HIROSHIMA MATH. J.
6 (1976), 377-402

Simplexes and Dirichlet Problems on Locally Compact Spaces

Hisako WATANABE

(Received January 20, 1976)

Introduction

Let Ω be a bounded open set in a euclidean space and f a continuous function defined on the boundary $d\Omega$. The classical Dirichlet problem asks for a continuous function u on the closure $\bar{\Omega}$ of Ω which is harmonic in Ω and equal to f on $d\Omega$. H. Bauer [1] considered an analogous abstract Dirichlet problem for a compact Hausdorff space X and a vector space B of real-valued, continuous functions on X which contains constant functions and separates points of X. He investigated conditions with which a continuous function f defined on the closure of the Choquet boundary $\delta(E)$ with respect to B can be extended to X as a function of B or a **B**-affine function. In the special case where X is a convex compact set in a locally convex real vector space and B is the vector space of the restrictions to X of all functions of the form $f+\alpha$ with a linear functional f and a constant function α, Bauer proved that $\mathscr{C}(\overline{\delta(\mathbf{B})})=\mathbf{B}|\overline{\delta(\mathbf{B})}$ if and only if B is a simplex and $\delta(\mathbf{B})$ is closed ([1, Satz 13]). Thus the abstract Dirichlet problem is deeply connected with the theory of simplexes (see [5] and [6]). Similar abstract Dirichlet problems on a compact set and their relations with the theory of simplexes have been discussed by many authors; e.g., [3] and [8].

In the case where X is a locally compact and σ-compact Hausdorff space, G. Mokobodzki and D. Sibony ([9], [10]) showed that the Choquet boundary with respect to a certain convex cone C of lower semicontinuous functions on X is not empty, using the notion of adapted cones due to G. Choquet [5].

Let P be an adapted convex cone consisting of non-negative continuous functions on X and C be a convex cone consisting of P-bounded continuous functions on X. We shall show that many results in [1], [3], [8] concerning simplexes and abstract Dirichlet problems, which are obtained for a compact space X, are also valid with respect to such a cone C in the case where X is a locally compact and σ-compact space. We shall then apply these results to Dirichlet problems for arbitrary open or closed sets in Bauer's axiomatic potential theory ([2]).

Most of the results in this paper were announced in [15] and [16]. Since the proofs in those papers are sketchy, we shall give details in the present paper.

Here we remark that recently J. Bliedtner and W. Hansen (Inventions math.

29 (1975), 83-110) showed that Corollary 4.2 in this paper is valid without Axiom D.

The author wishes to express her cordial thanks to Professor Seizó Itô for his kind remarks and suggestions to this research.

Chapter 1. The space $\mathbf{H_P}$

Throughout this paper X is a locally compact and σ-compact Hausdorff space. We denote by $\mathscr{C}(X)$ the set of all continuous real-valued functions on X, and by $\mathscr{C}^+(X)$ the set of all non-negative functions in $\mathscr{C}(X)$. Let $\mathscr{C}_K(X) = \{f \in \mathscr{C}(X); f$ *has* a compact support$\}$ and $\mathscr{C}_K^+(X) = \mathscr{C}_K(X) \cap \mathscr{C}^+(X)$.

The following lemma, which will be used later, is an immediate consequence of [4, Chap. IV, § 1, Théorème 1]:

LEMMA 1. *Let μ be a positive Radon measure on X and $\{f_\alpha\}$ a lower directed family of upper semi-continuous μ integrable functions. Suppose that there exist an index β and a continuous μ-integrable function g such that $f_\beta \leqq g$. Then*

$$(-\infty \leqq)\ \mu(\inf_\alpha f_\alpha) = \inf_\alpha \mu(f_\alpha).$$

§ 1.1. Adapted convex cone and the space $\mathbf{H_P}$

Let f and g be non-negative functions on X. We say that g *dominates* f *at infinity*, if for each $\varepsilon>0$ the set $\{x \in X;\ f(x) > \varepsilon g(x)\}$ is relatively compact.

We say that a convex cone $\mathbf{P}$ in $\mathscr{C}^+(X)$ is *adapted* if it satisfies the following conditions:

(p_1) for any $x \in X$ there exists $u \in \mathbf{P}$ satisfying $u(x)>0$,

(p_2) for any $u \in \mathbf{P}$ there exists $v \in \mathbf{P}$ such that v dominates u at infinity.

A linear subspace $\mathbf{B}$ of $\mathscr{C}(X)$ is said to be adapted if $\mathbf{B} = \mathbf{B}^+ - \mathbf{B}^+$, where $\mathbf{B}^+ = \mathbf{B} \cap \mathscr{C}^+(X)$, and $\mathbf{B}^+$ is an adapted cone.

Obviously, if $\mathbf{P}$ is an adapted convex cone and Y is a closed subset of X, then $\mathbf{P}|Y = \{f|Y; f \in \mathbf{P}\}$ is an adapted convex cone on Y.

Let $\mathbf{P}$ be an adapted convex cone in $\mathscr{C}^+(X)$. For $u \in \mathbf{P}$ we denote by $\mathbf{H}_u$ the Banach space of continuous functions f on X such that $|f| \leqq \lambda u$ for some $\lambda \geqq 0$ with the norm $\|f\|_u = \inf\{\lambda;\ |f| \leqq \lambda u\}$ and consider $\mathbf{H_P} = \bigcup_{u \in \mathbf{P}} \mathbf{H}_u$ with the topology of the inductive limit of Banach spaces $\{\mathbf{H}_u\}_{u \in \mathbf{P}}$. By ($p_1$), we see that $\mathscr{C}_K(X) \subset \mathbf{H_P}$.

PROPOSITION 1.1. *Let $\mathbf{P}$ be an adapted convex cone in $\mathscr{C}^+(X)$. Then for each $f \in \mathbf{H_P}$ we can find $u \in \mathbf{P}$ such that for each $\varepsilon>0$, there exists $h \in \mathscr{C}_K(X)$*

satisfying $\|f-h\|_u<\varepsilon$.

PROOF. Assume that $f\in \mathbf{H}_v$ for $v\in \mathbf{P}$. Since P is adapted, there exists u e P dominating v at infinity and $u\geqq v$; accordingly $f\in \mathbf{H}_u$. For any $\varepsilon>0$, there exists a compact set K such that $v\leqq \varepsilon u$ on $X-K$. We may find $g\in \mathscr{C}_K^+(X)$ satisfying $|f|\leqq g$ on X. Put $h=\max\{-g, \min\{f, g\}\}$. Then we have $h\in \mathscr{C}_K(X)$ and $\|f-h\|_u\leqq \|f\|_v\varepsilon$.

§ 1.2. Positive linear functionals on $\mathbf{H_P}$

Let P be an adapted convex cone in $\mathscr{C}^+(X)$. A positive Radon measure μ on X is said to be **P**-*integrable* if $\mu(f)<\infty$ for any $f\in \mathbf{P}$. The space of all **P**-integrable positive measures on X is denoted by $\mathfrak{M}_\mathbf{P}^+$. Since P is adapted, any positive linear functional on $\mathbf{H_P}$ is represented by a measure in $\mathfrak{M}_\mathbf{P}^+$ and $\mathfrak{M}_\mathbf{P}=\mathfrak{M}_\mathbf{P}^+-\mathfrak{M}_\mathbf{P}^+$ is the dual of $\mathbf{H_P}$ (cf. [9, § 3, Proposition 11]).

LEMMA 1.2. *Let* **B** *be a subspace of* $\mathbf{H_P}$ *containing* P. *Any positive linear functional L on* B *may be extended to a positive linear functional on* $\mathbf{H_P}$. // B *ıs dense in* $\mathbf{H_P}$, *the extension is unique.*

PROOF. For any $f\in \mathbf{H_P}$ we put $p(f)=\inf_{g\geqq f, g\in \mathbf{B}} L(g)$. Then we have $|p(f)|<\infty$. Since the mapping: $f\mapsto p(f)$ is a sublinear functional on $\mathbf{H_P}$ and $p(f)=L(f)$ on B, we may find a linear functional L' on $\mathbf{H_P}$ satisfying $L'(f)\leqq p(f)$ for all $f\in \mathbf{H_P}$ and $L'(f)=L(f)$ for each $f\in \mathbf{B}$ by the Hahn-Banach extension theorem. If $f\leqq 0$, then $p(f)\leqq 0$, and accordingly $L'(f)\leqq 0$. If B is dense in $\mathbf{H_P}$, then the above extension L' is unique, since any positive linear functional on $\mathbf{H_P}$ is continuous.

The following lemma is an extension of Hilfssatz 4 in [1].

LEMMA 1.3. *Let* P *be an adapted cone in* $\mathscr{C}^+(X)$,*E be a subspace of* $\mathbf{H_P}$ *which is a lattice in the natural order, and F be a positive linear functional on* $\mathbf{H_P}$ *which satisfies*

$$(*)\qquad F(f\wedge g)=\min\{F(f), F(g)\}$$

for any /, $g\in \mathbf{B}$ *and which is not identically zero on* B. *Suppose that for every x there exists* $f\in \mathbf{B}$ *such that* $f(x)\neq 0$. *Then there exist* $x\in X$ *and* $\lambda>0$ *such that* $F(f) = \lambda f(x)$ *for all* $f\in \mathbf{B}$.

