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A Semigroup Treatment of the Hamilton-Jacobi Equation in Several Space Variables

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1. Introduction

This paper is a sequel to our earlier paper [1] and presents a semigroup treatment of the Cauchy problem (hereafter called (CP)) for the Hamilton-Jacobi equation

(DE) $$u_t+f(u_x)=0, \qquad x \in R^n, \quad t>0.$$

Here $u(x, t)$ is a real-valued function, $f: R^n \to R^1$, $u_t = \partial u/\partial t$, and u_x denotes the gradient $(\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ in the space variables x.

Multi-dimensional equations of Hamilton-Jacobi type have been treated by several authors in several ways (for instance, see A. Douglis [8, p. 203 and Bibliography]). Speaking of (CP), global generalized solutions with uniformly Lipschitz continuous initial data have been obtained when $f: R^n \to R^1$ is strictly convex. Uniqueness theorems have been given by A. Douglis [7] and S. N. Kružkov [10]. Nevertheless, it seems worthwhile to add a new method for treating (CP), to which this paper is devoted. Besides, from the viewpoint of semigroup theory, the Hamilton-Jacobi equation (DE) provides an example of nonlinear semigroups in nonreflexive Banach spaces that are not differentiable.

In the present paper we shall, as before [1], choose $L^\infty(R^n)$ as the Banach space that may be associated with (CP), and construct a semigroup of contractions on the subspace of $L^\infty(R^n)$ consisting of all bounded and uniformly continuous functions on R^n. As we shall see, the semigroup approach enables us to treat (CP) under the assumption that $f: R^n \to R^1$ is merely convex and of class C^2 (Note that we do not assume the strict convexity of f). Moreover, as an intermediate step in the development, the existence and uniqueness of certain bounded (possibly generalized) solutions will be established for the equation

(1) $$u+f(u_x)=h(x), \qquad x \in R^n,$$

for given h.

We start, in Section 2, with a definition of the operator $A: v \to f(v_x)$ in $L^\infty(R^n)$ that may be associated with (CP). Section 3 concerns the existence and uniqueness of certain bounded generalized solutions of (1). Here the solutions are obtained as limits of bounded solutions of the regularized elliptic equations

(2) $$u+f(u_x)-\varepsilon\Delta u = h(x), \qquad x \in R^n,$$

as $\varepsilon \downarrow 0$. Various results concerning (2) are obtained as needed. Section 4 is devoted to the construction of a semigroup of contractions generated by A through the generation theorem of M. G. Crandall and T. M. Liggett [5] and to the study of its properties relating to (CP).

2. Definition of the operator $A: v \to f(v_x)$

Throughout the present paper we shall work in the Banach space $L^\infty(R^n)$ of all real-valued, bounded and measurable functions v on the real n-dimensional Euclidean space R^n with norm

$$\|v\|_\infty = \text{essential sup}\,\{|v(x)|;\, x \in R^n\}.$$

$W_k^\infty(R^n)$ denotes the subspace of $L^\infty(R^n)$ consisting of all measurable functions whose distribution derivatives of order at most k lie in $L^\infty(R^n)$. Thus, in particular, $W_1^\infty(R^n)$ is the subspace of all bounded and uniformly Lipschitz continuous functions on R^n. For $v \in W_1^\infty(R^n)$ we set

$$\|v_x\|_\infty = \Big(\sum_{i=1}^n \|\partial v/\partial x_i\|_\infty^2\Big)^{1/2}.$$

We shall assume that the function $f: R^n \to R^1$ appearing in (DE) is convex. (Note that a finite convex function on R^n is necessarily continuous.) Corresponding to this assumption, we need a subclass of $W_1^\infty(R^n)$:

$E(R^n)$ denotes the subset of $L^\infty(R^n)$ consisting of all bounded and uniformly Lipschitz continuous functions v on R^n such that v satisfies the following semiconcavity condition:

(SC) $$v(x+y)+v(x-y)-2v(x) \leq k|y|^2, \qquad x,\, y \in R^n,$$

for some constant $k \geq 0$.

A function $v: R^n \to R^1$ that satisfies the semiconcavity condition (SC) is called a semiconcave function, and we let $|v|_E$ denote the infimum of such constants k.

Our first task is to define an operator A_0 associated with (CP) in $L^\infty(R^n)$.

DEFINITION 2.1. *Let $f: R^n \to R^1$ be convex. A_0 is the operator in $L^\infty(R^n)$ defined by: $v \in D(A_0)$, $w = A_0 v$ if*

(i) *$v \in E(R^n)$, $w \in W_1^\infty(R^n)$, and $w = f(v_x)$, that is, the equality $w(x) = f(v_x(x))$ holds at almost all points of R^n, and*

(ii) *there is a positive number λ_0 depending upon v for which $v + \lambda_0 w \in$*

$E(R^n)$.

The next lemma will clarify our definition of the operator A_0.

