)$ if n is even, and contradicts the assumption that $y(t)$ is positive if n is odd. Suppose next that $M_{n-1}<0$. Then, there are numbers $t_2 \geqq t_1$ and $M'_{n-1}<0$ such that

$$(-1)^n Y_{n-1}(t) = (-1)^n r_{n-1}(t)Y'_{n-2}(t) \leqq M'_{n-1} \qquad \text{for} \quad t \geqq t_2. \tag{10}$$

From (10) we obtain

$$(-1)^n Y_{n-2}(t)-(-1)^n Y_{n-2}(t_2) \leqq M'_{n-1} \int_{t_2}^{t} \frac{ds}{r_{n-1}(s)},$$

and letting $t\to\infty$, we see that

$$\lim_{t\to\infty}(-1)^n Y_{n-2}(t) = -\infty.$$

It follows that

$$\lim_{t\to\infty}(-1)^n Y_{n-3}(t) = \cdots = \lim_{t\to\infty}(-1)^n Y_0(t) = -\infty,$$

which yields $\lim_{t\to\infty}(-1)^n y(t)=-\infty$, a contradiction. Therefore, the sole possibility is that $M_{n-1}=0$. We have thus shown that

$$\lim_{t\to\infty} Y_{n-1}(t) = 0 \quad \text{and} \quad (-1)^n Y_{n-1}(t) \geqq 0 \qquad \text{for} \quad t \geqq t_1.$$

The last inequality implies that $(-1)^{n-1}Y_{n-2}(t)$ is nonincreasing and tends to a limit $M_{n-2}\geqq-\infty$ as $t\to\infty$. Exactly as above it can be shown that $M_{n-2}=0$. Hence,

$$\lim_{t\to\infty} Y_{n-2}(t) = 0 \quad \text{and} \quad (-1)^{n-1} Y_{n-2}(t) \geqq 0 \qquad \text{for} \quad t \geqq t_1.$$

Repeating the same argument, we arrive at the desired conclusion in case $y(t)$ is positive. A parallel argument holds if we assume that $y(t)$ is a bounded negative solution of (1). This completes the proof.

On the basis of Lemma 2 we can prove in a similar way the following lemma.

LEMMA 4. *Suppose* (6) *holds. Let* $y(t)$ *be a nonoscillatory solution of* (3) *and define the functions* $Y_i(t)$ *by* (7). *Then,*

$$(-1)^{n-1+i} Y_0(t) Y_i(t) \geqq 0 \textit{ eventually and } \lim_{t\to\infty} Y_i(t) = 0,\ i = 1,\dots, n-1.$$

PROOF OF THEOREM 1. Suppose to the contrary that there exists a bounded nonoscillatory solution $y(t)$ of (1). Lemma 3 (i) implies that $y(t)y'(t)\geqq 0$ eventually, so that $y(t)$ goes monotonically to a nonzero limit as $t\to\infty$. But this is inconsistent with the conclusion of Lemma 1.

PROOF OF THEOREM 2. Let $y(t)$ be a bounded nonoscillatory solution of (2). By Lemma 1 $y(t)$ tends to zero as $t\to\infty$. From Lemma 3 (ii) it follows that $y(t)y'(t)\leqq 0$ eventually and $\lim_{t\to\infty} Y_i(t)=0$, $i=1,\dots, n-1$, where the functions $Y_i(t)$ are defined by (7). Thus, $y(t)$ approaches monotonically to zero as $t\to\infty$ together with $Y_i(t)$, $i=1,\dots, n-1$.