PROOF. First we shall show that there exists a point x e X satisfying

$$(1.1)\qquad F^{-1}(0)\cap \mathbf{B}=\{f\in \mathbf{B}; f(x)=0\}.$$

Assume that for each x e X, there exists f_xe $F^{-1}(0)$ n B satisfying $f_x(x)\neq 0$. By condition (*), if $f\in F^{-1}(0)\cap B$, then $\max\{f, 0\}$ and $\min\{f, 0\}$ both belong to $F^{-1}(0)\cap \mathbf{B}$. Hence we may assume $f_x\geqq 0$. For any $f\in \mathbf{B}^+=B\cap \mathscr{C}^+(X)$, there exists $g\in P$ such that the closure K of $\{z\in X; f(z)>\varepsilon g(x)\}$ is compact for any $\varepsilon>0$, since P is adapted. By the continuity of f_z and the compactness of K, we may find a finite number of points $z_i\in K$ $(i=1,\dots, n)$ such that $f_0=\sum_{i=1}^{n} f_{z_i}>0$ on K. Then $f_0\in F^{-1}(0)$. For sufficiently large $\alpha>0$ we have $f\leqq \alpha f_0$ on K, whence $f\leqq \alpha f_0+\varepsilon g$ on X. Thus we have $0\leqq F(f)\leqq \alpha F(f_0)+\varepsilon F(g)=\varepsilon F(g)$. Hence $F(f)=0$. Since B is a lattice, F is identically zero on B, which is contrary to the assumption. Thus there exists $x\in X$ satisfying

$$F^{-1}(0)\cap \mathbf{B}\subset\{f\in \mathbf{B}; f(x)=0\}.$$

We note that $\{f\in B; f(x)=0\}$ does not coincide with B by the assumption. Since the linear space $F^{-1}(0)\cap B$ is maximal in B, we have the relation (1.1). Further, taking $h\in \mathbf{B}^+$ with $F(h)\neq 0$, we have $(F(f)/F(h))h-f\in F^{-1}(0)\cap \mathbf{B}$ for any $f\in \mathbf{B}$, whence $(F(f)/F(h))h(x)-f(x)=0$ from (1.1) and $h(x)\neq 0$. Putting $\lambda=F(h)/h(x)>0$, we have $F(f)=\lambda f(x)$ for any $f\in \mathbf{B}$.

§ 1.3. Totality of a convex cone in $\mathbf{H}_\mathbf{P}$

A convex cone C in $\mathscr{C}(X)$ is said to be *linearly separating* if for any different two elements x, y of X and any $\lambda\geqq 0$, there exists $f\in \mathbf{C}$ such that $f(x)\neq\lambda f(y)$. A convex cone $C\subset\mathscr{C}(X)$ is said to be *min-stable* if $f, g\in C$ implies $\min\{f, g\}\in C$.

PROPOSITION 1.2. *Let* P *be an adapted convex cone in* $\mathscr{C}^+(X)$. *Assume that a linear subspace* B *of* $\mathbf{H}_\mathbf{P}$ *containing* P *is linearly separating and min-stable. Then for each* $f\in\mathscr{C}_K(X)$ *we can find* $v\in \mathbf{P}$ *such that for each* $\varepsilon>0$, *there exists* $g\in\mathscr{C}_K(X)\cap B$ *satisfying* $\|f-g\|_v<\varepsilon$.

PROOF (cf. the proof of [11, 2éme partie, Théorème 12]). (I) We prove first that for each $x\in X$ there exists $\varphi\in \mathbf{B}^+$ such that its support is compact and $\varphi(x)>0$. Suppose there exists $x\in X$ such that every $\varphi\in \mathbf{B}^+$ with compact support is zero at x. Let V be an arbitrary relatively compact open set containing x. Then, since B is min-stable, by considering $\varphi=v-\min\{u, v\}$ we see that $u\geqq v$ on CV implies $u(x)\geqq v(x)$ for $u, v\in \mathbf{B}$. Put $\mathbf{B}_1=\mathbf{B}|CV$. Then $\mathbf{B}_1\supset \mathbf{P}|CV$ and the mapping $F: u\mapsto u(x)$ is a positive linear functional on $\mathbf{B}_1$ which is not identically equal to zero. Hence F may be extended to a positive linear functional Φ on $\mathbf{H}_\mathbf{P}(CV)$ by Lemma 1.2, where in general

$$\mathbf{H}_\mathbf{P}(Y)=\{f\in\mathscr{C}(Y);\ |f|\leqq\lambda v \text{ on } Y \text{ for some } \lambda\geqq 0 \text{ and } v\in \mathbf{P}\}$$

for $Y\subset X$. Further $\mathbf{B}_1$ is a lattice on which the relation

$$\Phi(\min\{u, v\}) = \min\{u(x), v(x)\} = \min\{\Phi(u), \Phi(v)\}$$

holds. Hence by Lemma 1.3, there exist $y \in CV$ and $\lambda > 0$ such that $u(x) = \Phi(u) = \lambda u(y)$ for any $u \in \mathbf{B}$. Since $x \neq y$, this contradicts the assumption that B is linearly separating.

(II) Let $f \in \mathscr{C}_K(X)$. From the consideration in (I) it follows that there exists $\varphi \in \mathscr{C}_K(X) \cap \mathbf{B}^+$ satisfying $\varphi \geqq |f|$ on the support S_f of f. Choose $\imath \in \mathbf{P}$ satisfying $\imath \geqq 1$ on S_φ. Let $\varepsilon > 0$ be given. By the Stone-Weierstrass theorem we find $h \in E$ such that $|f - h| < \varepsilon$ on S_φ. Put $g = \max\{\min\{h, \varphi\}, -\varphi\}$. Then we have $\|f - g\|_\imath < \varepsilon$.

By Propositions 1.1 and 1.2 we have the following corollaries.

COROLLARY 1.1 ([cf. 11, 2éme partie, Théorème 12]). *Let* B *be an adapted linear subspace of* $\mathscr{C}(X)$. *If it is min-stable and linearly separating, then* B *is dense in* $\mathbf{H}_{\mathbf{B}^+}$.

COROLLARY 1.2 ([14, Proposition 9]). *If* P *is an adapted cone in* $\mathscr{C}^+(X)$ *and* C *is a min-stable and linearly separating convex cone such that* $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H}_\mathbf{P}$, *then* C *is total in* $\mathbf{H}_\mathbf{P}$, *i.e., the linear space* $\mathbf{C} - \mathbf{C}$ *is dense in* $\mathbf{H}_\mathbf{P}$.

Chapter 2. Simplexes

§ 2.1. Extremal measures

Let P be an adapted convex cone in $\mathscr{C}^+(X)$ and S be a subset of X. An extended real-valued function f on S is said to be *upper* (resp. *lower*) ***P-bounded*** if there exists $u \in \mathbf{P}$ satisfying $f \leqq u$ (resp. $-u \leqq f$) on S.

Let C be a convex cone of lower P-bounded and lower semicontinuous functions on X satisfying $\mathbf{C} \supset \mathbf{P}$. For two measures $\mu, \nu \in \mathfrak{M}_\mathbf{P}^+$, we write

$$\mu \prec_\mathbf{C} \nu \quad \text{or simply } \mu \prec \nu$$

if $\nu(f) \leqq \mu(f)$ for any $f \in \mathbf{C}$. A measure $\mu \in \mathfrak{M}_\mathbf{P}^+$ is said to be ***C-extremal*** (or simply *extremal*) if any measure $\nu \in \mathfrak{M}_\mathbf{P}^+$ with $\mu \prec_\mathbf{C} \nu$ satisfies

$$\nu(f) = \mu(f)$$

for all $f \in \mathbf{C}$, i.e., if $\nu \in \mathfrak{M}_\mathbf{P}^+$ and $\mu \prec_\mathbf{C} \nu$, then $\nu \prec_\mathbf{C} \mu$.

We obtain the following proposition and corollary, the proofs of which are the same as those in the case where X is compact (cf. [5, 12.6 Theorem] and [6, Théorème 3]).

PROPOSITION 2.1. *Let* C *be a convex cone of lower* P*-bounded and lower*

semicontinuous** functions on* X ***satisfying $\mathbf{C}\supset\mathbf{P}$. *Then for* $\mu\in\mathfrak{M}_{\mathbf{P}}^{+}$, $\mathfrak{M}_{\mu}=\{\nu \in\mathfrak{M}_{\mathbf{P}}^{+};\ \mu\prec_{\mathbf{C}}\nu\}$ *is compact in the weak topology* $\sigma(\mathfrak{M}_{\mathbf{P}},\ \mathbf{H}_{\mathbf{P}})$ *on* $\mathfrak{M}_{\mathbf{P}}$.

COROLLARY **2.1.** *For any* $\mu\in\mathfrak{M}_{\mathbf{P}}^{+}$, *there exists a* ***C-extremal*** *measure* $\nu \in\mathfrak{M}_{\mathbf{P}}^{+}$ *satisfying* $\mu\ \prec_{\mathbf{C}}\nu$.