LEMMA 2.1. *Let $f \in C^2$ and let A_0 be given by Definition 2.1. If $v \in W_3^\infty(R^n)$, then $v \in D(A_0)$ and $A_0 v = f(v_x)$.*

PROOF. Let $f \in C^2$. If $v \in W_3^\infty(R^n)$, then both v and $w = f(v_x)$ lie in $W_2^\infty(R^n)$. It is easily shown that every function in $W_2^\infty(R^n)$ is semiconcave, and that a linear combination of semiconcave functions with positive coefficients again is semiconcave. Hence, by definition, $v \in D(A_0)$ and $A_0 v = f(v_x)$, which proves the lemma.

We are now in a position to define an operator A in $L^\infty(R^n)$ that may be multi-valued for general convex f.

DEFINITION 2.2. *A is the closure of A_0, i.e., $v \in D(A)$ and $w \in Av$ if there is a sequence $\{v^m\} \subset D(A_0)$ such that $v^m \to v$, $A_0 v^m \to w$ in $L^\infty(R^n)$.*

3. The equation $u + f(u_x) = h$

Our object in this section is to establish the existence and uniqueness of certain bounded generalized solutions of the equation

$$u + f(u_x) = h(x), \qquad x \in R^n, \tag{3.1}$$

where h is a given function, under the assumption that $f: R^n \to R^1$ is convex. For the sake of simplicity the normalization

$$f(0) = 0 \tag{3.2}$$

will be assumed throughout this section, for this can always be achieved by introducing the new unknown $\bar{u} = u + f(0)$.

DEFINITION 3.1. *Let $f: R^n \to R^1$ be convex and let A_0 be given by Definition 2.1. Let $h \in W_1^\infty(R^n)$. Then a function $u \in E(R^n)$ is called a bounded generalized solution of (3.1) provided $u \in D(A_0)$ and $u + A_0 u = h$.*

Our results concerning equation (3.1) are stated in the following three theorems.

THEOREM 3.1 (*Existence*). *Let $f: R^n \to R^1$ be convex, and the normalization (3.2) be assumed. Then we have $R(I + A_0) \supset E(R^n)$, i.e., for each $h \in E(R^n)$ there is a bounded generalized solution u of (3.1) such that*

$$\|u\|_\infty \le \|h\|_\infty, \qquad \|u_x\|_\infty \le \|h_x\|_\infty, \tag{3.3}$$

and, for the semiconcavity constant of u,

$$|u|_E \leq |h|_E. \tag{3.4}$$

THEOREM 3.2 (*Uniqueness*). *Let $f: R^n \to R^1$ be convex and of class C^2. Then there exists at most one bounded generalized solution of* (3.1).

THEOREM 3.3. *Under the assumptions of Theorem* 3.2, *let $u, v \in D(A_0)$ satisfy the equations*

$$u + \lambda A_0 u = h, \qquad v + \lambda A_0 v = g, \tag{3.5}$$

respectively, where λ is an arbitrary number such that $0 < \lambda \leq 1$. If both $u + A_0 u$ and $v + A_0 v$ lie in $E(R^n)$, then:

(i) $\|u - v\|_\infty \leq \|h - g\|_\infty$.

(ii) *If $g \geq h$, then $v \geq u$.*

First we shall take up the problem of uniqueness of bounded generalized solutions. The following proof has been given in our recent paper [2].

PROOF OF THEOREM 3.2. To prove the theorem by contradiction let u and v be two bounded generalized solutions of (3.1). For u and v, let U denote a common absolute bound in R^n, let P be a common Lipschitz constant, and let k be a common semiconcavity constant. We set

$$K_1 = \sup\{(\sum_{i=1}^{n} (f_{p_i}(p))^2)^{1/2};\ |p| \leq P\}$$

and

$$K_2 = \sup\{\sum_{i=1}^{n} f_{p_i p_i}(p);\ |p| \leq P\}.$$

Since

$$u + f(u_x) = h(x), \qquad v + f(v_x) = h(x),$$

a. e. in R^n, the difference $w = u - v$ satisfies the equation

$$w + \sum_{i=1}^{n} G_i w_{x_i} = 0$$

a. e. in R^n, where

$$G_i = G_i(u, v) = \int_0^1 f_{p_i}(v_x + \theta(u_x - v_x))d\theta, \qquad i = 1,\dots, n.$$

If we set $W = w^q$, where q is an even integer, we have

(3.6) $$qW+\sum_{i=1}^{n}G_iW_{x_i}=0$$

a. e. in R^n.

By convolving u and v with mollifying kernels, we can find two approximating sequences $\{u^m\}$ and $\{v^m\}$ of C^2 functions, each having the same absolute bound U, Lipschitz constant P and semiconcavity constant k as u and v, such that the sequences $\{u_x^m\}$ and $\{v_x^m\}$ converge a. e. in R^n to u_x and v_x respectively. If we set

$$G_i^m=G_i(u^m,v^m),\qquad i=1,\dots,n,$$

then equation (3.6) can be written as

(3.7) $$qW+\sum_{i=1}^{n}(G_i^mW)_{x_i}=\sum_{i=1}^{n}(G_i^m-G_i)W_{x_i}+W\sum_{i=1}^{n}(G_i^m)_{x_i}.$$

Let r be an arbitrary positive number, and we integrate the both sides of (3.7) over the ball $|x|\le r$. We thus get

