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On the Existence of Boundary Values of Beppo Levi Functions Defined in the Upper Half Space of R^n

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1. Introduction and statement of results

Let R^n ($n \geqq 2$) be the n-dimensional Euclidean space and its points be denoted by x, y, etc., or $x=(x_1, x_2,\dots, x_n)=(x', x_n)$, $y=(y_1, y_2,\dots, y_n)=(y', y_n)$, etc. For a positive number α such that $\alpha<n$, the Riesz potential of order α of a measure μ on R^n is defined by

$$U_\alpha^\mu(x) = \int |x-y|^{\alpha-n} d\mu(y).$$

If μ has a density f (that is, $d\mu=fdx$, where f is locally integrable), we may write U_α^f instead of U_α^μ. The Riesz capacity $C_\alpha(E)$ of a Borel set E in R^n may be defined as follows:

$$C_\alpha(E) = \sup \mu(R^n),$$

where the supremum is taken over all positive measures μ concentrated on E such that $U_\alpha^\mu(x) \leqq 1$ for every $x \in S_\mu$ (S_μ is the support of μ).

Our main theorem is the following:

THEOREM 1. *Let α and p be numbers such that $\alpha \geqq 0$ and $1+\alpha<p<n+\alpha$. Let f be a function which is defined and continuous in the upper half space $R_+^n=\{x=(x', x_n); x_n>0\}$. Suppose that all partial derivatives of f of first order exist a.e. on R_+^n and that for any bounded open set Ω in R_+^n*

$$\iint_\Omega |\mathrm{grad} f(x', x_n)|^p x_n^\alpha dx' dx_n < \infty. \tag{1}$$

Then $\lim_{x_n \downarrow 0} f(x', x_n)$ exists and is finite except for $(x', 0)$ in a Borel set E in $R_0^n =\{(y', 0); y' \in R^{n-1}\}$ such that $C_{p-\alpha}(E)=0$ if $p \leqq 2$ and $C_{p-\alpha-\varepsilon}(E)=0$ for any $\varepsilon>0$ with $p-\alpha-\varepsilon>0$ if $p>2$.

In the case $p=2$ this theorem was shown by H. Wallin [7]. He also showed that his result is the best possible as to the size of the exceptional set. We shall generalize this result in the following theorem:

THEOREM 2. *Let α and p be as in* Theorem 1. *Let E be a set in R_0^n such* *that* $C_{p-\alpha}(E)=0$ *if* $p\geqq 2$ *and* $C_{p-\alpha+\varepsilon}(E)=0$ *for some* $\varepsilon>0$ *with* $p-\alpha+\varepsilon<n$ *if* $p<2$. *Then there exists a function* f *of class* C^∞ *in* R_+^n *such that*

$$\iint_{R_+^n} |\mathrm{grad} f(x', x_n)|^p x_n^\alpha dx' dx_n < \infty$$

and $\lim_{x_n \downarrow 0} f(x', x_n)=\infty$ *for any* $(x', 0)\in E$.

We see that there is a gap between Theorem 1 and Theorem 2 in the case $p\neq 2$.

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2. Lemmas

To prove Theorem 1, we prepare several lemmas.

LEMMA 1. *Let* β *and* γ *be numbers such that*

$$0\leqq \gamma < 1 \quad \text{and} \quad \gamma < \beta < n.$$

Let η *be a positive number. Then*

$$\int_{|x-y|\leqq \eta} |x-y|^{\beta-n}|y_n|^{-\gamma}dy \leqq M\eta^{\beta-\gamma}$$

for some constant $M>0$ *independent of* x *and* η.

PROOF. We may assume that $x=(0, x_n)$, $x_n\geqq 0$. We shall show that the integral assumes its maximum when $x_n=0$. We set

$$E_1 = \{y;\ |x-y| \leqq \eta,\ |y| > \eta\},$$

$$E_2 = \left\{y;\ |x-y| \leqq \eta,\ y_n > \frac{x_n}{2}\right\},$$

$$E_3 = \left\{y;\ |x-y| \leqq \eta,\ y_n < \frac{x_n}{2}\right\}$$

and

$$E_4 = \{y;\ |x-y| > \eta,\ |y| \leqq \eta\}.$$

Then we note

$$\int_{E_1} |x-y|^{\beta-n}|y_n|^{-\gamma}dy \leqq \int_{E_4} |y|^{\beta-n}|y_n|^{-\gamma}dy.$$

and

$$\int_{E_2}\{|x-y|^{\beta-n}-|y|^{\beta-n}\}|y_n|^{-\gamma}dy \leqq \int_{E_3}\{|y|^{\beta-n}-|x-y|^{\beta-n}\}|y_n|^{-\gamma}dy.$$