An upper or lower P-bounded semicontinuous function / on a closed set S is said to be *C-concave* or simply *concave* on 5 if for any $x\in S$ and any measure $\mu\in\mathfrak{M}_{\mathbf{P}}^{+}$ such that $\mu(\mathbf{X}-\mathbf{S})=0$ and $\varepsilon_{x}\prec_{\mathbf{C}}\mu$, the relation $\mu(f)\leqq f(x)$ holds, where ε_{x} is the unit measure at x. The set of all lower P-bounded and lower semicontinuous C-concave functions on X is denoted by $\hat{\mathrm{C}}$. This is a **min-stable** convex cone containing C. A function / on S is said to be **C-*affine*** or simply ***affine*** on S if f **and** $-f$ **are** both C-concave on 5.

Let μ be a measure in $\mathfrak{M}_{\mathbf{P}}^{+}$ and S a closed subset of X. For an upper P-bounded function / defined on a set containing S the extended real number

$$\inf\{\mu(g);\ g\in\mathbf{C},\ g\geqq f\ \text{on}\ S\}$$

is denoted by

$$Q_{\mu}^{S,\mathbf{C}}(f)=Q_{\mu}^{\mathbf{C}}(f)=Q_{\mu}^{S}(f)=Q_{\mu}(f).$$

We write $Q_{x}(f)$ for $Q_{\varepsilon_{x}}(f)$. The function: $x\mapsto Q_{x}(f)$ is denoted by Qf. The mapping $f\mapsto Q_{\mu}(f)$ is sublinear. $\mu\prec\nu$ implies $Q_{\mu}(f)\geqq Q_{\nu}(f)$. Obviously, $Q^{S}f\geqq f$ on 5. If $f\in$ C, then $Q^{S}f\leqq f$ on X, and hence $Q^{S}f=f$ on S.

PROPOSITION 2.2. // C ***is a** min-stable convex cone such that* $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H}_{\mathbf{P}}$, *then* Qf *is an upper* ***P-bounded,*** *upper semicontinuous C-concave function on* X *and* $Q_{\mu}(f)=\mu(Qf)$ ***for any*** $\mu\in\mathfrak{M}_{\mathbf{P}}$.

PROOF. It is easy to see that Qf is upper P-bounded, upper semicontinuous and C-concave. The equality $Q_{\mu}(f)=\mu(Qf)$ follows from Lemma 1.1.

A closed subset S of X is said to be **C-*determining*** or simply *determining* if any function in C non-negative on S is non-negative on X. If S is C-determining **and** $f\in$ -C, then $Q_{\mu}^{S}(f)\geqq\mu(f)$ and hence $Q^{S}f\geqq f$ on X.

The following lemma and Corollary 2.2 give an extension of Lemma 1.1 in **[3].**

LEMMA 2.1. *Let f be an upper P-bounded and upper semicontinuous function on a determining closed set S. Then for any* $\mu\in\mathfrak{M}_{\mathbf{P}}^{+}$, *there exists a measure* $\nu\in\mathfrak{M}_{\mathbf{P}}$ *such that* $\mu\prec\nu$, $\nu(X-S)=0$ *and* ν(/) $=Q_{\mu}^{S}(f)$.

PROOF. We first assume $f\in\mathbf{H}_{\mathbf{P}}$. Then we have $|Q_{\mu}^{S}(f)|<+\infty$. In fact, obviously $Q_{\mu}^{S}(f)<+\infty$. **Since** $f\in\mathbf{H}_{\mathbf{P}}$, there is $v\in\mathbf{P}$ such that $f\geqq-v$. Since

S is determining, we see that $Q_\mu^S(f) \geqq -\mu(v) > -\infty$. As the mapping: $g \mapsto Q_\mu^S(g)$ from $\mathbf{H_P}(S)$ into R is sublinear, we may find, by the Hahn-Banach extension theorem, a linear functional ν_f on $\mathbf{H_P}(S)$ such that $\nu_f \leqq Q_\mu^S$ on $\mathbf{H_P}(S)$ and $\nu_f(f) = Q_\mu^S(f)$. Obviously, if $g \in \mathbf{H_P}(S)$ is non-positive, then $Q_\mu^S(g) \leqq 0$. Hence we may regard ν_f as an element of $\mathfrak{M}_\mathbf{P}^+$ with $\nu_f(X-S) = 0$. It follows that

$$\nu_f(g) = \sup_{\substack{h \leq g \text{ on } S \\ h \in \mathbf{H_P}}} \nu_f(h) \leqq \sup_{\substack{h \leqq g \text{ on } S \\ h \in \mathbf{H_P}}} Q_\mu^S(h) \leqq \mu(g)$$

for any $g \in \mathbf{C}$, whence $\mu \prec \nu_f$.

Next, let / be an upper P-bounded and upper semicontinuous function on S. We denote by $\mathscr{G}$ the lower directed family $\{g \in \mathbf{H_P}\ \ g \geqq f \text{ on } S\}$. By the preceding consideration, we can choose, for any $g\ E\ \mathscr{G}$, a measure ν_g e $\mathfrak{M}_\mathbf{P}^+$ such that $\mu \prec \nu_g$, $\nu_g(X-S) = 0$ and $\nu_g(g) = Q_\mu^S(g)$. Since $\{\lambda \in \mathfrak{M}_\mathbf{P}^+;\ \mu \prec \lambda\}$ is compact in the topology $\sigma(\mathfrak{M}_\mathbf{P}, \mathbf{H_P})$ by Proposition 2.1, there is $\nu \in \mathfrak{M}_\mathbf{P}^+$ such that a cofinal subfamily of $\{\nu_g\}_{g \in \mathscr{G}}$ converges to $\nu \in \mathfrak{M}_\mathbf{P}^+$ with $\mu \prec \nu$. Obviously $\nu(X-S) = 0$. We also have

$$Q_\mu^S(f) \leqq \inf_{gey} Q_\mu^S(g) = \inf_{g \in \mathscr{G}} \nu_g(g) \leqq \inf_{g' \in \mathscr{G}} \inf_{\substack{g \in \mathscr{G} \\ g \leqq g'}} \nu_g(g')$$

$$\leqq \inf_{\text{flfeaf}} \nu(g') = \nu(f)\ \text{g} \inf_{\substack{v \in \mathbf{C} \\ \iota \geqq f \text{ on } S}} \nu(v) \leqq \inf_{\substack{v \in \mathbf{C} \\ v \geqq f \text{ on } S}} \mu(v) = Q_\mu^S(f).$$

COROLLARY 2.2. *Letf and S be as in Lemma* 2.1. *Then,*

$$Q_\mu^S(f) = \sup\{\nu(f);\ \nu \in \mathfrak{M}_\mathbf{P}^+,\ \nu(X-S) = 0,\ \mu \prec \nu\}.$$

PROPOSITION 2.3. *Let C be a min-stable convex cone such that* $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$. *Then for any* μ, $\nu \in \mathfrak{M}_\mathbf{P}^+$ *the relation* $\mu \prec_\mathbf{C} \nu$ *ıs equivalent to the relation* $\mu \prec_{\hat{\mathbf{C}}} \nu$.

PROOF. By the definition of C the relation $\varepsilon_x \prec_\mathbf{C} \mu$ and the relation $\varepsilon_x \prec_{\hat{\mathbf{C}}} \mu$ are equivalent for any x e X and μ e $\mathfrak{M}_\mathbf{P}^+$. By Corollary 2.2 we have

$$Q_x^\mathbf{C}(f) = \sup\{\mu(f);\ \varepsilon_x \prec_\mathbf{C} \mu\} = \sup\{\mu(f);\ \varepsilon_x \prec_{\hat{\mathbf{C}}} \mu\} = Q_x^{\hat{\mathbf{C}}}(f)$$

for any xeX and any $f \in \mathbf{H_P}$. Suppose $\mu \prec_\mathbf{C} \nu$ For any $v \in \hat{\mathbf{C}}$

$$\nu(v) = \sup\{\nu(f)\ f \in \mathbf{H_P}, / \leqq v\} \leqq \sup\{Q_\nu^\mathbf{C}(f)\ f \in \mathbf{H_P}, / \leqq v\}$$

$$\leqq \sup\{Q_\mu^\mathbf{C}(f)\ f \in \text{Hp}, / \leqq v\} = \sup\{\mu(Q^\mathbf{C} f)\ f \in \mathbf{H_P}, / \leqq v\}$$

$$= \sup\{\mu(Q^{\hat{\mathbf{C}}} f)\ f \in \text{Hp}, / \leqq v\} \leqq \mu(v),$$

where the last inequality follows from the relation $Q^{\hat{\mathbf{C}}} f \leqq v$. This implies $\mu \prec_{\hat{\mathbf{C}}} \nu$. On the other hand $\mu \prec_{\hat{\mathbf{C}}} \nu$ obviously implies $\mu \prec_\mathbf{C} \nu$. Hence we have the conelusion of Proposition 2.3.