(3.8) $$q\int_{|x|\le r}Wdx+\int_{|x|=r}W\sum_{i=1}^{n}G_i^m\cos(n,x_i)dS$$
$$=\int_{|x|\le r}\sum_{i=1}^{n}(G_i^m-G_i)W_{x_i}dx+\int_{|x|\le r}W\sum_{i=1}^{n}(G_i^m)_{x_i}dx,$$

where n is the outer normal to the sphere S: $|x|=r$ and dS is the surface element. On the other hand, we have

$$\int_{|x|=r}W\sum_{i=1}^{n}G_i^m\cos(n,x_i)dS\ge -K_1\int_{|x|=r}WdS$$

and

$$\int_{|x|\le r}W\sum_{i=1}^{n}(G_i^m)_{x_i}dx\le kK_2\int_{|x|\le r}Wdx,$$

since

$$\sum_{i=1}^{n}(G_i^m)_{x_i}$$
$$=\sum_{i,j=1}^{n}\left(u_{x_ix_j}^m\int_0^1\theta f_{p_ip_j}(\cdots)d\theta+v_{x_ix_j}^m\int_0^1(1-\theta)f_{p_ip_j}(\cdots)d\theta\right)$$

where $(\cdots)=(v_x^m+\theta(u_x^m-v_x^m))$. (Note that, by virtue of the convexity of f and the semiconcavity condition (SC),

$$\sum_{i,j=1}^{n}u_{x_ix_j}^m f_{p_ip_j}(\cdots)=\mathrm{tr}\,[(M-kI)F]+k\sum_{i=1}^{n}f_{p_ip_i}(\cdots)$$

$$\leq k\sum_{i=1}^{n} f_{p_i p_i}(\cdots),$$

M and F denoting the matrices $(u^m_{x_i x_j})$ and $(f_{p_i p_j}(\cdots))$ respectively.)

Substituting these inequalities into (3.8), we get

$$q\int_{|x|\leq r} W dx - K_1 \int_{|x|=r} W dS$$
$$\leq \int_{|x|\leq r} \sum_{i=1}^{n} (G^m_i - G_i) W_{x_i} dx + kK_2 \int_{|x|\leq r} W dx$$

and, hence, by letting m tend to infinity and using the bounded convergence theorem

$$q\int_{|x|\leq r} W dx - K_1 \int_{|x|=r} W dS \leq kK_2 \int_{|x|\leq r} W dx. \tag{3.9}$$

If we set

$$I(r) = \int_{|x|\leq r} W dx \qquad \text{for} \quad r > 0$$

and choose an even integer q so large that $q > kK_2$, then inequality (3.9) can be written as a differential inequality for $I(r)$

$$aI(r) - dI(r)/dr \leq 0, \qquad r > 0, \tag{3.10}$$

where $a = (q - kK_2)/K_1$ is a positive constant.

Now suppose that there is a positive number r_0 for which $I(r_0) > 0$. Then the differential inequality (3.10) gives a lower bound $I(r_0)\exp(a(r - r_0))$ for the growth order of $I(r)$ as r tends to infinity. But this is a contradiction, since the integral $I(r)$ increases at most polynomially with r because of the boundedness of $W = w^q$. Therefore, $I(r) = 0$ for $r \geq 0$ and, hence, the difference $w = u - v$ must vanish identically on R^n. This completes the proof.

Next we shall proceed to the problem of existence of bounded generalized solutions and their properties. As was stated in the introduction, the bounded generalized solution of (3.1) will be obtained as a limit of bounded solutions of the regularized elliptic equations

$$u + f(u_x) - \varepsilon \Delta u = h(x), \qquad x \in R^n, \tag{3.11}$$

as $\varepsilon \downarrow 0$. Consequently, in order to prove Theorems 3.1 and 3.3, it will suffice to prove the corresponding results for bounded solutions of (3.11). To this end, we shall use a result of T. Kusano [11, Th. 1, p. 2] that is a variant of the maximum principle. We shall state it in a form suitable for our later use and give a proof for the sake of completeness.

LEMMA 3.1. *Let* $a_i: R^n \to R^1$, $i=1,\dots,n$, *be bounded. If* $v \in C^2(R^n)$ *is bounded from above and satisfies the inequality*

$$Lv = v + \sum_{i=1}^{n} a_i(x) v_{x_i} - \varepsilon \Delta v \leq 0, \qquad x \in R^n,$$

where ε *is an arbitrary positive number, then* $v \leq 0$ *on* R^n.

PROOF. To prove the lemma by contradiction, suppose there is a point x^0 such that $v(x^0)>0$. Set

$$w = (v(x^0) - \eta) \prod_{i=1}^{n} \cosh[k(x_i - x_i^0)],$$

where $\eta (0<\eta<v(x^0))$ and k are positive constants. A simple calculation shows that we can choose a sufficiently small k in such a way that $Lw>0$ on R^n. Then we have $L(v-w)<0$ and, hence, $v-w$ can not have a positive maximum at finite points of R^n. But this contradicts the fact that $v-w>0$ at x^0 and $v-w \to -\infty$ as $|x| \to \infty$.