Hence

$$\int_{|x-y|\leqq\eta}|x-y|^{\beta-n}|y_n|^{-\gamma}dy \leqq \int_{|y|\leqq\eta}|y|^{\beta-n}|y_n|^{-\gamma}dy$$
$$= M\eta^{\beta-\gamma},$$

where $M=\int_{|x|\leqq 1}|x|^{\beta-n}|x_n|^{-\gamma}dx<\infty$.

LEMMA 2. *Let β and γ be numbers such that*

$$\beta+\gamma>0 \quad \textit{and} \quad 0\leqq\gamma<1.$$

Let η be a positive number. Then

$$\int_{|x-y|\geqq\eta}|x-y|^{-\beta-n}|y_n|^{-\gamma}dy \leqq M\eta^{-\beta-\gamma}$$

for some constant $M>0$ independent of x and η.

PROOF. Again we may assume that $x=(0, x_n)$, $x_n\geqq 0$. By change of variables $z=y/\eta$,

$$\int_{|x-y|\geqq\eta}|x-y|^{-\beta-n}|y_n|^{-\gamma}dy = \eta^{-\beta-\gamma}\int_{|x^*-z|\geqq 1}|x^*-z|^{-\beta-n}|z_n|^{-\gamma}dz,$$

where $x^*=x/\eta$. We can easily verify that $\int_{|x^*-z|\geqq 1}|x^*-z|^{-\beta-n}|z_n|^{-\gamma}dz$ is bounded, dividing the domain of integration into three parts, that is, (a) $|x^*-z|<\frac{1}{2}|z|$ (this implies $|x^*-z|\leqq|z_n|/\sqrt{3}$), (b) $|z|<1$, (c) $|z|\geqq 1$, $|x^*-z|\geqq\frac{1}{2}|z|$.

LEMMA 3 (cf. [7; Lemma 4]). *Let β and γ be numbers such that $0\leqq\gamma<1$ and $\gamma<\beta<\frac{n+\gamma}{2}$. Then*

$$\int|x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\gamma}dy \leqq M|x-z|^{2\beta-\gamma-n}$$

for some constant M independent of x and z.

PROOF. Set $\eta=|x-z|/2$. Noting that $|y-z|\geqq\eta$ if $|x-y|\leqq\eta$, we have

$$I_1 \equiv \int_{|x-y|\geqq\eta}|x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\gamma}dy$$
$$\leqq \eta^{\beta-n}\int_{|x-y|\geqq\eta}|x-y|^{\beta-n}|y_n|^{-\gamma}dy.$$

Lemma 1 gives

$$I_1 \leqq M_1 \eta^{2\beta-\gamma-n}$$

for some constant M_1 independent of x and z. Similarly

$$I_2 \equiv \int_{|z-y|\leqq\eta} |x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\gamma}dy \leqq M_1\eta^{2\beta-\gamma-n}.$$

On the other hand we have

$$I_3 \equiv \int_{|x-y|>\eta,\ |z-y|>\eta, |y-x|<|y-z|} |x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\gamma}dy$$

$$\leqq \int_{|x-y|>\eta} |x-y|^{2(\beta-n)}|y_n|^{-\gamma}dy \leqq M_2\eta^{2\beta-\gamma-n}$$

for some constant M_2 independent of x and z, because of Lemma 2. Similarly

$$I_4 \equiv \int_{|x-y|>\eta, |z-y|>\eta,\ |y-x|\geqq|y-z|} |x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\gamma}dy$$

$$\leqq M_2\eta^{2\beta-\gamma-n}.$$

Hence we obtain

$$\int |x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\gamma}dy = I_1+I_2+I_3+I_4$$

$$\leqq 2(M_1+M_2)\eta^{2\beta-\gamma-n}.$$

From Lemma 3 we derive the following lemma, which will be used to show Theorem 1 in case $p \leqq 2$.