COROLLARY 2.3. *Iff is an upper P-bounded and upper* ***semicontinuous*** *function and μ is a* ***measure*** *in* $\mathfrak{M}_{\mathbf{P}}^{+}$, ***then***

$$\text{fiîω} = Q_{\mu}^{\mathbf{C}}(f).$$

PROOF. By Corollary 2.2 and Proposition 2.3 we have

$$Q_{\mu}^{\mathbf{C}}(f) = \sup\{\nu(f);\ \mu \prec_{\mathbf{C}} \nu\} = \sup\{\nu(f);\ \mu \prec_{\hat{\mathbf{C}}} \nu\} = Q_{\mu}^{\hat{\mathbf{C}}}(f).$$

COROLLARY 2.4. *For* μ, $\nu \in \mathfrak{M}_{\mathbf{P}}^{+}$, *if* $\mu(f) = \nu(f)$ *for all* $f \in \mathbf{C}$, *then so is for any* $f \in \hat{\mathbf{C}}$. ***Hence,*** *a C-extremal measure is* $\hat{\mathbf{C}}$***-extremal.***

PROPOSITION 2.4. *Let* C *be a convex cone of* ***lower*** *P-bounded and lower semicontinuous functions which* ***contains*** P, *S be a C-determining closed set in X. Then a measure* $\mu \in \mathfrak{M}_{\mathbf{P}}^{+}$ is *C-extremal if and only if*

$$Q_{\mu}^{S,\mathbf{C}}(f) = \mu(f) \quad \textit{for any} \quad f \in \text{-C}. \tag{2.1}$$

// C *ıs a* ***min-stable*** *convex cone such* ***that*** $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H}_{\mathbf{P}}$ *and if* μ *is* **C*****-extremal,*** *then* (2.1) *holds for any* $f \in$ -$\hat{\mathrm{C}}$.

PROOF. Let μ be a C-extremal measure in $\mathfrak{M}_{\mathbf{P}}^{+}$. **From** Lemma 2.1 it follows that for each /∈ —C there exists a measure $\nu \in \mathfrak{M}_{\mathbf{P}}^{+}$ satisfying $\mu \prec_{\mathbf{C}} \nu$ and $\nu(f) = Q_{\mu}(f)$ Since μ is C-extremal, we have v(/) $= \mu(f)$. Hence $Q_{\mu}(f) = \mu(f)$. If C is min-stable and $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H}_{\mathbf{P}}$, then μ is $\hat{\mathrm{C}}$-extremal by Corollary 2.4. Hence the above arguments hold for $f \in -\hat{\mathrm{C}}$.

Conversely, suppose that a measure $\mu \in \mathfrak{M}_{\mathbf{P}}^{+}$ satisfies

$$Q_{\mu}(f) = \mu(f)$$

for **each** $f \in$ —C. Any measure $\mu \in \mathfrak{M}_{\mathbf{P}}^{+}$ with $\mu \prec_{\mathbf{C}} \nu$ satisfies $\mu(f) \leqq \nu(f)$ for all $f \in$ -C. Since μ(/) $= Q_{\mu}(f) \geqq \nu(f)$, we have v(/) $= \mu(f)$ for all $f \in$ -C. Hence μ is C-extremal.

PROPOSITION 2.5. ***Let*** *C be a linearly separating min-stable convex cone such that* $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H}_{\mathbf{P}}$. *If S is a C-determining set and* $\mu \in \mathfrak{M}_{\mathbf{P}}^{+}$ is **C*****-extremal,*** ***then*** $\mu(X - S) = 0$.

PROOF. By Lemma **2.1,** there exists v e $\mathfrak{M}_{\mathbf{P}}^{+}$ such that $\mu \prec \nu$ and $\nu(X - S) = 0$. Since μ is C-extremal, μ(/) $= \nu(f)$ for all $f \in \mathbf{C}$. From Corollary 1.2, it follows that μ = v. Hence $\mu(X - S) = 0$.

§ 2.2. Concave functions and determining sets

In this section P is an adapted convex cone in $\mathscr{C}^+(X)$ and C a min-stable convex cone such that $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$.

PROPOSITION 2.6. *Let S be a determining closed set and f be an upper* **P**-*bounded and upper semicontinuous function on S. Then for any concave function g on a closed set T containing S such that $g \geqq f$ on S we have*

$$g \geqq Q^S f \quad on \quad T.$$

PROOF. Let $x \in T$. By Lemma 2.1 we find a measure $\mu \in \mathfrak{M}_\mathbf{P}^+$ such that $\varepsilon_x \prec \mu$, $\mu(X-S)=0$ and $\mu(f) = Q_x^S(f)$ Then for any concave function g on T such that $g \geqq f$ on S we have

$$Q_x^S(f) = \mu(f) \leqq \mu(g) \leqq g(x).$$

Immediately we derive

COROLLARY 2.5. *Let S be a determining closed set and f an upper* P-*bounded and upper semicontinuous concave function on S. Then*

$$f = Q^S f \quad on \quad S.$$

COROLLARY 2.6. *Let S be a determining closed set and f a P-bounded affine function on X. Then f is continuous on X if its restriction to S is continuous.*

PROOF. By Proposition 2.6 we have $f \geqq Q^S f$ on X and $-f \geqq Q^S(-f)$ on X. Since S is a determining set, we see that $Q^S f \geqq -Q^S(-f)$. Hence $f = Q^S f = -Q^S(-f)$. Therefore f is continuous.

COROLLARY 2.7. *Let S be a determining closed set and f be an upper* P-*bounded and upper semicontinuous concave function on X. Then f is non-negative on X if it is non-negative on S.*

PROOF. If f is non-negative on S, then $Q^S f$ is non-negative on X. Since $Q^S f \leqq f$ on X by Proposition 2.6, f is also non-negative on X.

§2.3. Simplexes

Let C be a convex cone satisfying $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$. The pair (X, C) is called a *simplex* if, given any $x \in X$ and any two **C-extremal** measures μ, $\nu \in \mathfrak{M}_\mathbf{P}^+$ such that $\varepsilon_x \prec \mu$ and $\varepsilon_x \prec \nu$, we have

$$\mu(f) = \nu(f)$$

for all $f \in \mathbf{C}$.

We denote by $\mathbf{A}=\mathbf{A}(\mathbf{C})$ the set of all upper P-bounded and upper semi-continuous **C**-affine functions on X. We have the following theorem (cf. [3, Theorem 3.1]):

THEOREM 2.1. *Let C be a min-stable convex cone satisfying $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$ and S a $\mathbf{C}$-determining closed set. Then the following assertions are equivalent:*

a) *(X, C) is a simplex,*

b) *for any $f \in -\hat{\mathbf{C}}$, $Q^S f$ is* **C***-affine,*

c) *for any extremal measure μ with $\varepsilon_x \prec_{\mathbf{C}} \mu$ and any $f \in -\hat{\mathbf{C}}$*

$$Q_x^S(f) = \mu(f),$$

d) $Q^S(f+g) = Q^S f + Q^S g$ *for any two $f, g \in -\hat{\mathrm{C}}$,*

e) *for any $f \in -\hat{\mathrm{C}}$ and any C-concave function g on S such that $f \leqq g$ on S there exists $h \in \mathbf{A}(\mathbf{C})$ satisfying*

$$f \leqq h \leqq g \quad \text{on } S.$$

PROOF. a)$\Rightarrow$b): Let $f \in -\hat{\mathrm{C}}$ and let $x \in X$ and $\mu \in \mathfrak{M}_{\mathbf{P}}^+$ satisfying $\varepsilon_x \prec \mu$ be given. Then we have $Q_\mu^S(f) \leqq Q_x^S(f)$ By Lemma 2.1 we find a measure ν satisfying $\varepsilon_x \prec \nu$ and $Q_x^S(f) = \nu(f)$. Let ν' (resp. μ') be an extremal measure satisfying $\nu \prec \nu'$ (resp. $\mu \prec \mu'$) Since (X, C) is a simplex, we have $\nu'(g) = \mu'(g)$ for all $g \in \mathbf{C}$, and hence $\nu'(f) = \mu'(f)$ by Corollary 2.4. Hence, using Propositions 2.3 and 2.4, we obtain

$$Q_x^S(f) = \nu(f) \leqq \nu'(f) = \mu'(f) = Q_{\mu'}^S(f) \leqq Q_\mu^S(f).$$

Thus $Q_x^S(f) = Q_\mu^S(f) = \mu(Q^S f)$ by **Proposition** 2.2. Hence $Q^S f$ is **affine.**

b)$\Rightarrow$c): For each extremal measure $\mu \in \mathfrak{M}_{\mathbf{P}}^+$ with $\varepsilon_x \prec \mu$ and each $f \in -\hat{\mathrm{C}}$, the relation $\mu(f) = Q_\mu^S(f) = \mu(Q^S f)$ holds by Propositions 2.4 and 2.2. Since $Q^S f$ is affine, the equality

$$\mu(Q^S f) = Q_x^S(f)$$

holds and hence $\mu(f) = Q_x^S(f)$

c)$\Rightarrow$d): For any $f, g \in -\hat{\mathbf{C}}$ and an extremal measure μ satisfying $\varepsilon_x \prec \mu$, we have

$$Q_x^S(f+g) = \mu(f+g) = \mu(f) + \mu(g) = Q_x^S(f) + Q_x^S(g).$$

d)$\Rightarrow$a): We define $L(f-g) = -Q_x^S(-f) + Q_x^S(-g)$ for any $f, g \in \mathbf{C}$. Then