PROPOSITION 3.1. *Let* $f: R^n \to R^1$ *be continuous and let* $u, v \in C^2(R^n) \cap W_1^\infty(R^n)$ *satisfy the equations*

$$u + f(u_x) - \varepsilon \Delta u = h, \qquad v + f(v_x) - \varepsilon \Delta v = g, \tag{3.12}$$

respectively, where ε *is an arbitrary positive number. If* $h, g \in L^\infty(R^n)$, *then*:

(i) $\|u-v\|_\infty \leq \|h-g\|_\infty$.

(ii) *If* $g \geq h$, *then* $v \geq u$.

PROOF. We shall only give a proof of the first part (i), since the second part (ii) can be proved quite similarly.

First Step. Let $f \in C^1$. Then the difference $w=u-v$ satisfies the equation

$$\begin{aligned} Lw &\equiv w + f(u_x) - f(v_x) - \varepsilon \Delta w \\ &= w + \sum_{i=1}^{n} f_{p_i}(v_x + \theta(u_x - v_x)) w_{x_i} - \varepsilon \Delta w \\ &= w + \sum_{i=1}^{n} a_i(x) w_{x_i} - \varepsilon \Delta w = h - g, \end{aligned}$$

where $0<\theta=\theta(x)<1$ and the $a_i: R^n \to R^1$ are bounded. Hence an application of Lemma 3.1 yields

$$\|w\|_\infty = \|u-v\|_\infty \leq \|h-g\|_\infty,$$

since

$$L(\pm w-\|h-g\|_\infty)=\pm(h-g)-\|h-g\|_\infty\le 0.$$

Second Step. Let $f\in C$ and let P be a common Lipschitz constant for u and v. Convolving f with mollifying kernels, we can find a sequence $\{f_m\}$ of C^1 functions satisfying

$$|f_m(p)-f(p)|\le 1/m, \qquad m=1, 2,\dots$$

for every p such that $|p|\le P$. Then, since

$$u+f_m(u_x)-\varepsilon\Delta u = h+f_m(u_x)-f(u_x),$$

$$v+f_m(v_x)-\varepsilon\Delta v = g+f_m(v_x)-f(v_x),$$

and

$$\|h-g+f_m(u_x)-f(u_x)-(f_m(v_x)-f(v_x))\|_\infty\le\|h-g\|_\infty+2/m,$$

we have by the result of the first step

$$\|u-v\|_\infty\le\|h-g\|_\infty+2/m$$

and, hence, by letting m tend to infinity, we obtain (i). Thus the proof is complete.

Immediate consequences of Proposition 3.1 are:

COROLLARY 3.1. *Let $f\colon R^n\to R^1$ be continuous. Then for each $h\in L^\infty(R^n)$ there exists at most one bounded solution $u\in C^2(R^n)\cap W_1^\infty(R^n)$ of* (3.11).

COROLLARY 3.2. *Under the assumption of Corollary* 3.1, *let $u\in C^2(R^n)\cap W_1^\infty(R^n)$ satisfy* (3.11). *If $h\in W_1^\infty(R^n)$, then $\|u_x\|_\infty\le\|h_x\|_\infty$. Moreover, if the normalization* (3.2) *is assumed, then $\|u\|_\infty\le\|h\|_\infty$.*

PROPOSITION 3.2. *Let $f\colon R^n\to R^1$ be convex and let $u\in C^2(R^n)\cap W_1^\infty(R^n)$ satisfy* (3.11). *If $h\in E(R^n)$, then $|u|_E\le|h|_E$.*

PROOF. The convexity of f implies that to each point p^0 in R^n there corresponds a real vector $(a_1,\dots, a_n)$ such that the inequality

$$f(p)-f(p^0)\ge\sum_{i=1}^n a_i(p_i-p_i^0) \tag{3.13}$$

holds for all points p of R^n. We note that when p varies on a bounded subset of R^n the set of corresponding vectors $(a_1,\dots, a_n)$ forms a bounded set in R^n.

Let y be a fixed vector, and we set

$$v(x) = u(x+y)+u(x-y)-2u(x).$$

Then, since we have

$$f(u_x(x+y))-f(u_x(x)) \geq \sum_{i=1}^{n} a_i(x)(u_{x_i}(x+y)-u_{x_i}(x))$$

and

$$f(u_x(x-y))-f(u_x(x)) \geq \sum_{i=1}^{n} a_i(x)(u_{x_i}(x-y)-u_{x_i}(x)),$$

where the $a_i(x)$ form a vector corresponding to the point $p^0=u_x(x)$ for which (3.13) holds, the function v satisfies the differential inequality

$$\begin{aligned} Lv &= v+\sum_{i=1}^{n} a_i(x)v_{x_i}-\varepsilon\Delta v \\ &\leq h(x+y)+h(x-y)-2h(x) \leq |h|_E|y|^2 \end{aligned}$$

on the whole of R^n. Here the $a_i\colon R^n\to R^1$ are bounded, since $p^0=u_x(x)$ varies on a subset of the ball $|p^0|\leq\|h_x\|_\infty$ in view of Corollary 3.2. Hence an application of Lemma 3.1 again yields

$$v(x) = u(x+y)+u(x-y)-2u(x) \leq |h|_E|y|^2$$

for each fixed y. But this implies that $u\in E(R^n)$ and $|u|_E\leq|h|_E$. The proof is complete.