LEMMA 4 (cf. [7; Lemma 3]). *Let α and p be non-negative numbers such that*

$$1+\alpha < p \leqq 2.$$

Let g be a non-negative function in $L^p(R^n)$, and set

$$G_\lambda = \{x \in R^n;\ U_1^g(x) > \lambda\}, \qquad \lambda > 0.$$

Then there is a constant $M>0$ independent of g and λ such that

$$C_{p-\alpha}(G_\lambda) \leqq M\lambda^{-p}\int |x_n|^\alpha g(x)^p dx.$$

PROOF. Let μ be a positive measure such that $S_\mu \subset G_\lambda$, S_μ is compact and

$\int|x-y|^{p-\alpha-n}d\mu(y)\leqq 1$ for all $x\in S_\mu$. Then by using Hölder's inequality, we have for $p'=p/(p-1)$

$$\{\lambda\cdot\mu(R^n)\}^{p'}\leqq\left|\int\left\{\int|x-y|^{1-n}g(y)dy\right\}d\mu(x)\right|^{p'}$$

$$\leqq\left\{\int|y_n|^\alpha g(y)^p dy\right\}^{p'/p}\int|y_n|^{-p'\alpha/p}\left\{\int|x-y|^{1-n}d\mu(x)\right\}^{p'}dy.$$

Set $\beta=(p-\alpha)/2+\alpha/2(p-1)$ and note

$$1-n=(p-\alpha-n)(1-2/p')+(\beta-n)\cdot 2/p'.$$

By Hölder's inequality and the fact that $U^\mu_{p-\alpha}\leqq 1$ on R^n (Frostman's maximum principle),

$$\int|x-y|^{1-n}d\mu(x)\leqq\left\{\int|x-y|^{\beta-n}d\mu(x)\right\}^{2/p'}.$$

Hence

$$\{\lambda\cdot\mu(R^n)\}^{p'}$$

$$\leqq\left\{\int|y_n|^\alpha g(y)^p dy\right\}^{p'/p}\int|y_n|^{-\alpha/(p-1)}\left\{\int|x-y|^{\beta-n}d\mu(x)\right\}^2 dy$$

$$\leqq\left\{\int|y_n|^\alpha g(y)^p dy\right\}^{p'/p}\iint d\mu(x)d\mu(z)\int|x-y|^{\beta-n}|z-y|^{\beta-n}|y_n|^{-\alpha/(p-1)}dy.$$

By Lemma 3 the integral with respect to y is majorated by

$$\text{const.}\,|x-z|^{p-\alpha-n}.$$

Therefore we have

$$\{\lambda\cdot\mu(R^n)\}^{p'}\leqq M'\left\{\int|y_n|^\alpha g(y)^p dy\right\}^{\frac{p'}{p}}\mu(R^n),$$

and hence

$$\mu(R^n)\leqq M\lambda^{-p}\int|y_n|^\alpha g(y)^p dy$$

for some constants M' and M independent of μ and λ. This leads to the conclusion of the lemma.

To show Theorem 1 in case $p>2$, we need

LEMMA 5. *Let α and p be non-negative numbers such that*

$$2 < p < n+\alpha.$$

Let $0<\varepsilon<p-\alpha$ and $R>0$. Then there is a constant $M>0$ with the following property: if g is a non-negative function in $L^p(R^n)$ whose support is contained in the closed ball centered at the origin of R^n with radius R and if G_λ, $\lambda>0$, is as in Lemma 4, *then*

$$C_{p-\alpha-\varepsilon}(G_\lambda) \leqq M\lambda^{-p}\int |x_n|^\alpha g(x)^p dx.$$