L is well-defined on $C-\mathbf{C}$ and is a positive linear functional. Hence L may be extended to a positive linear functional L' on $\mathbf{H_P}$ by Lemma 1.2. Since P is adapted, there exists a measure $\mu\in\mathfrak{M}_P^+$ satisfying $\mu(f)=L'(f)$ on $\mathbf{H_P}$; see § 1.2. Hence

$$\mu(f)=L'(f)=-L(-f)=Q_x^S(f)\quad \text{for all } f\in -C.$$

Since $\nu(f)\leqq Q_x^S(f)=\mu(f)$ for an extremal measure ν with $\varepsilon_x\prec\nu$, the relation $\nu\prec\mu$ holds and accordingly $\nu(f)=\mu(f)$ for all $f\in -C$. Thus (X, C) is a simplex.

b)⇒e): For any $f\in -\hat{C}$ and any concave function g on S satisfying $f\leqq g$ on S, the relation $f\leqq Q^S f\leqq g$ holds on S by Proposition 2.6. Since $Q^S f$ is affine by assumption, it suffices to put $h=Q^S f$.

e)⇒b): Let $f\in -\hat{C}$, $x\in X$ and $\mu\in\mathfrak{M}_P^+$ with $\varepsilon_x\prec\mu$. Let ν be a measure in $\mathfrak{M}_P^+$ such that $\nu(X-S)=0$ and $\mu\prec\nu$; such a measure exists on account of Lemma 2.1. We observe that $Q_x^S(f)\geqq Q_\mu^S(f)\geqq Q_\nu^S(f)$. By e) and Proposition 2.6 we have

$$\begin{aligned} Q_\nu^S(f) &= \inf\{\nu(g);\ g\in\mathbf{C},\ g\geqq f \text{ on } S\} \\ &\geqq \inf\{\nu(h);\ h\in\mathbf{A},\ h\geqq f \text{ on } S\} \\ &= \inf\{h(x);\ h\in\mathbf{A},\ h\geqq f \text{ on } S\}\geqq Q_x^S(f), \end{aligned}$$

so that $\mu(Q^S f)=Q_\mu^S(f)=Q_x^S(f)$, which shows that $Q^S f$ is affine.

Remark that in the proof of d)⇒a), we used the equality in d) only for $f, g\in -C$. Therefore, we immediately obtain

THEOREM 2.1′. *Let* C *and* S *be as in Theorem* 2.1. *Then the following assertions are equivalent:*

a) (X, C) *is a simplex,*

b) *for any* $f\in -C$, $Q^S f$ *is affine,*

c) *for any extremal measure* μ *with* $\varepsilon_x\prec\mu$ *and any* $f\in -C$,

$$Q_x^S(f)=\mu(f),$$

d) $Q^S(f+g)=Q^S f+Q^S g$ *for any two* $f, g\in -\mathbf{C}$.

If C is a linearly separating and min-stable convex cone such that $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H_P}$ and (X, C) is a simplex, then an extremal measure μ satisfying $\varepsilon_x\prec\mu$ is unique for each $x\in X$, since $C-\mathbf{C}$ is dense in $\mathbf{H_P}$ by Corollary 1.2. The unique extremal measure is denoted by μ_x. We have the following proposition which is an extension of Theorem 12 in [6].

PROPOSITION 2.7. *Let C be a min-stable and linearly separating convex* *cone satisfying* $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H}_{\mathbf{P}}$. *If* (X,C) *is a simplex, then the function:* $x\mapsto\mu_x(f)$ *defined on* X *is Borel measurable for each* $f\in\mathbf{H}_{\mathbf{P}}$.

PROOF. Since (X, C) is a simplex, $\mu_x(f)=(Qf)(x)$ for each $f\in -\mathbf{C}$ by Theorem 2.1. Hence the function: $x\mapsto\mu_x(f)$ is upper semicontinuous by Proposition 2.2. It follows that $x\mapsto\mu_x(f)$ is Borel measurable for each $f\in\mathbf{C}-\mathbf{C}$.

Let $g\in\mathscr{C}_K(X)$. By Proposition 1.2 there exist $v\in\mathbf{P}$ and a sequence $\{f_n\}\subset\mathbf{C}-\mathbf{C}$ such that

$$|g-f_n|\leqq(1/n)v \qquad (n=1, 2,\dots).$$

Since μ_x is positive, it follows that

$$|\mu_x(g)-\mu_x(f_n)|\leqq(1/n)\mu_x(v).$$

Hence $\lim_{n\to\infty}\mu_x(f_n)=\mu_x(g)$ for every $x\in X$. This implies that the function: $x\mapsto\mu_x(g)$ is Borel measurable for each $g\in\mathscr{C}_K(X)$.

Similarly we can show that the function: $x\mapsto\mu_x(\varphi)$ is Borel measurable for each $\varphi\in\mathbf{H}_{\mathbf{P}}$ because by Proposition 1.1 we find $u\in\mathbf{P}$ and a sequence $\{g_n\}\subset\mathscr{C}_K(X)$ such that

$$|g_n-\varphi|\leqq(1/n)u \qquad (n=1, 2,\dots).$$

Chapter 3. Dilations and abstract Dirichlet problems

§3.1. The Choquet boundaries

Let P be an adapted convex cone in $\mathscr{C}^+(X)$ and C a convex cone satisfying $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H}_{\mathbf{P}}$. A closed subset $A\subset X$ is said to be **C**-*stable* or simply *stable* if the assumptions $\varepsilon_x\prec_{\mathbf{C}}\mu$ for $x\in A$ and $\mu\in\mathfrak{M}_{\mathbf{P}}^+$ imply $\mu(X-A)=0$. Every compact stable set contains a minimal compact stable set. The open set $\bigcup_{v\in\mathbf{C}}\{x\in X;\ v(x)<0\}$ is denoted by $X^-(C)=X^-$. Denote by $\delta(\mathbf{C})$ the set of all points $x\in X^-$ each of which is an element of a minimal compact stable set. We shall call it the *Choquet boundary* with respect to C. It is known that if $X^-(\mathbf{C})$ is not empty, then the Choquet boundary is not empty and its closure is a determining set; see [9, §4, Proposition 2].

Now suppose that C is linearly separating. Then a minimal compact stable set consists of only one point (cf. [9, §4, Lemma 5] and [3, p. 23]). It follows that $x\in\delta(\mathbf{C})$ if and only if ε_x is the unique measure μ satisfying $\varepsilon_x\prec\mu$. (Note that if $x\notin X^-$, then $\varepsilon_x\prec 0$.) Furthermore, $\overline{\delta(\mathbf{C})}$ is the smallest determining set ([9, §4, Proposition 7]). By Proposition 2.5, if $\mu\in\mathfrak{M}_{\mathbf{P}}^+$ is an extremal measure, then $\mu(X-\overline{\delta(\mathbf{C})})=0$.

PROPOSITION 3.1. *Let* C *be a linearly separating, min-stable convex cone such that* $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H_P}$. *Then the following assertions are equivalent:*

(a) $x\in\delta(\mathbf{C})$,

(b) $Q_x^{\mathbf{C}}(h)=h(x)$ *for any* $h\in\mathbf{H_P}$,

(c) *there exists a subset* $\mathbf{C}_1$ *of* $-$C *which is total in* $\mathbf{H_P}$ *and satisfies* $Q_x^{\mathbf{C}}(h)=h(x)$ *for any* $h\in\mathbf{C}_1$.

PROOF. (a)$\Rightarrow$(b): Suppose $x\in\delta(\mathbf{C})$. Since ε_x is the unique extremal measure μ satisfying $\varepsilon_x\prec\mu$, $Q_x^{\mathbf{C}}(h)=h(x)$ for any $h\in\mathbf{H_P}$ by Corollary 2.2.

(b)$\Rightarrow$(c): It suffices to put $\mathbf{C}_1=-$C by virture of Corollary 1.2.

(c)$\Rightarrow$(a): Assume that $Q_x^{\mathbf{C}}(h)=h(x)$ for any $h\in\mathbf{C}_1$. For any $g\in\mathbf{C}$ satisfying $g\geqq h$ and any $\mu\in\mathfrak{M}_{\mathbf{P}}^+$ satisfying $\varepsilon_x\prec\mu$, we have $\mu(h)\leqq\mu(g)\leqq g(x)$. Hence $\mu(h)\leqq Q_x^{\mathbf{C}}(h)=h(x)$. On the other hand, since fte-C, it follows that $\mu(h)\geqq h(x)$, whence $\mu(h)=h(x)$ for any fteC^ Since $\mathbf{C}_1$ is total in $\mathbf{H_P}$, we have $\mu=\varepsilon_x$. Thus x is an element of $\delta(\mathbf{C})$.

LEMMA 3.1. *IfX has a countable base, then* $\mathbf{H_P}$ is *separable.*

PROOF. Since X has a countable base, there is a countable subfamily $\mathscr{D}$ of $\mathscr{C}_K(X)$such that for any $\varphi\in\mathscr{C}_K(X)$, any relatively compact open set ω containing the support of φ and $\varepsilon>0$, we find $\psi\in\mathscr{D}$ such that $S_\psi\subset\omega$ and $\backslash\varphi-\psi|<\varepsilon$ on X. Then $\mathscr{D}$ is dense in $\mathbf{H_P}$ by virtue of Proposition 1.1.

PROPOSITION 3.2. *// X has a countable base and* C *is a linearly separating, min-stable convex cone such that* $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H_P}$, *then* $\delta(\mathbf{C})$ is *a* G_δ*-set and* $\mu(X-\delta(\mathbf{C}))=0$ *for any extremal measure* $\mu\in\mathfrak{M}_{\mathbf{P}}^+$.

