HIROSHIMA MATH. J.
6 (1976), 55–59

On the Primary Decomposition of Differential Ideals

Shizuka SATÔ

(Received August 10, 1975)

Let R be a commutative ring containing an identity. Then a derivation on R is an abelian group homomorphism $D: R \to R$ such that for all a, b in R,

$$D(ab) = aD(b) + bD(a).$$

A higher derivation of rank m on R is a sequence of abelian group homomorphisms $\delta_i: R \to R$, $i=0, 1,\dots, m$ such that

(1) δ_0 is the identity mapping,

(2) for all x, y in R and $i \geqq 1$,

$$\delta_i(xy) = \sum_{j+k=i} \delta_j(x)\delta_k(y).$$

We shall let $Der(R)$ denote the collection of all derivations on R and let $H_m(R)$ denote the collection of all higher derivations of rank m on R. If m is infinite, we shall call this sequence merely a higher derivation. Let $\mathfrak{a}$ be an ideal of R and let T, G be subsets of $Der(R)$, $H_m(R)$ respectively. We shall say that $\mathfrak{a}$ is a T-ideal if $D(\mathfrak{a}) \subseteqq \mathfrak{a}$ for all $D \in T$. Similarly, we shall say that $\mathfrak{a}$ is a G-ideal if for all $\delta = \{\delta_i\} \in G$, $\delta_i(\mathfrak{a}) \subseteqq \mathfrak{a}$ for all $i=0, 1,\dots, m$. Let x be an element of R. We shall denote by $[x]$ the smallest G-ideal that contains the element x. This is well defined since the intersection of any number of G-ideals is again a G-ideal.

On the primary decomposition of differential ideals, the following is known:

THEOREM A (Theorem 1, [1]). *Let R be a Noetherian ring and let $\mathfrak{a}$ be an ideal of R with associated prime ideals $\mathfrak{p}_1, \mathfrak{p}_2,\dots, \mathfrak{p}_s$. Let $\delta = \{\delta_i\}$ be a higher derivation such that $\mathfrak{a}$ is a δ-ideal. Then $\mathfrak{p}_1, \mathfrak{p}_2,\dots, \mathfrak{p}_s$ are also δ-ideals and $\mathfrak{a}$ can be written as an irredundant intersection $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ of primary δ-ideals $\mathfrak{q}_i$.*

In this short paper, we wish to generalize Theorem A to the case of a set of higher derivations of rank m. Since in [1] they use essentially the fact that the rank is infinite, the method can not be used for the case of finite rank. So, we shall take up new techniques. We shall begin with the definition of G-primary G-ideals.

DEFINITION 1. *Let $\mathfrak{q}$ be a G-ideal of R. $\mathfrak{q}$ is called a G-primary G-ideal* *if the conditions* $\mathfrak{q}\supset\mathfrak{a}\mathfrak{b}$, $\mathfrak{q}\not\supset\mathfrak{a}$ *for G-ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of R always imply* $\mathfrak{q}\supset\mathfrak{b}^n$ *for some integer n.*

From Definition 1, it is immediate that primary G-ideals are G-primary. But the converse is not necessarily true. Here we shall give an example.

EXAMPLE 1. Let $R=k[X_1, X_2,\ldots]$ be a polynomial ring in infinitely many variables X_i $(i=1, 2,\ldots)$ over a field k of positive characteristic p. Let D be a k-derivation of R such that $D(X_i)=X_i$ for every i. Let $\mathfrak{a}$ be a D-ideal generated by the set $\{X_1^{2p}, X_i^i, X_1^pX_i^2, i=2, 3,\ldots\}$. Then $\mathfrak{a}$ is not a D-primary D-ideal but a primary D-ideal. For, since $\sqrt{\mathfrak{a}}=(X_i, i=1, 2, 3,\ldots)$, $\sqrt{\mathfrak{a}}$ is a maximal ideal and hence $\mathfrak{a}$ is primary. On the other hand, let $\mathfrak{b}=(X_1^p)$, $\mathfrak{c}=(X_1^p, X_i^2, i=2, 3,\ldots)$. Then $\mathfrak{b}$, $\mathfrak{c}^n$ (for any integer n) are D-ideals not contained in $\mathfrak{a}$. But $\mathfrak{b}\mathfrak{c}\subset\mathfrak{a}$. Therefore $\mathfrak{a}$ is not a D-primary D-ideal.

DEFINITION 2. *Let* $\mathfrak{q}$ *be a G-ideal of R.* $\mathfrak{q}$ *is called a G-irreducible G-ideal if* $\mathfrak{q}$ *can not be represented as an intersection of two G-ideals strictly containing* $\mathfrak{q}$.