PROOF. Set

$$t = \frac{n-(p-\alpha-\varepsilon)}{p(n-1)}.$$

Then $0<t<1$. By Hölder's inequality we have for a positive measure μ on R^n

$$\text{(2)} \qquad \int |x-y|^{1-n} d\mu(x)$$

$$\leqq \left\{\int |x-y|^{pt(1-n)} d\mu(x)\right\}^{1/p} \left\{\int |x-y|^{p'(1-t)(1-n)} d\mu(x)\right\}^{1/p'},$$

where $1/p+1/p'=1$. Now let μ be a positive measure such that $S_\mu \subset G_\lambda$, S_μ is compact and $U^\mu_{p-\alpha-\varepsilon}(x) \leqq 1$ for every $x \in S_\mu$. In a way similar to that in the proof of Lemma 4, we see that

$$\{\lambda\mu(R^n)\}^{p'}$$

$$\leqq \left\{\int |y_n|^\alpha g(y)^p dy\right\}^{p'/p} \int_{|y|\leqq R} |y_n|^{-p'\alpha/p} \left\{\int |x-y|^{1-n} d\mu(x)\right\}^{p'} dy.$$

Noting that $pt(1-n)=p-\alpha-\varepsilon-n$ and $p'(1-t)(1-n)=(\alpha+\varepsilon)/(p-1)-n$, we have by (2)

$$\{\lambda\mu(R^n)\}^{p'}$$

$$\leqq \left\{\int |y_n|^\alpha g(y)^p dy\right\}^{p'/p} \left\{\sup_{y\in R^n} \int |x-y|^{p-\alpha-\varepsilon-n} d\mu(x)\right\}^{p'/p}$$

$$\int d\mu(x) \int_{|y|\leqq R} |x-y|^{\beta-n} |y_n|^{-\alpha/(p-1)} dy,$$

where $\beta=(\alpha+\varepsilon)/(p-1)$. Denote the last integral with respect to y by I. Obviously, it assumes its maximum at $x=0$ (cf. Lemma 1). Then

$$I \leqq \int_{|y|\leqq R} |y|^{\beta-n} |y_n|^{-\alpha/(p-1)} dy < \infty.$$

Since $U^{\mu}_{p-\alpha-\varepsilon}$ is bounded on R^n, we obtain

$$\mu(R^n) \leqq M\lambda^{-p}\int |y_n|^\alpha g(y)^p dy$$

for a suitable constant M independent of λ and g, which implies the conclusion of the lemma.

3. Proof of Theorem 1

Let f be a function as in Theorem 1. Choose a number r such that $1<r<p/(\alpha+1)$. Then, by (1) and Hölder's inequality we see that for any bounded open set Ω in R^n_+, $\int_\Omega |\mathrm{grad} f|^r dx<\infty$. Hence by [5; Theorem 5.6] there exists an extension $\hat{f}$ of f to R^n so that $\hat{f}$ is locally r-precise in R^n and symmetric with respect to R^n_0 (see [5] for the definition of locally r-precise functions). Let us show that $\hat{f}$ is locally L^p on R^n. Let $l_{x'}$ be the line through $(x', 0)$ which is parallel to the x_n-axis. Since $\hat{f}$ is absolutely continuous along $l_{x'}$ for a.e. x',

$$f(x', x_n) = -\int_{x_n}^{R} \frac{\partial f}{\partial y_n}(x', y_n)\, dy_n + f(x', R),\ 0 < x_n < R, \text{ for a.e. } x'.$$

Noting that $\int_{|x'|<R} |f(x', R)|^p dx'<\infty$ because f is continuous in R^n_+, we have by (1) and Hölder's inequality that

$$\iint_{|x'|<R,\ 0<x_n<R} |f(x', x_n)|^p dx' dx_n < \infty \qquad \text{for any} \quad R > 0,$$

which implies that $\hat{f}\in L^p_{\mathrm{loc}}(R^n)$. Hence we may suppose that $\mathrm{supp}\hat{f}$ is compact. By [4] we have the following integral representation of $\hat{f}$:

$$\hat{f}(x) = \sum_{i=1}^{n} a_i \int \frac{x_i - y_i}{|x-y|^n} \frac{\partial \hat{f}}{\partial y_i}(y)\, dy \quad \text{a.e.,} \tag{3}$$

where a_i are constants. Let $f_{i,j}$, $i=1, 2,\dots, n$; $j=1, 2,\dots$, be continuous functions on R^n with compact supports and set

$$g_j(y) = \sum_{i=1}^{n} |a_i| \left| f_{i,j}(y) - \frac{\partial \hat{f}}{\partial y_i}(y) \right|,\ y\in R^n,\ j = 1, 2,\dots$$