For the proof, see [9, § 4, Proposition 10] or [14, p. 360]. Note that if $\mathscr{D}$ is as in the proof of Lemma 3.1, then Proposition 3.1 implies

$$\delta(\mathbf{C})=\bigcap_{f\in\mathscr{D}}\{x\in X;\ Q_x(f)=f(x)\}.$$

§3.2. Dilations

In this section, we suppose that X has a countable base and C is a linearly separating, min-stable convex cone satisfying $\mathbf{P}\subset\mathbf{C}\subset\mathbf{H_P}$.

A mapping D from X into $\mathfrak{M}_{\mathbf{P}}^+$ is called a **C**-*dilation* or simply a *dilation* on X if $\varepsilon_x\prec_{\mathbf{C}}D(x)$ for any $x\in X$ and the function: $x\mapsto(Df)(x)=D(x)(f)$ is Borel measurable for each $f\in\mathbf{H_P}$. Given a dilation D on X, a point $x\in X$ is said to be *D-regular* if $D(x)=\varepsilon_x$. The set of D-regular points is denoted by $\delta_r^D(\mathbf{C})$. Obviously, $\delta(\mathbf{C})\subset\delta_r^D(\mathbf{C})$. A dilation D is said to be *weakly affine* if there exists a linearly separating min-stable convex cone $\mathbf{C}_1$ such that $\mathbf{P}\subset\mathbf{C}_1\subset\mathbf{C}$ and for any $v\in-\mathbf{C}_1$, Dv is the limit of a decreasing net of functions in A(C).

In the case where X is a compact set and C is a linear subspace of $\mathscr{C}(X)$ separating points of X and containing constant functions, the above definition is equivalent to the definition in [8, p. 101] on account of the following Propositions 3.3 and 3.4, which are similar to [8, Theorem 2.5].

PROPOSITION 3.3. *Suppose that there is a weakly affine* **C**-*dilation D. Then* (X, C) *is a simplex and, for* $x \in X$, $D(x)$ *is the unique extremal measure* μ *satisfying* $\varepsilon_x \prec \mu$. *In particular,* $\delta(\mathbf{C}) = \delta_r^P(\mathbf{C})$.

PROOF. Let μ and ν be extremal measures in $\mathfrak{M}_{\mathbf{P}}^+$ satisfying $\varepsilon_x \prec \mu$ and $\varepsilon_x \prec \nu$. Since D is a weakly affine dilation, there exists a min-stable and linearly separating convex cone $\mathbf{C}_1$ such that $\mathbf{P} \subset \mathbf{C}_1 \subset \mathbf{C}$ and for any $v \in -\mathbf{C}_1$, Dv is the limit of a decreasing net in A. By Lemma 1.1, we have $\mu(Dv) = \nu(Dv)$. Since μ and ν are carried by $\delta(\mathbf{C})$ by Proposition 3.2 and $Dv = v$ on $\delta(\mathbf{C})$, we have $\mu(v) = \nu(v)$. Since $\mathbf{C}_1 - \mathbf{C}_1$ is dense in $\mathbf{H}_{\mathbf{P}}$ by Corollary 1.2, we have $\mu = \nu$ and hence (X, C) is a simplex.

Let $x \in X$ and μ_x be the unique extremal measure μ with $\varepsilon_x \prec \mu$. Let $v \in -\mathbf{C}_1$. Then we have, by Theorem 2.1 and Corollary 2.2,

$$\mu_x(v) = Q_x^{\mathbf{C}}(v) = \sup\{\mu(v); \mu \in \mathfrak{M}_{\mathbf{P}}^+,\ \varepsilon_x \prec \mu\} \geqq (Dv)(x).$$

To prove the converse inequality, let $a \in \mathbf{A}$ satisfy $Dv \leqq a$. Then we have $v \leqq a$ since $v(y) \leqq D(y)(v)$ for any $y \in X$. Hence $Q_x^{\mathbf{C}}(v) \leqq Q_x^{\mathbf{C}}(a) = a(x)$. Taking the infimum of such $a \in \mathrm{A}$, we see that

$$\mu_x(v) = Q_x^{\mathbf{C}}(v) \leqq (Dv)(x).$$

Hence we have

$$\mu_x(v) = (Dv)(x) = D(x)(v)$$

for $v \in -\mathbf{C}_1$. Since $\mathbf{C}_1$ is total in $\mathbf{H}_{\mathbf{P}}$, it follows that $\mu_x = D(x)$.

PROPOSITION 3.4. *If* (X, C) *is a simplex, then there exists a weakly affine* **C**-*dilation.*

PROOF. For each $x \in X$, let μ_x be the extremal measure satisfying $\varepsilon_x \prec \mu_x$ and let $D(x) = \mu_x$. Then D is a dilation since the mapping: $x \mapsto \mu_x(f)$ is Borel measurable by Proposition 2.7 for each $f \in \mathbf{H}_{\mathbf{P}}$. Further, the relation

$$\mu_x(v) = Q_x^{\mathbf{C}}(v) = \inf\{h(x);\ h \in \mathrm{A},\ h \geqq v\}$$

follows for any $v \in -\mathrm{C}$ from Theorem 2.1 and the fact that $\mu_x(h) = h(x)$ for any $h \in A$. Suppose that $h_1, h_2 \in \mathbf{A}$ satisfy $h_1 \geqq v$ and $h_2 \geqq v$. Then the function $\min\{h_1, h_2\}$ is concave and satisfies $\min\{h_1, h_2\} \geqq v$, whence there exists $h \in A$

satisfying $v \leqq h \leqq \min\{h_1, h_2\}$ by Theorem 2.1. Therefore, Dv is the limit of a decreasing net of functions in A. Thus D is a weakly affine dilation.

§ 3.3. Bauer's simplex

Let C be a min-stable convex cone satisfying $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$. If (X, C) is a simplex and $\delta(\mathbf{C})$ is closed, then *(X,* C) is called a ***Bauer's** simplex.*

We have the following theorem which is well-known in the case of a compact space X (cf. [1, Satz 13] and [12, Proposition **9.10**]).

THEOREM 3.1. *Let C be a **linearly separating**, **min-stable** convex cone satisfying* $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$. *Suppose* $X^-(\mathbf{C}) \neq \phi$. *Then the **following** assertions are **equivalent**:*

(a) *(X,* C) is *a **Bauer's simplex**,*

(b) *any **P-bounded** continuous function on* $\overline{\delta(C)}$ *is **uniquely** extended to an element of* A(C) n $\mathbf{H_P}$,

(c) (X, C) is *a simplex and the function:* $x \mapsto \mu_x(f)$ *is continuous for any* $f \in \mathbf{H_P}$, *where* μ_x *is the extremal measure satisfying* $\varepsilon_x \prec \mu_x$.

PROOF. (a)$\Rightarrow$(b): Put $S = \delta(\mathbf{C})$. Then S is a determining set. Let h be a P-bounded continuous function on S. Choose $\iota \in \mathbf{P}$ such that $|h| \leqq v$ on S. Put $f(x) = h(x)$ for $x \in S$ and $f(x) = -v(x)$ for $x \in X - S$. Then / is P-bounded and upper semicontinuous on X. If x *e* S, then any measure μ e $\mathfrak{M}_\mathbf{P}^+$ satisfying $\varepsilon_x \prec \mu$ is equal to ε_x, so that $\mu(f) = f(x)$, i.e., f is affine on S. If $x \notin S$, then $\varepsilon_x \prec \mu$ implies $\mu(f) \geqq -\mu(v) \geqq -v(x) = f(x)$. Therefore $-f \in \hat{\mathbf{C}}$. By Theorem 2.1, $Q^S f$ e A. It is easy to see that $Q^S f$ is P-bounded. By Corollary 2.5, $Q^S f = f = h$ on S; and by Corollary 2.6, $Q^S f$ is continuous on X. Hence $Q^S f$ is an extension of h and $Q^S f \in$ A n $\mathbf{H_P}$. The uniqueness follows from Corollary 2.7.

(b)$\Rightarrow$(c): Put $S = \overline{\delta(\mathbf{C})}$. For each $f \in \mathbf{H_P}$ we denote by h_f the unique extension of $f|S$ to an element of A n $\mathbf{H_P}$. If μ and v are extremal measures satisfying $\varepsilon_x \prec \mu$ and $\varepsilon_x \prec \nu$, then the supports S_μ and S_ν are both contained in S by Proposition 2.5. Hence

$$\mu(f) = \mu(h_f) = h_f(x) = \nu(h_f) = \nu(f) \tag{3.1}$$

for all $f \in \mathbf{H_P}$. Thus (X, C) is a simplex. By (3.1), we have $\mu_x(f) = h_f(x)$ for any $f \in \mathbf{H_P}$. Since $h_f \in \mathbf{H_P}$, the mapping: $x \mapsto \mu_x(f)$ is continuous.