LEMMA 1. *If* $\mathfrak{a}$, $\mathfrak{b}$ *are G-ideals of R, then* $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$ *and* $(\mathfrak{a}:\mathfrak{b})$ *are G-ideals.*

PROOF. It is straightforward that $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$ are G-ideals. Let $x\in(\mathfrak{a}:\mathfrak{b})$. We shall prove by the induction on i that for any $\delta=\{\delta_i\}\in G$, $\delta_i(x)\in(\mathfrak{a}:\mathfrak{b})$, $i=0, 1,\ldots, m$. The case $i=1$ is immediate from $\mathfrak{a}\in\delta_1(xb)=x\delta_1(b)+\delta_1(x)b$ for any $b\in\mathfrak{b}$. By the definition, we have

$$\delta_i(xb) = \delta_i(x)b+\sum_{k=0}^{i-1}\delta_k(x)\delta_{i-k}(b).$$

By the induction assumption and $\delta_{i-k}(b)\in\mathfrak{b}$, we obtain $\delta_i(x)b\in\mathfrak{a}$ for any $b\in\mathfrak{b}$. Therefore $\delta_i(x)\in(\mathfrak{a}:\mathfrak{b})$ and $(\mathfrak{a}:\mathfrak{b})$ is a G-ideal. Q.E.D.

LEMMA 2. *Let* $\mathfrak{q}$ *be a G-ideal of a Noetherian ring R and let x be an element of R such that* $x\notin\sqrt{\mathfrak{q}}$. *Then for some integer k,* $(\mathfrak{q}:x^k)=(\mathfrak{q}:x^i)$ *for all* $i\geqq k$ *and* $(\mathfrak{q}:x^k)$ *is a G-ideal.*

PROOF. Let $\mathfrak{q}=\bigcap_{i=1}^{n}\mathfrak{q}_i$ be an irredundant primary decomposition. We may suppose that $x\notin\sqrt{\mathfrak{q}_i}$ $(i=1, 2,\ldots, t)$, $x\in\sqrt{\mathfrak{q}_j}$ $(j=t+1,\ldots, m)$. Then $(\mathfrak{q}:x^k)=\bigcap_{i=1}^{t}\mathfrak{q}_i$ for some k and this is the first part of this assertion. Next, let $y\in(\mathfrak{q}:x^k)$. We shall prove by the induction on i that for any $\delta=\{\delta_i\}\in G$, $\delta_i(y)\in(\mathfrak{q}:x^k)$ $(i=0, 1,\ldots, m)$. The case $i=1$ follows from $\delta_1(x^ky)=\delta_1(x^k)y+\delta_1(y)x^k$ and $(\mathfrak{q}:x^k)=(\mathfrak{q}:x^i)$ for $i\geqq k$. The equation

$$\delta_i(x^ky) = \delta_i(y)x^k+\sum_{j=0}^{i-1}\delta_j(y)\delta_{i-j}(x^k)$$

implies $\delta_i(y)x^{k^2} \in \mathfrak{q}$ by the induction assumption. Therefore we have $\delta_i(y) \in (\mathfrak{q}: x^{k2}) = (\mathfrak{q}: x^k)$ and this completes the proof. Q. E. D.

LEMMA 3. *Let* $\mathfrak{q}$ *be a G-primary G-ideal of a Noetherian ring R. Then* $\mathfrak{q}$ *is a primary G-ideal.*

PROOF. Let $xy \in \mathfrak{q}$ and $x \notin \sqrt{\mathfrak{q}}$ for x, y in R. Then by Lemma 2, $(\mathfrak{q}: x^k)$ for some integer k is a G-ideal. Thus $[y] \subset (\mathfrak{q}: x^k)$ and hence $x^k \in (\mathfrak{q}: [y])$. By Lemma 1, we have $[x^k] \subset (\mathfrak{q}: [y])$ and $[x^k][y] \subset \mathfrak{q}$. Since $\mathfrak{q}$ is G-primary and $x \notin \sqrt{\mathfrak{q}}$, we obtain $y \in \mathfrak{q}$ and $\mathfrak{q}$ is primary. Q. E. D.