We can choose the functions $f_{i,j}$ so that $\int |y_n|^\alpha g_j(y)^p dy \leqq 2^{-2pj}$. We define the continuous function v_j in R_n, $j=1, 2,\dots$, by

$$v_j(x) = \sum_{i=1}^{n} a_i \int \frac{x_i - y_i}{|x-y|^n} f_{i,j}(y)\, dy.$$

Set $\omega_j=\{x\in R^n;\ U_1^{g_j}(x)>2^{-j}\}$. First we consider the case $p\leqq 2$. From Lemma 4 it follows that $C_{p-\alpha}(\omega_j)\leqq M2^{-pj}$. If we set $E_k=\cup_{j=k}^{\infty}\omega_j$, then we see that $C_{p-\alpha}(E_k)\to 0$ as $k\to\infty$ and that v_j is uniformly convergent to v on R^n-E_k, $k=1, 2,\ldots$, where v is defined by the right-hand side of (3). In general, denote by E^* the projection of a set E in R^n to the hyperplane R_0^n. Setting

$$E_0 = \bigcap_{k=1}^{\infty} E_k^*,$$

we have $C_{p-\alpha}(E_0)=0$, by the fact that the Riesz capacity does not increase with respect to a transformation which does not increase the distance. Setting $E^0=\{x\in R_+^n;\ f(x)\neq v(x)\}^*$, we note $C_{p-\alpha}(E^0)=0$. Let $E=E_0\cup E^0$. Then $C_{p-\alpha}(E)=0$. If $(x', 0)\notin E$, then f is equal to v on $l_{x'}\cap R_+^n$ and v is continuous on $l_{x'}$. Consequently

$$\lim_{x_n\downarrow 0} f(x', x_n)$$

exists and is finite for $(x', 0)\notin E$. Thus the case $p\leqq 2$ is proved.

Next we consider the case $p>2$. In this case we may assume that the supports of functions g_j are all included in a fixed closed ball. Then note that $C_{p-\alpha-\varepsilon}(E)=0$ for any ε, $0<\varepsilon<p-\alpha$, on account of Lemma 5. In the same way as above we can show Theorem 1 in case $p>2$. Thus our theorem is proved.

REMARK. The above proof shows that Theorem 1 is valid if f is a locally p-precise function on R_+^n and (1) is satisfied for any bounded open set Ω in R_+^n.

4. Proof of Theorem 2

To prove Theorem 2, we need the following lemma.

LEMMA 6. *Let g be a non-negative function in $L^p(R^n)$ and set*

$$f(x) = \int |x-y|^{1-n}|y_n|^{-\alpha/p}g(y)dy,$$

where $\alpha\geqq 0$ and $1+\alpha<p<n+\alpha$. Then

$$\left\{\int |x_n|^{\alpha}|\mathrm{grad} f|^p dx\right\}^{1/p} \leqq M\|g\|_p$$

for some constant $M>0$ independent of g, where the derivatives are considered in the sense of distribution.

PROOF. Noting that $(1+|y|)^{1-n}|y_n|^{-\alpha/p}\in L^{p'}(R^n)$, $p'=p/(p-1)$, we have

(4) $$\int (1+|y|)^{1-n}|y_n|^{-\alpha/p}g(y)dy < \infty.$$

We set $\kappa_\varepsilon(x)=(|x|^2+\varepsilon^2)^{(1-n)/2}$, $\varepsilon>0$, and define

$$F_\varepsilon(x) = \int \kappa_\varepsilon(x-y)|y_n|^{-\alpha/p}g(y)dy.$$

From (4) we see that $F_\varepsilon \in C^\infty(R^n)$ and

$$\frac{\partial F_\varepsilon}{\partial x_i}(x) = \int \frac{\partial \kappa_\varepsilon}{\partial x_i}(x-y)|y_n|^{-\alpha/p}g(y)dy,\ i=1,2,\dots,n.$$