(c)$\Rightarrow$(a): If $x \in \delta(\mathbf{C})$, then $\mu_x(f) = f(x)$ for $f \in \mathbf{H_P}$. Since the mapping: $x \mapsto \mu_x(f)$ is continuous, the equality $\mu_x(f) = f(x)$ also holds for any $x \in \overline{\delta(\mathbf{C})}$. Let $x \in \overline{\delta(\mathbf{C})}$ and μ be any measure in $\mathfrak{M}_\mathbf{P}^+$ satisfying $\varepsilon_x \prec \mu$. Then we have

$$g(x) = \mu_x(g) \leqq \mu(g) \leqq g(x)$$

for $g \in \mathbf{C}$, since μ_x is the unique extremal measure satisfying $\varepsilon_x \prec \mu_x$. Hence $\mu(g) = g(x)$ for any $g \in \mathbf{C}$. Since C is total in $\mathbf{H_P}$, we have $\mu = \varepsilon_x$ and hence $\overline{\delta(\mathbf{C})} \subset \delta(\mathbf{C})$. Thus $\delta(C)$ is closed.

§ 3.4. Lattices of affine functions

PROPOSITION 3.5. *Let* C *be a min-stable convex cone satisfying* $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$. *Suppose that there is a linear space* B *of* **C**-*affinecontinuousfunctions in* Hp *which is a lattice in the natural order and is linearly separating. Then* $\delta(\mathbf{C})$ *is non-empty and if* $x \in X$ *satisfies the equality*

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

for each pair of $fg \in \mathbf{B}$, *then* x *is a point of* $\overline{\delta(\mathbf{C})}$.

PROOF, Since B is a linear space and linearly separating, there is $g \in \mathrm{B}$ such that $g(x) < 0$ for some $x \in X$. Since g is affine, Corollary 2.2 implies $Q_x(g) = g(x) < 0$, and hence there is $\iota \in \mathbf{C}$ such that $v(x) < 0$. Thus, $X^-(\mathbf{C}) \neq \phi$, so that $\delta(\mathbf{C}) \neq \phi$. Put $S = \overline{\delta(\mathbf{C})}$. Since $\mathbf{C} \supset \mathbf{P}$, we have $|Q_x^S(g)| < \infty$ for $g \in \mathbf{H_P}(S)$ (cf. the proof of Lemma 2.1). By Corollary 2.2 again, we see that $Q_x^S(g) = g(x)$ for any g eB|5. Evidently the mapping: $g \to Q_x^S(g)$ is sublinear on $\mathbf{H_P}(S)$ and particularly linear on $\mathbf{B}|S$. By the Hahn-Banach extension theorem, there exists a linear functional F on $\mathbf{H_P}(S)$ satisfying $F \leqq Q_x^S$ and $F(g) = g(x)$ on B|S. If $g \leqq 0$, we have $Q_x^S(g) \leqq 0$, whence $F(g) \leqq 0$. Thus F is positive. Further, B|S is a lattice and

$$F(/\wedge g) = (/\wedge \text{flf})\ (\text{x}) = \min\{f(x), g(x)\} = \min\{F(f), F(g)\}$$

for f, g e B. Hence F satisfies the assumptions of Lemma 1.3 with $X = S$. Consequently there exist $\lambda > 0$ and y e S satisfying

$$F(f) = \lambda f(y)$$

for any $f \in \mathbf{B}$. Since B is linearly separating, we have $x = y$ and hence $x \in \overline{\delta(\mathbf{C})}$.

PROPOSITION 3.6. *Let* C *be a linearly separating, min-stable convex cone with* $\mathbf{P} \subset \mathbf{C} \subset \mathbf{H_P}$. *Assume that a linear space* B *of C-affine continuous functions in* $\mathbf{H_P}$ *is a lattice in the natural order. If* x *is a point of* $\overline{\delta(\mathbf{C})}$ *and satisfies*

$$Q_x^{\mathbf{C}}(\max\{f, g\}) = \inf\{h(x); h \geqq \max\{f, g\}, h \in \mathbf{B}\} \tag{3.2}$$

for any /, g e B, *then*

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \tag{3.3}$$

for any f, $g \in \mathbf{B}$.

PROOF. Let $x \in \delta(\mathbf{C})$. Since B is a linear space, it suffices to establish the following relation:

$$(3.4) \qquad (f \vee g)(x) = \max\{f(x), g(x)\}$$

for any /, $g \in \mathbf{B}$. Obviously $(f \vee g)(x) \geqq \max\{f(x), g(x)\}$ holds. Putting $\varphi = \max\{/, g\}$, we have

$$\varphi(x) = Q_x^{\mathbf{C}}(\varphi) = \inf\{h(x);\ h \geqq \varphi,\ h \in \mathbf{B}\}$$

by Proposition 3.1 and (3.2). Therefore, for any $\varepsilon > 0$ there exists $h \in \mathbf{B}$ satisfying $h \geqq \varphi$ and $\varphi(x) + \varepsilon > h(x)$. Since $h \geqq f$, $h \geqq g$ and fteB, we have

$$\varphi(x) + \varepsilon > h(x) \geqq (f \vee g)(x),$$

whence $\varphi(x) \geqq (fg)(x)$. Hence (3.4), and so (3.3), holds for $x \in \delta(\mathbf{C})$. By continuity, (3.3) holds for x e $\overline{\delta(\mathbf{C})}$.

By Propositions 3.5 and 3.6 we have the following corollary.

COROLLARY 3.1. *Let C and* B *be as in* ***Proposition*** 3.6. *Assume that* B *is linearly separating and* (3.2) *holds for any f,* g eB *and* $x \in X$. ***Then,*** $x \in X$ *is an element of* $\overline{\delta(\mathbf{C})}$ *if and only if*

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

for any f, g *in* B.

Now, let B be an adapted space in $\mathscr{C}(X)$. We write

$$\mathbf{C}(\mathbf{B}) = \{\min\{f_1, \ldots, f_n\};\ f_i \in \mathbf{B},\ n \geqq 2\}.$$

Then C(B) is a min-stable convex cone which contains the adapted cone $\mathbf{B}^+$ and which is contained in $\mathbf{H}_{\mathbf{B}^+}$. For $x \in X$ and $\mu \in \mathfrak{M}^+_{\mathbf{B}^+}$ the relation $\varepsilon_x \prec_{\mathbf{C}(\mathbf{B})} \mu$ is equivalent to the relation $\varepsilon_x \prec_{\mathbf{B}} \mu$. It follows that $\delta(\mathbf{C}(\mathbf{B})) = \delta(\mathbf{B})$ and, by Corollary 2.2, $Q_x^{\mathbf{C}(\mathbf{B})}(g) = Q_x^{\mathbf{B}}(g)$ for $g \in \mathbf{H}_{\mathbf{B}^+}$.

The following theorem is an extension of Satz 10 in [1].

THEOREM 3.2. *Let* B *be an adapted space* ***which*** *is linearly separating and closed under the compact convergence topology. Then the* ***following*** *two* ***assertions*** *are equivalent:*

(a) B *is a lattice in the natural* ***order,***

(b) *any function in* $\mathbf{H}_{\mathbf{B}^+}(\overline{\delta(\mathbf{B})})$ *can be extended to an element* ***of*** $\mathbf{B}$.

Proof. (a) $\Rightarrow$ (b): Since $Q_x^{\mathbf{C}(\mathbf{B})}(\varphi) = Q_x^{\mathbf{B}}(\varphi) = \inf\{h(x)\ h$ e B, $h \geqq \varphi\}$ for any $\varphi \in \mathbf{H}_{\mathbf{B}^+}$, the previous corollary implies

$$\overline{\delta(\mathbf{B})} = \overline{\delta(\mathbf{B}(\mathbf{C}))}$$

$$= \{x \in X;\ (f \wedge g)(x) = \min\{f(x), g(x)\} \quad \text{for any } f, g \in \mathbf{B}\}.$$

Put $S = \overline{\delta(\mathbf{B})}$ and $\mathbf{B}_1 = \mathbf{B}|\overline{\delta(\mathbf{B})}$. Then S is a B-determining set and $\mathbf{B}_1$ is min-stable and linearly separating. Let $f \in \mathscr{C}_K(S)$. By Proposition 1.2 there exist $v \in \mathbf{B}^+$ and a sequence $\{g_n\} \subset \mathbf{B}$ such that

$$|f - g_n| \leqq (1/n)v \qquad (n = 1, 2, \dots)$$

on S. Since

$$|g_n - g_m| \leqq ((1/n) + (1/m))v \quad \text{on} \quad S \quad \text{for any} \quad n, m \in \mathrm{N},$$

the same inequality holds also on the whole X. Consequently the sequence $\{g_n\}$ in B converges uniformly on any compact set and $g = \lim g_n$ belongs to B by our assumption. It is obvious that $g = f$ on S. Consequently any function in $\mathscr{C}_K(S)$ can be extended to an element of B.

Similarly we may show that any function in $\mathbf{H}_{\mathbf{B}^+}(S)$ can be extended to an element of B by using Proposition 1.1.

(b)$\Rightarrow$(a): Since $\overline{\delta(\mathbf{B})}$ is a B-determining set, the extension of f in $\mathbf{H}_{\mathbf{B}^+}(\overline{\delta(\mathbf{B})})$ to an element of B is unique, which we denote by h_f. Let $f, g \in \mathrm{B}$ and $\varphi = \min\{f|\overline{\delta(\mathbf{B})}, g|\overline{\delta(\mathbf{B})}\}$. Evidently we have $f \wedge g = h_\varphi$, and hence infer that B is a lattice.