Existence theorems for bounded solutions on the whole space R^n of second order elliptic equations have been given by K. Akô and T. Kusano [3] under the assumption that there exist a bounded superfunction and a bounded subfunction. Using their result, we can prove the

PROPOSITION 3.3. *Let $f\colon R^n\to R^1$ be locally Hölder continuous and the normalization* (3.2) *be assumed. Then for each $h\in W_1^\infty(R^n)$ there is a bounded solution $u\in C^2(R^n)\cap W_1^\infty(R^n)$ of* (3.11) *such that $\|u\|_\infty\leq\|h\|_\infty$ and $\|u_x\|_\infty\leq\|h_x\|_\infty$.*

PROOF. Let $F\colon R^n\to R^1$ be a bounded and locally Hölder continuous function such that $F(p)=f(p)$ when $|p|\leq\|h_x\|_\infty$, and consider the equation

$$u+F(u_x)-\varepsilon\Delta u = h(x), \qquad x\in R^n. \tag{3.14}$$

Then the constant functions $\|h\|_\infty$ and $-\|h\|_\infty$ are respectively a bounded superfunction and a bounded subfunction of this equation. By a result of [3, Th. 1, p. 30], there is a bounded solution $u\in C^2(R^n)$ of the equation (3.14). For this solution u we have

$$\|u\|_\infty \leq \|h\|_\infty \quad \text{and} \quad \|u_x\|_\infty \leq \|h_x\|_\infty$$

by virtue of Corollary 3.2. Thus, in particular, u is also a bounded solution in $C^2(R^n) \cap W_1^\infty(R^n)$ of the original equation (3.11). Thus the proof is complete.

PROPOSITION 3.4. *Let $f: R^n \to R^1$ be convex and the normalization* (3.2) *be assumed. Then for each $h \in E(R^n)$ there is a bounded solution $u \in C^2(R^n) \cap E(R^n)$ such that*

$$\|u\|_\infty \leq \|h\|_\infty, \qquad \|u_x\|_\infty \leq \|h_x\|_\infty, \tag{3.15}$$

and, for the semiconcavity constant of u,

$$|u|_E \leq |h|_E. \tag{3.16}$$

PROOF. This follows immediately from Propositions 3.2 and 3.3, since every convex function $f: R^n \to R^1$ is locally Lipschitz continuous.

To prove the existence theorem for bounded generalized solutions of (3.1), we need a lemma concerning the convergence of a sequence of semiconcave functions. The following result has been observed by A. Douglis (cf. [7], [8]).

LEMMA 3.2. *Let u^m, $m=1, 2,\ldots$, be a sequence of functions in $E(R^n)$ such that*

$$\|u^m\|_\infty \leq U, \qquad \|u_x^m\|_\infty \leq P, \qquad |u^m|_E \leq K, \qquad m = 1, 2,\ldots \tag{3.17}$$

where U, P, and K are uniform positive constants. If the sequence $\{u^m\}$ converges to a limit u uniformly on compact sets, then the limit u satisfies the three inequalities (3.17) *with u^m replaced by u. Moreover, the sequence $\{u_x^m\}$ converges to u_x at almost all points of R^n.*

PROOF. This lemma is a version of a result stated in the work of A. Douglis (for instance, see [7, Th. 2.3, p. 11]).

PROOF OF THEOREM 3.1. Given $h \in E(R^n)$, let $u^m \in C^2(R^n) \cap E(R^n)$ be the unique bounded solution of the equation

$$u + f(u_x) - (1/m)\Delta u = h(x), \qquad x \in R^n,$$

guaranteed by Proposition 3.4. Then the estimates (3.15) and (3.16) imply that the sequence $\{u^m\}$ satisfies (3.17) with $U = \|h\|_\infty$, $P = \|h_x\|_\infty$, and $K = |h|_E$. Arzela's theorem asserts that there is a subsequence $\{u^{m(i)}\}$ of $\{u^m\}$ converging uniformly on compact sets to a limit u. By Lemma 3.2, the subsequence $\{u_x^{m(i)}\}$ then converges to u_x a.e. in R^n. We denote this convergence in $W_1^\infty(R^n)$ by $\twoheadrightarrow$, $u^{m(i)} \twoheadrightarrow u$. Obviously, the limit u enjoys the properties (3.3) and (3.4).

We shall show that the limit u satisfies (3.1) a.e. in R^n. To see this, let $\varphi \in C_0^\infty(R^n)$. Multiplying the equation satisfied by u^m by φ and integrating we have

$$\int_{R^n} \{(u^m + f(u_x^m))\varphi + (1/m)\sum_{i=1}^n u_{x_i}^m \varphi_{x_i}\} dx = \int_{R^n} h\varphi dx.$$

Letting m tend to infinity through the subsequence $\{m(i)\}$ and using the convergence $u^{m(i)} \twoheadrightarrow u$, we obtain

$$\int_{R^n} (u + f(u_x))\varphi dx = \int_{R^n} h\varphi dx,$$

since $\int_{R^n} \sum_{i=1}^n u_{x_i}^m \varphi_{x_i} dx$ is bounded in m by (3.15). But this implies that $u + f(u_x) = h$ a.e. in R^n, since $\varphi \in C_0^\infty(R^n)$ is arbitrary.