We set $\kappa_\varepsilon * g(x)=\int \kappa_\varepsilon(x-y)g(y)dy$ for $\varepsilon>0$. In the proof of [4; Lemma 3.2], it is shown that $\|D_i(\kappa_\varepsilon * g)\|_p \leqq M_1\|g\|_p$ for any i, where $D_i=\partial/\partial x_i$ and M_1 is a constant independent of g. On the other hand,

$$(5)\quad \left| |x_n|^{\alpha/p}D_iF_\varepsilon(x)-D_i(\kappa_\varepsilon * g)\right|$$

$$\leqq M_2\int |x-y|^{-n}|1-(|x_n|/|y_n|)^{\alpha/p}|g(y)dy$$

$$= M_2\int \frac{|1-(|x_n|/|y_n|)^{\alpha/p}|}{|x_n-y_n|}\int \frac{|x_n-y_n|}{\{|x'-y'|^2+(x_n-y_n)^2\}^{n/2}}g(y',y_n)dy'dy_n.$$

We set

$$G(x';x_n,y_n) = \int \frac{|x_n-y_n|}{\{|x'-y'|^2+(x_n-y_n)^2\}^{n/2}}g(y',y_n)\,dy'.$$

Then we note that for some constant $M_3>0$ (independent of x_n and y_n)

$$\int G(x';x_n,y_n)^p dx' \leqq M_3\int g(y',y_n)^p dy'$$

(see [6; Theorem 1, (a) in Chap. III and Theorem 1, (c) in Chap. I]). Hence by using Minkowski's inequality ([6; Appendix A.1]), we have

$$\iint\left\{\int \frac{|1-(|x_n|/|y_n|)^{\alpha/p}|}{|x_n-y_n|}G(x';x_n,y_n)dy_n\right\}^p dx'dx_n$$

$$\leqq M_3\int\left|\int \frac{|1-(|x_n|/|y_n|)^{\alpha/p}|}{|x_n-y_n|}\left\{\int g(y',y_n)^p dy'\right\}^{1/p}dy_n\right|^p dx_n.$$

Applying Appendix A.3 in [6] with $K(x_n,y_n)=|1-(|x_n|/|y_n|)^{\alpha/p}|/|x_n-y_n|$, we see that the above integral is not greater than $M_3A_K^p\|g\|_p^p$, where $A_K=\int_{-\infty}^{\infty}K(1,y_n)|y_n|^{-1/p}dy_n<\infty$. This shows that (5) belongs to $L^p(R^n)$ and its L^p norm is not greater than $M_4\|g\|_p$, $M_4=M_2M_3{}^{1/p}A_K$. Hence for $M=M_1+M_4$

$$\| |x_n|^{\alpha/p}D_iF_\varepsilon\|_p$$

$$(6) \qquad \leqq \| |x_n|^{\alpha/p} D_i F_\varepsilon - D_i(\kappa_\varepsilon * g)\|_p + \|D_i(\kappa_\varepsilon * g)\|_p$$

$$\leqq M\|g\|_p.$$

Let r be any number such that $1<r<p/(\alpha+1)$ and let $\varphi \in C_0^\infty(R^n)$. Then by (6) and Hölder's inequality we see that $\{\varphi D_i F_\varepsilon;\ \varepsilon>0\}$ is bounded in $L^r(R^n)$. We shall show that $D_i f \in L^r_{loc}(R^n)$ (in the sense of distribution). For any φ and $\psi \in C_0^\infty(R^n)$ we have

$$<\varphi D_i f,\ \psi> \ = \ <D_i f,\ \varphi\psi> \ = \ -\int f(x) D_i(\varphi\psi)(x) dx.$$

Since $F_\varepsilon(x)$ increases to $f(x)$ as $\varepsilon \downarrow 0$ for any $x \in R^n$ and $f \in L^r_{loc}(R^n)$, the right-hand side is equal to $-\lim_{\varepsilon\downarrow 0}\int F_\varepsilon(x) D_i(\varphi\psi)(x)dx = \lim_{\varepsilon\downarrow 0}\int \varphi(x) D_i f_\varepsilon(x)\psi(x)dx$. From the boundedness of $\{\varphi D_i F_\varepsilon;\ \varepsilon>0\}$ in $L^r(R^n)$ we see that there is a constant A such that $|<\varphi D_i f,\ \psi>| \leqq A\|\psi\|_{r'}$, where $1/r+1/r'=1$. It follows that $\varphi D_i f \in L^r(R^n)$, and hence $D_i f$ is a function (as distribution). Let $\{\varphi_j\}$ be a sequence in $C_0^\infty(R^n)$ such that $\varphi_j(x) \geqq 0$ for any $x \in R^n$ and $\varphi_j(x)$ increases to $|x_n|^{\alpha/p}$ as $j \to \infty$. Then as seen in the above,