PROOF OF THEOREM 3. Let n be even. If $y(t)$ is a nonoscillatory solution of (3), then $y(t)y'(t) \geqq 0$ eventually by Lemma 4, so $y(t)$ tends monotonically to a nonzero limit as $t \to \infty$. This, however, contradicts the conclusion of Lemma 2. It follows that all solutions of (3) are oscillatory if n is even. Now, let n be odd and let $y(t)$ be a nonoscillatory solution of (3). According to Lemma 2 $y(t)$ tends to zero as $t \to \infty$. Since, by Lemma 4, $y(t)y'(t) \leqq 0$ for all large t, $y(t)$ is monotonic. Again by Lemma 4 the functions $Y_i(t)$, $i=1,\dots, n-1$, defined by (7) also tend monotonically to zero as $t \to \infty$.

Consider the functional differential equations

$$[r(t)y^{(n-m)}(t)]^{(m)} + (-1)^n y(g(t))F([y(g(t))]^2, t) = 0, \tag{11}$$

$$[r(t)y^{(n-m)}(t)]^{(m)} + (-1)^{n+1} y(g(t))F([y(g(t))]^2, t) = 0, \tag{12}$$

$$[r(t)y^{(n-m)}(t)]^{(m)} + y(g(t))F([y(g(t))]^2, t) = 0, \tag{13}$$

where $0<m<n$, $g(t)$ and $F(z, t)$ are as in the equations (1), (2), (3), and $r(t)$ is a positive continuous function such that

$$\int_\tau^\infty \frac{dt}{r(t)} = \infty .$$

It is easy to see that (11), (12) and (13) are the special cases of (1), (2) and (3), respectively.

COROLLARY 1. *Suppose* (5) *holds. Then, all bounded solutions of* (11) *are oscillatory.*

COROLLARY 2. *Suppose* (5) *holds. Then, every bounded solution of* (12) *either oscillates or tends monotonically to zero as* $t \to \infty$.

COROLLARY 3. *Suppose* (6) *holds. Then, if n is even, every solution of* (13) *is oscillatory, and if n is odd, every solution of* (13) *either oscillates or tends monotonically to zero as* $t \to \infty$.

REMARK 1. Of practical importance are the particular cases of the equations (1), (2), (3) in which $yF(y^2, t)$ has the form

$$yF(y^2, t) \equiv p(t)|y|^\alpha \operatorname{sgn} y, \qquad \alpha > 0,$$

where $p(t)$ is a positive continuous function on $[\tau, \infty)$. For these cases the integral conditions (5) and (6) read

$$\int^\infty R_{n-1}(t)p(t)dt = \infty \quad \text{and} \quad \int^\infty p(t)dt = \infty ,$$

respectively.

REMARK 2. Theorems 1 and 2 are extensions of recent results of Lovelady [4, Theorems 1 and 2] regarding linear ordinary differential equations of the forms (1) and (2). Since (5) and (6) do not involve the functional argument $g(t)$, our criteria are applicable not only to equations with a delayed argument but also to equations with an advanced argument.

REMARK 3. Corollary 3 improves a result of Terry [6, Corollary 2.7] who examines the particular case of (13) in which $n=2m$, $r(t)$ is bounded both from above and below by positive constants, $g(t)$ is a delayed argument with bounded delay, and $F(z, t)$ is nondecreasing in z.

REMARK 4. Consider the even order equation (3). According to Theorem 1 all bounded solutions of (3) are oscillatory if the condition (5) is satisfied. This condition, however, is not sufficient in order that all solutions of (3) be oscillatory, as the following examples show. Consider the second order equations

$$(t^{-2}y'(t))'+2t^{-4}y(t) = 0,$$

$$(t^{-2}y'(t))'+2t^{-4}(y(t^{1/3}))^3 = 0,$$

$$(t^{-2}y'(t))'+2t^{-4}(y(t^3))^{1/3} = 0.$$

All of these equations have $y(t)=t$ as a nonoscillatory solution, though the condition (5) is satisfied. This is why we have demanded a much more stringent condition (6) for the oscillation of all solutions of (3). It might be a question of interest to study the gap between (5) and (6) in connection with the nonlinearity of the equations and the growth of the deviating arguments involved.

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