LEMMA 4. *Let* $\mathfrak{q}$ *be a G-irreducible G-ideal of a Noetherian ring R. Then* $\mathfrak{q}$ *is G-primary.*

PROOF. Let $\mathfrak{a}$, $\mathfrak{b}$ be G-ideals of R such that $\mathfrak{a}\mathfrak{b} \subseteqq \mathfrak{q}$ and $\mathfrak{a}^n \nsubseteqq \mathfrak{q}$ for any integer n. Let $\mathfrak{q} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be an irredundant primary decomposition. From $\mathfrak{a} \nsubseteqq \sqrt{\mathfrak{q}}$, we may suppose that $\mathfrak{a} \nsubseteqq \sqrt{\mathfrak{q}_i}$ $(1 \leqq i \leqq t)$ and $\mathfrak{a} \subseteqq \sqrt{\mathfrak{q}_j}$ $(t+1 \leqq j \leqq n)$. If $t = n$, we have $(\mathfrak{q}: \mathfrak{a}) = \bigcap_{i=1}^{n} \mathfrak{q}_i = \mathfrak{q}$ and $\mathfrak{b} \subseteqq \mathfrak{q}$. Suppose $t \neq n$. Then since R is Noetherian, there is an integer M such that $\mathfrak{a}^M \subseteqq \mathfrak{q}_j$ $(t+1 \leqq j \leqq n)$. Therefore,

$$\mathfrak{q} \subseteqq (\mathfrak{q}: \mathfrak{a}^M) \cap (\mathfrak{q}+\mathfrak{a}^M) \subseteqq \left(\bigcap_{i=1}^{t} \mathfrak{q}_i\right) \cap \left(\bigcap_{j=t+1}^{n} \mathfrak{q}_j\right) = \mathfrak{q}$$

and hence $\mathfrak{q} = (\mathfrak{q}: \mathfrak{a}^M) \cap (\mathfrak{q}+\mathfrak{a}^M)$. On the other hand, we have $(\mathfrak{q}: \mathfrak{a}^M) \supseteqq (\mathfrak{q}+\mathfrak{b})$. So, we have $\mathfrak{q} = (\mathfrak{q}+\mathfrak{b}) \subseteqq (\mathfrak{q}+\mathfrak{a}^M)$. From Lemma 1, $(\mathfrak{q}+\mathfrak{b})$ and $(\mathfrak{q}+\mathfrak{a}^M)$ are G-ideals. Since $\mathfrak{q}$ is G-irreducible and $\mathfrak{a}^M \nsubseteqq \mathfrak{q}$, we must have $(\mathfrak{q}+\mathfrak{b}) = \mathfrak{q}$ and hence $\mathfrak{b} \subseteqq \mathfrak{q}$. Thus $\mathfrak{q}$ is G-primary. Q. E. D.

THEOREM. *Let R be a Noetherian ring. Then every G-ideal can be represented as the intersection of a finite number of primary G-ideals of R.*

PROOF. The routine technique shows that every G-ideal can be represented as the intersection of a finite number of G-irreducible G-ideals. Then this theorem follows from Lemma 3 and 4. Q. E. D.

COROLLARY 1. *Let R be a Noetherian ring. Then every T-ideal can be represented as the intersection of a finite number of primary T-ideals of R.*

Let $\mathfrak{q}$ be a T-ideal (G-ideal). Then the radical $\sqrt{\mathfrak{q}}$ is not necessarily a T-ideal (G-ideal). But we shall show that if R is a ring of characteristic 0, a certain T-ideal has this property and furthermore we shall show that for a family G of higher derivations the radical of a G-ideal is a G-ideal.

LEMMA 5. *Let R be a ring of characteristic* 0 *and* $\mathfrak{a}$ *be a T-ideal of R.*

If $x^n \in \mathfrak{a}$ for an integer n, then we have

$$(n!/(n-i)!)x^{n-i}D(x)^{2i-1} \in \mathfrak{a} \qquad (0 \leqq i \leqq n)$$

for any derivation D in T. In particular, $n!D(x)^{2n-1} \in \mathfrak{a}$.

PROOF. We shall prove the Lemma by the induction on i. The case $i=1$ is immediate. Now assume the Lemma for the case $i=k$. Since $\mathfrak{a}$ is a T-ideal, we have

$$(n!/(n-k)!)D(x^{n-k}D(x)^{2k-1}) \in \mathfrak{a}$$

for any derivation D in T. Hence,

$$(n!/(n-k-1)!)x^{n-k-1}D(x)^{2k} + (n!/(n-k)!)(2k-1)x^{n-k}D(x)^{2k-2}D^2(x) \in \mathfrak{a}.$$

Therefore multiplying $D(x)$ on the left hand side, we have $(n!/(n-k-1)!)x^{n-k-1}D(x)^{2k+1} \in \mathfrak{a}$. This completes the proof. Q.E.D.