It remains to show that $u \in D(A_0)$ and $u + A_0 u = h$. But this fact is clear, since $u \in E(R^n)$, $f(u_x) = w \equiv h - u \in W_1^\infty(R^n)$ a.e. in R^n, and $u + w = h \in E(R^n)$. Thus the proof of Theorem 3.1 has been completed.

Proof of Theorem 3.3. First we note that a linear combination of semiconcave functions with positive coefficients again is semiconcave. Hence, the functions

$$h = u + \lambda A_0 u = (1-\lambda)u + \lambda(u + A_0 u)$$

and

$$g = v + \lambda A_0 v = (1-\lambda)v + \lambda(v + A_0 v)$$

are semiconcave for $0 < \lambda \leq 1$.

Now the proof of (i) and (ii) can be carried out as follows: From Theorem 3.2 it follows immediately that $u, v \in D(A_0)$ are unique bounded generalized solutions of (3.5) respectively, where $h, g \in E(R^n)$. Hence, by what was shown in the proof of Theorem 3.1, u, v can be obtained as respective limits of bounded solutions $u^m, v^m \in C^2(R^n) \cap E(R^n)$ of the equations

$$u + \lambda f(u_x) - (1/m)\Delta u = h, \qquad v + \lambda f(v_x) - (1/m)\Delta v = g$$

as m tends to infinity. Since $u^m \twoheadrightarrow u$, $v^m \twoheadrightarrow v$ in $W_1^\infty(R^n)$, Proposition 3.1 can be used to prove (i) and (ii). The proof is complete.

4. The semigroup of contractions associated with (CP)

The Cauchy problem (CP) consists of (DE) and the initial condition

(IC) $$u(x, 0) = u^0(x), \qquad x \in R^n,$$

where u^0 is a given function on R^n.

It is assumed throughout the section that f: $R^n \to R^1$ is of class C^2 and satisfies the convexity condition:

The matrix $(f_{ij}(p))$, where $f_{ij} = \partial^2 f/\partial p_i \partial p_j$ $(i, j = 1, \dots, n)$, is nonnegative, i.e.,

$$\sum_{i,j=1}^{n} f_{ij}(p)\lambda_i\lambda_j \geq 0$$

for each $p \in R^n$ and each real $\lambda_1, \dots, \lambda_n$. In addition, the normalization (3.2) will be assumed, for this can always be achieved by introducing the new unknown $\bar{u} = u + f(0)t$.

We shall choose $L^\infty(R^n)$ as the Banach space associated with (CP) and regard the unknown function u as a map: $[0, \infty) \ni t \to u(\cdot, t) \in L^\infty(R^n)$. Let A be given by Definition 2.2. Then (CP) can be rewritten in the abstract form

(ACP) $$du/dt + Au \ni 0, \qquad u(0) = u^0$$

(Note that A may be multi-valued for general convex f).

In order to apply the abstract theory to (ACP), we shall state the generation theorem of M. G. Crandall and T. M. Liggett [5] in a form suitable for our later use. Let X be a real Banach space and A be an operator in X (that is allowed to be multi-valued). A is said to be accretive in X if

$$\|(u + \lambda w) - (v + \lambda z)\| \geq \|u - v\|$$

for $\lambda > 0$, u, $v \in D(A)$, $w \in Au$, and $z \in Av$, where $\|\cdot\|$ denotes the norm in X. For $\lambda > 0$, let $D_\lambda = D(J_\lambda) = R(I + \lambda A)$, $J_\lambda = (I + \lambda A)^{-1}$, and $A_\lambda = \lambda^{-1}(I - J_\lambda)$. Set $\mathscr{D} = \bigcup_{\kappa > 0}(\bigcap_{0 < \lambda < \kappa} D_\lambda)$, and define, if $\mathscr{D} \supset D(A)$,

$$\hat{D}(A) = \{v \in \mathscr{D};\ |Av| < \infty\},$$

where we have set for $v \in \mathscr{D}$

$$|Av| = \lim_{\lambda \downarrow 0} \|A_\lambda v\|.$$

The following generation theorem is a result of M. G. Crandall and T. M. Liggett [5].

GENERATION THEOREM. *Let A be an accretive operator in a real Banach space X. If $R(I + \lambda A) \supset \overline{D(A)}$ for all sufficiently small positive λ, then*

(4.1) $$\lim_{n \to \infty} \left(I + \frac{t}{n} A\right)^{-n} u^0$$

exists for $u^0 \in \overline{D(A)}$ *and* $t>0$. *Moreover, if* $S(t)u^0$ *is defined as the limit in* (4.1), *then* $S(t)$ *is a semigroup of contractions on* $\overline{D(A)}$:

(i) *We have* $S(t): \overline{D(A)} \to \overline{D(A)}$ *for* $t \geq 0$; $S(t)S(\tau)=S(t+\tau)$ *for* $t, \tau \geq 0$; $\|S(t)v - S(t)w\| \leq \|v-w\|$ *for* $v, w \in \overline{D(A)}$ *and* $t \geq 0$; $S(0)=I$ *and* $S(t)v$ *is continuous in* (t, v).