$$<\varphi_j D_i f,\ \psi> \ = \lim_{\varepsilon\to 0}\int \varphi_j(x) D_i F_\varepsilon(x)\psi(x)dx$$

holds for any $\psi \in C_0^\infty(R^n)$. The absolute value of the right-hand side is not greater than

$$\limsup_{\varepsilon\to 0}\left\{\int |x_n|^\alpha |D_i F_\varepsilon|^p dx\right\}^{1/p}\|\psi\|_{p'} \leqq M\|g\|_p\|\psi\|_{p'},$$

where $p'=p/(p-1)$. Hence $\|\varphi_j D_i f\|_p \leqq M\|g\|_p$. Since $\varphi_j|D_i f|$ increases to $|x_n|^{\alpha/p}|D_i f|$ as $j\to\infty$, we have by Lebesgue's monotone convergence theorem

$$\| |x_n|^{\alpha/p} D_i f\|_p \leqq M\|g\|_p, \qquad i = 1, 2,\dots, n,$$

which imply the required inequality for f.

We shall introduce the capacity $C_{\beta,p}$ $(0<\beta<n,\ 1<p<\infty)$, which is a special case of the capacity $C_{k;\mu;p}$ studied by N. G. Meyers [3], and which is defined as follows:

$$C_{\beta,p}(E) = \inf \|f\|_p^p, \qquad E \subset R^n,$$

where the infimum is taken over all non-negative functions f in $L^p(R^n)$ such that $U_\beta^f(x) \geqq 1$ for all $x \in E$.

Theorem 2 is a consequence of the following theorem in view of a result of B. Fuglede [1; Theorem A] (see also [2]).

THEOREM 2'. *Let* α *and* p *be as in* Theorem 1. *Let* E *be a set in* R_0^n *such* *that* $C_{1-\alpha/p,p}(E)=0$. *Then there exists a function* f *as in* Theorem 2.

PROOF. By our assumption that $C_{1-\alpha/p,p}(E)=0$, we can construct a non-negative function g in $L^p(R^n)$ such that $U^g_{1-\alpha/p}(x)=\infty$ for all $x\in E$. We set $f(x)=\int|x-y|^{1-n}|y_n|^{-\alpha/p}g(y)dy$. Then Lemma 6 implies that $\int|x_n|^\alpha|\text{grad} f|^p dx<\infty$. Noting that $|x-y|\geqq|y_n|$ for all $y\in R^n$ and all $x\in R^n_0$, we have $f(x)=\infty$ for all $x\in E$. We consider a mollified function as given by M. Ohtsuka [5]. He has shown that there exists a function $\beta\in C^\infty(R^n_+)$ having the following properties ([5; Lemma 2.10]):

i) $0<\beta<1$, ii) $|\text{grad}\,\beta|<1/2$, iii) $2\beta(x)<x_n$,

iv) $\omega(x)\leqq 2\omega(y)$ for any pair (x, y) such that $x\in R^n_+$ and $|x-y|<\beta(x)$, where $\omega(x)=x_n^\alpha$, $x\in R^n_+$.

Choosing a non-negative function ψ in $C_0^\infty(R^n)$ such that $\psi(x)=0$ if $|x|>1$ and $\int\psi(x)dx=1$, we define the mollified function F of f as follows:

$$F(x) = \int f(x+\beta(x)y)\psi(y)dy, \qquad x\in R^n_+.$$

Then $F\in C^\infty(R^n_+)$ and $\int_{R^n_+}x_n^\alpha|\text{grad}\,F|^p dx<\infty$ ([5; Theorem 4.4]). Since f is lower semi-continuous, $f(x)\to\infty$ as $x\to(x', 0)\in E$. Hence we easily see that $\lim_{x_n\downarrow 0}F(x', x_n)=\infty$ for $(x', 0)\in E$. Thus F is the required function.

Added in proof. After submitting this paper for publication, I found that A.A. Bagarshakyan (Sibirsk. Mat. Ž. 15 (1974), 1011–1020) had obtained a result similar to our Theorem 1, in which he characterizes the exceptional set for u in Theorem 1 by using a capacity different from the Riesz capacity.

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