PROPOSITION 1. *Let R be a ring of characteristic 0 and let $\mathfrak{q}$ be a primary T-ideal of R not containing non zero integers. Then $\sqrt{\mathfrak{q}}$ is a prime T-ideal.*

PROOF. Let $x \in \sqrt{\mathfrak{q}}$. Then there is an integer n such that $x^n \in \mathfrak{q}$. By Lemma 5, we have $n!\,D(x)^{2n-1} \in \mathfrak{q}$. Since $\mathfrak{q}$ does not contain non zero integers and $\mathfrak{q}$ is primary, we have $D(x)^{2n-1} \in \mathfrak{q}$ and $\sqrt{\mathfrak{q}}$ is a T-ideal. Q.E.D.

REMARK 1. Theorem 1 in [4] is an immediate consequence of Corollary 1 and Proposition 1.

DEFINITION 3. *A T-ideal (G-ideal) $\mathfrak{p}$ is called a T-prime T-ideal (G-prime G-ideal) if the relation $\mathfrak{a}\mathfrak{b} \subseteqq \mathfrak{p}$ for T-ideals (G-ideals) $\mathfrak{a}$, $\mathfrak{b}$ of R implies either $\mathfrak{a} \subseteqq \mathfrak{p}$ or $\mathfrak{b} \subseteqq \mathfrak{p}$.*

Let $r(\mathfrak{q})$ be a T-ideal (G-ideal) generated by the T-ideals (G-ideals) $\mathfrak{a}$ such that $\mathfrak{a}^n \subseteqq \mathfrak{q}$ for some integer n. From Definition it follows that for a primary T-ideal (G-ideal) $\mathfrak{q}$, $r(\mathfrak{q})$ is a T-prime T-ideal (G-prime G-ideal).

PROPOSITION 2. *Let R be a Noetherian ring and let G be a family of higher derivations of R. Then a G-prime G-ideals $\mathfrak{p}$ is a prime ideal.*

REMARK. Let p be the characteristic of R. (1) the case $p>0$. It is sufficient to prove $\mathfrak{p} = \sqrt{\mathfrak{p}}$. Let $x^m \in \mathfrak{p}$ for an integer m and let t be an integer such that $p^t \geqq m$. For any $\delta \in G$, we have

$$\mathfrak{p} \in \delta_{i_p}(x^{p^t}) = \delta_i(x)^{p^t} \qquad \text{for every} \quad i.$$

Therefore $\sqrt{\mathfrak{p}}$ is a G-ideal. Since R is Noetherian, we have $\mathfrak{p}^N \subseteqq \mathfrak{p}$ for some integer N. By the assumption, we have $\mathfrak{p} = \sqrt{\mathfrak{p}}$. (2) the case $p=0$. Let S be the set of non zero integers. Then δ is uniquely extended to a higher derivation $\bar{\delta}$ of R_S. Furthermore $\mathfrak{p}R_S$ is a $\bar{\delta}$-prime $\bar{\delta}$-ideal. Since R_S contains the rational number field, there is a bijection from a set of higher derivations to a set of infinite sequences of derivations ([2]). Therefore $\mathfrak{p}R_S$ is a T-prime T-ideal for some set T of derivations. Hence by Proposition 1 $\mathfrak{p}R_S$ is a prime ideal and also $\mathfrak{p}=\mathfrak{p}R_S \cap R$ is a prime ideal. Q. E. D.

REMARK 2. Theorem 1 in [1] is an immediate consequence Theorem and Proposition 2.

PROPOSITION 3. *Let R be a Noetherian ring such that* (0) *is the only one $H_m(R)$-ideal in R for $m<\infty$. Then if the characteristic p of R is positive, R is either a field or a primary ring in which every element is nilpotent. If the characteristic of R is* 0, *R is an integral domain.*

PROOF. The case $p\neq 0$. Suppose R is not a field. It follows that (0) is primary by Theorem and the assumption. Let x be a non-unit of R. Then (x^{p^n}) is a $H_m(R)$-ideal for some integer n and so we must have $x^{p^n}=0$. Thus the set of non-unit elements is an ideal and hence R is a local ring. Since any non-unit element is nilpotent, R is primary. The case $p=0$. Since (0) is the only one $H_m(R)$-ideal, R contains the rational number field. In this case $H_m(R)$-ideals are Der (R)-ideals and vice versa ([2]). Therefore, by Corollary of Theorem 1 in [4], R is an integral domain. Q. E. D.

References

[1] W. C. Brown and W. E. Kuan: Ideals and higher derivations in commutative rings, Canad. J. Math. **24** (1972), 400–415.

[2] N. Heerema: Derivations and embeddings of a field in its power series ring, Proc. Amer. Math. Soc. **11** (1960), 188–194.

[3] I. Kaplansky: An introduction to differential algebra, Herman, 1957.

[4] A. Seidenberg: Differential ideals in rings of finitely generated type, Amer. J. Math. **89** (1967), 22–42.

Department of Mathematics,
Faculty of Engineering,
ôita University, ôita