(ii) *If* $v \in \hat{D}(A)$, *then* $S(t)v$ *is locally Lipschitz continuous in* t.

(iii) *For each* $\varepsilon>0$ *and each* $u^0 \in \overline{D(A)}$, *the problem*

$$(4.2) \qquad \begin{cases} \varepsilon^{-1}(u^\varepsilon(t) - u^\varepsilon(t-\varepsilon)) + Au^\varepsilon(t) \ni 0, & t > 0, \\ u^\varepsilon(t) = u^0, & t \leq 0, \end{cases}$$

has a unique solution $u^\varepsilon(t)$ *on* $[0, \infty)$ *and* $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t) = S(t)u^0$ *uniformly in* t *on compact sets.*

We have to verify the hypotheses of the Generation Theorem for the A of Definition 2.2.

First we shall establish the accretiveness of the operator A_0.

Proposition 4.1. A_0 *is accretive in* $L^\infty(R^n)$, *i.e., we have*

$$(4.3) \qquad \|(u+\lambda A_0 u) - (v+\lambda A_0 v)\|_\infty \geq \|u-v\|_\infty$$

for each $\lambda>0$ *and each* $u, v \in D(A_0)$.

Proof. Let $u, v \in D(A_0)$. By definition, it follows immediately that there is a positive number λ_0 for which both $u+\lambda_0 A_0 u$ and $v+\lambda_0 A_0 v$ lie in $E(R^n)$. The same argument as in the proof of Theorem 3.3 then shows that (4.3) holds for $0<\lambda \leq \lambda_0$. Using a result of T. Kato [9, Lemma 1.1, p. 509] that, in a real Banach space X, $\|x\| \leq \|x+\alpha y\|$ for every $\alpha>0$ if and only if there is $f \in Fx$ such that $f(y) \geq 0$ (F being the duality map from X to its dual space X^*), we can conclude that (4.3) holds for every $\lambda>0$. The proof is complete.

From Theorem 3.3 and Proposition 4.1 we easily have:

Proposition 4.2. *If* $u, v \in D(A)$, $w \in Au$, *and* $z \in Av$ *satisfy the equations*

$$(4.4) \qquad u+\lambda w = h, \qquad v+\lambda z = g,$$

where $\lambda>0$, *then*:

(i) A *is accretive in* $L^\infty(R^n)$, *i.e., we have for* (4.4)

$$\|u-v\|_\infty \leq \|h-g\|_\infty.$$

(ii) *If* $g \geq h$, *then* $v \geq u$.

In what follows, $BU(R^n)$ denotes the closed linear subspace of $L^\infty(R^n)$ consisting of all bounded and uniformly continuous functions on R^n.

Now we shall give another definition of bounded generalized solutions of (3.1).

DEFINITION 4.1. *Let* $h \in BU(R^n)$. *Then a function* $u \in BU(R^n)$ *is a bounded generalized solution of* (3.1) *provided* $u \in D(A)$ *and* $h \in u + Au$.

It follows from Theorem 3.1 that $R(I+\lambda A)=BU(R^n)$ for $\lambda>0$, since $R(I+\lambda A_0)\supset E(R^n)$ is dense in $BU(R^n)$ and A is the closure of A_0 (Note that $R(I+\lambda A)$ is closed for $\lambda>0$ when A is closed and accretive). By Lemma 2.1 we have $\overline{D(A)}=BU(R^n)$. Therefore we have proved the

THEOREM 4.1. *Let* $f\colon R^n \to R^1$ *be convex and of class* C^2. *Then the operator* A *of Definition* 2.2 *satisfies the assumptions of the Generation Theorem with* $\overline{D(A)}=BU(R^n)$. *In particular,* $u=(I+A)^{-1}h$ *is the unique bounded generalized solution of* (3.1) *for* $h \in BU(R^n)$.

According to Theorem 4.1 and the Generation Theorem, a semigroup of contractions $S(t)$ on $BU(R^n)$ is determined by the operator A. Concerning the properties of this semigroup we have first the

THEOREM 4.2. *Let* $f\colon R^n \to R^1$ *be convex and of class* C^2, *and let* $S(t)$ *be the semigroup of contractions on* $BU(R^n)$ *obtained from* A *through the Generation Theorem. Let* $u, v \in BU(R^n)$ *and* $t \geq 0$. *Then*:

(i) *If* $y \in R^n$, *then*

$$\sup_{x \in R^n} |S(t)v(x+y)-S(t)v(x)| \leq \sup_{x \in R^n} |v(x+y)-v(x)|.$$

Moreover, if $v \in E(R^n)$, *then* $S(t)v \in E(R^n)$ *and*

$$\|S(t)v\|_\infty \leq \|v\|_\infty, \quad \|(S(t)v)_x\|_\infty \leq \|v_x\|_\infty, \quad |S(t)v|_E \leq |v|_E$$

(Note that the normalization (3.2) *is assumed).*

(ii) *If* $v \geq u$, *then* $S(t)v \geq S(t)u$.

PROOF. The solution $u^\varepsilon(t)$ of (4.2) is given by $u^\varepsilon(t)=(I+\varepsilon A)^{-[t/\varepsilon]-1}u^0$, where $[t/\varepsilon]$ is the greatest integer in t/ε. Since $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t)=S(t)u^0$ uniformly in t on compact sets, the proofs of (i) and (ii) follow immediately from Proposition 4.2, (i), (ii), and Theorem 3.1.

When $f\colon R^n \to R^1$ is convex, a Lipschitz continuous function $u(x, t)$ defined on $R^n \times [0, \infty)$ is called a generalized solution of (CP) if: i) u satisfies (DE) a.e. as well as (IC); ii) for each level $t>0$, u satisfies a semiconcavity condition

$$u(x+y, t)+u(x-y, t)-2u(x, t) \leq k(t)|y|^2, \qquad x, y \in R^n,$$

where $k(t)\leq k_\delta$ for $t \geq \delta > 0$. Below we shall show that the semigroup $S(t)$ obtained above provides a bounded generalized solution $S(t)u^0$ of (CP) if u^0 lies in $E(R^n)$.

THEOREM 4.3. *Let $f: R^n \to R^1$ be convex and of class C^2, and let $S(t)$ be the semigroup of contractions on $BU(R^n)$ obtained from A through the Generation Theorem. If $u^0 \in E(R^n)$, then*:

(i) $\|S(t)u^0\|_\infty \le \|u^0\|_\infty$, $\|(S(t)u^0)_x\|_\infty \le \|u_x^0\|_\infty$, *and, for the semiconcavity constant of $S(t)u^0$, $|S(t)u^0|_E \le |u^0|_E$.*

(ii) *$S(t)u^0(x)$ is Lipschitz continuous on $R^n \times [0, \infty)$ and satisfies* (DE) *a.e.*.

PROOF. It suffices to prove (ii). For $u^0 \in E(R^n)$, let $u^\varepsilon(t)$ satisfy

$$(4.5) \qquad \begin{cases} \varepsilon^{-1}(u^\varepsilon(t) - u^\varepsilon(t-\varepsilon)) + A_0 u^\varepsilon(t) = 0, & t > 0, \\ u^\varepsilon(t) = u^0, & t \le 0. \end{cases}$$

Then $u^\varepsilon(t) = (I + \varepsilon A_0)^{-[t/\varepsilon]-1}u^0$ for $t \ge 0$ and, by Theorem 3.1, we have

$$(4.6) \qquad \|u^\varepsilon(t)\|_\infty \le \|u^0\|_\infty, \quad \|(u^\varepsilon(t))_x\|_\infty \le \|u_x^0\|_\infty, \quad |u^\varepsilon(t)|_E \le |u^0|_E$$

for $t \ge 0$. Next we note that $E(R^n) \subset \hat{D}(A)$. Indeed, if $v \in E(R^n)$, then $u = J_\lambda v$ satisfies $u + \lambda A_0 u = v$ and so $A_\lambda v = \lambda^{-1}(I - J_\lambda)v = A_0 u = f(u_x)$ for $\lambda > 0$ (cf. Definition 2.1). Hence we have

$$\|A_\lambda v\|_\infty \le \sup\{|f(p)|;\ |p| \le \|v_x\|_\infty\}$$

for $\lambda > 0$, which implies $v \in \hat{D}(A)$. Therefore, according to the Generation Theorem, (ii), $S(t)u^0$ is locally Lipschitz continuous in t.

By Definition 2.1, $u^\varepsilon(t)$ satisfies the equation

$$(4.7) \qquad \varepsilon^{-1}(u^\varepsilon(t) - u^\varepsilon(t-\varepsilon)) + f((u^\varepsilon(t))_x) = 0$$

a.e. in R^n for each $t \ge 0$. Let $T > 0$. Since $S(t)u^0$ is Lipschitz continuous for $0 \le t \le T$ and $S(t)u^0 \in W_1^\infty(R^n)$ for each $t \ge 0$, $S(t)u^0(x)$ is Lipschitz continuous in (x, t) and, hence, (totally) differentiable a.e. in $R^n \times [0, T]$. Moreover, by Lemma 3.2, the sequence $\{(u^\varepsilon(t))_x\}$ converges a.e. in $R^n \times [0, T]$ to $(S(t)u^0)_x$ as $\varepsilon \downarrow 0$. Multiply (4.7) by $\varphi \in C_0^\infty(R^n \times (0, T))$ and integrate over $R^n \times [0, T]$. Integrating by parts and letting $\varepsilon \downarrow 0$ yield

$$\int_0^T \int_{R^n} \{-(S(t)u^0)\varphi_t + f((S(t)u^0)_x)\varphi\} dxdt = 0,$$

which can be rewritten as

$$\int_0^T \int_{R^n} \{(S(t)u^0)_t + f((S(t)u^0)_x)\}\varphi dxdt = 0.$$

But this implies that $S(t)u^0(x)$ satisfies (DE) a.e. on $R^n\times(0, T)$, which in turn shows that $S(t)u^0(x)$ is uniformly Lipschitz continuous on $R^n\times[0, \infty)$ by Theorem 4.2, (i). Thus the proof is complete.

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