Chapter 4. Applications to potential theory

§ 4.1. Adapted cone of potentials

Let Ω be a harmonic space satisfying Bauer's axioms I, II, III and IV in [2, p. 11]. By definition Ω is a locally compact Hausdorff space with a countable base. A non-negative superharmonic function s is called a potential if the greatest subharmonic minorant of s is equal to 0. We call Ω a *strong harmonic space* if for any $x \in \Omega$ there exists a potential f with $f(x) > 0$.

Hereafter we assume that Ω is a strong harmonic space and use notations and terminologies in [2]. For a set E in Ω, let ∂E be the topological boundary of E.

Let E be a subset of Ω and f a non-negative function defined on E. We put

$$R_f^E = \inf\{g; g \text{ is non-negative hyperharmonic on } \Omega, g \geqq f \text{ on } E\}$$

and

$$\hat{R}_f^E(x) = \liminf_{y \to x} R_f^E(y).$$

By using the functions of the form R_f^Ω, we can show that there exists a continuous potential p_0 such that $p_0(x) > 0$ for all $x \in \Omega$ (cf. [2, Korollar 2.5.10] and [7, Proposition 2.2.2]).

Let P be the convex cone of all continuous potentials. Then P satisfies condition $(\mathbf{p}_1)$ in § 1.1 by the above consideration. By [7, Proposition 2.2.4], we see that P also satisfies condition $(\mathbf{p}_2)$, so that P is an adapted convex cone. Furthermore, P is min-stable and linearly separating by virtue of [2, Satz 2.5.3 and Satz 2.5.8].

We have the following minimum principle ([2, Korollar 2.4.3]):

PROPOSITION 4.1. *Let u be a hyperharmonic function in an open set U in Ω. If*

$$\liminf_{\substack{x \to z \\ x \in U}} u(x) \geqq 0 \qquad \textit{for all} \quad z \in \partial U$$

and if $u \geqq -v$ on U for some $v \in \mathrm{P}$, then $u \geqq 0$ on U.

Using this proposition and the potential $\mathbf{p}_0$ mentioned above, we obtain

PROPOSITION 4.2. *Let E be a closed set in Ω and u be a hyperharmonic function on an open set containing E. If*

$$\liminf_{\substack{x \to z \\ x \in CE}} u(x) \geqq 0 \qquad \textit{for all} \quad z \in \partial E$$

and if $u \geqq -v$ on E for some $v \in \mathrm{P}$, then $u \geqq 0$ on E.

§ 4.2. Balayaged measures and harmonic measures

Now, $\mathfrak{M}_{\mathbf{P}}^+$ is the space of all **P-integrable** measures on Ω.

PROPOSITION 4.3 (cf. [2, Satz 3.4.1], [7, Prop. 7.1.2]). *For each $\mu \in \mathfrak{M}_{\mathbf{P}}^+$ and each subset E of Ω, there exists a unique measure μ^E on $\bar{E}$ such that*

$$\mu^E(v) = \mu(\hat{R}_v^E)$$

for any $v \in \mathbf{P}$.

PROOF. Since $\bar{\mathbf{P}}|E$ is a min-stable, linearly separating adapted convex cone in $\mathscr{C}^+(E)$, $\mathscr{N} = \mathbf{P}|\bar{E} - \mathbf{P}|\bar{E}$ is dense in $\mathbf{H}_{\mathbf{P}}(\bar{E})$ by Corollary 1.2. Since the mapping $u \mapsto \hat{R}_u^E$ on $\mathbf{P}|\bar{E}$ is additive ([2, Satz 3.2.3]) and $\hat{R}_u^E$ is P-integrable for any $u \in \mathbf{P}|\bar{E}$, $L(d) = \mu(\hat{R}_u^E - \hat{R}_v^E)$ is well-defined for $d = u - v \in \mathscr{N}$. It is easy to see that L is a positive linear functional on $\mathscr{N}$. Hence L is uniquely extended to a positive linear functional on $\mathbf{H_P}(\bar{E})$ by Lemma 1.2. Hence there exists a measure μ^E on $\bar{E}$ such that $\mu^E(f)=L(f)$ for any $f\in\mathbf{H_P}$. In particular we have $\mu^E(v) = \mu(\hat{R}_v^E)$ for $v\in\mathbf{P}$.

The measure μ^E is called the *balayaged measure* of μ on E.

Let U be an open set in Ω. For $x\in U$ we call $(\varepsilon_x)^{CU}$ the *harmonic measure* with respect to c and U, and denote it by μ_x^U. Since $\hat{R}_v^{CU}=\hat{R}_{\cdot\cdot}^{\hat{c}U}$on (7 for any $v \in\mathbf{P}$ as is easily seen (see [16, Lemma 1]), μ_x^U is supported by dU (cf. [2, Satz 3.4.3]).

§ 4.3. Dirichlet problem for an open set U

We consider the Dirichlet problem for an open set U in Ω with $\partial U\neq\phi$. Let f be an extended real-valued function on ∂U. We denote by $\mathfrak{H}_f^U$ the family of all hyperharmonic functions v in U satisfying the following conditions:

1) $\liminf_{U\ni x\to z} v(x)\geqq f(z)$ for any $z\in\partial U$,
2) $v\geqq -p$ for some $p\in\mathbf{P}$.

The constant $+\infty$ belongs to $\mathfrak{H}_f^U$ and hence *fyy^φ*. We define

$$\bar{H}_f^U=\inf\{u;\ u\in\bar{\mathfrak{H}}_f^U\}$$

and $\underline{H}_f^U=-\bar{H}_{-f}^U$. By Proposition 4.1, $\underline{H}_f^U\leqq\bar{H}_f^U$. If $\underline{H}_f^U=\bar{H}_f^U$and it is harmonic in U, then we say that f is *resolutive* and we write

$$H_f^U=\underline{H}_f^U=\bar{H}_f^U.$$

The following proposition is easily proved (see the proof of [2, Satz 4.1.5] and [7, p. 18, Theorem 2.4.1 and Proposition 5.3.3]):

PROPOSITION 4.4. (a) */// and g are resolutivefunctions on dU, then $f+g$* (when it has a meaning everywhere on *dU) and* λf (λ: real) *are resolutive and*

$$H_{f+g}^U=H_f^U+H_g^U,\qquad H_{\lambda f}^U=\lambda H_f^U;$$

(b) *If $f\leqq g$ on ∂Uthen* $\bar{H}_f^U\leqq\bar{H}_g^U$*and* $\underline{H}_f^U\leqq\underline{H}_g^U$;

(c) *For any veP, its restriction to dU is resolutive and*

$$H_v^U=\hat{R}_v^{CU}=R_v^{CU}\qquad\text{on}\quad U.$$

By Propositions 1.1, 1.2, 4.3 and 4.4, we obtain (cf. [2, Satz 4.1.7])

PROPOSITION 4.5. *Any $f\in\mathbf{H_P}(\partial U)$ is resolutive and satisfies*

$$H_f^U(x)=\mu_x^U(f)\qquad\textit{for any}\quad x\in U.$$

A point $x_0 \in \partial U$ is said to be *regular* for U if

$$\lim_{U \ni x \to x_0} H_\varphi^U(x) = \varphi(x_0) \tag{4.1}$$

for any $\varphi \in \mathbf{H_P}(\partial U)$, or equivalently

$$\lim_{U \ni x \to x_0} \mu_x^U = \varepsilon_{x_0} \quad \text{in the topology} \quad \sigma(\mathfrak{M}_\mathbf{P}(\partial U), \mathbf{H_P}(\partial U)).$$

LEMMA 4.1. *Let U be an open set in Ω with $\partial U \neq \phi$. A point $x_0 \in \partial U$ is regular if and only if*

$$\liminf_{U \ni y \to x_0} H_v^U(y) \geqq v(x_0)$$

holds for any $v \in \mathbf{P}$.

PROOF. The "only if" part is obvious. Assume that $\liminf_{U \ni y \to x_0} H_v^U(y) \geqq v(x_0)$ for any $v \in \mathbf{P}$. Since $H_v^U(y) = \hat{R}_v^{CU}(y) \leqq v(y)$ for any $y \in U$, $\limsup_{U \ni y \to x_0} H_v^U(y) \leqq v(x_0)$. Consequently we have

$$\lim_{U \ni y \to x_0} H_g^U(y) = g(x_0) \qquad \text{for any} \quad g \in \mathbf{P} - \mathbf{P}.$$

Let $\varphi \in \mathbf{H_P}(\partial U)$. By Propositions 1.1, 1.2 and 4.4, we can find a sequence $\{g_n\}$ in $\mathbf{P} - \mathbf{P}$ such that $H_{g_n}^U$ converges to H_φ^U uniformly on a neighborhood of x_0. Hence we have (4.1) for $\varphi \in \mathbf{H_P}(\partial U)$.

LEMMA 4.2. *Let U be an open set in Ω with $\partial U \neq \phi$ and $z \in \partial U$. Assume that V is a neighborhood of z. Then z is a regular point of U if and only if z is a regular point of $U \cap V$.*

PROOF. Let $v \in \mathbf{P}$. Then $H_v^U \leqq H_v^{U \cap V} \leqq v$ on $U \cap V$. Hence by Lemma 4.1, if z is regular for U, then so is for $U \cap V$. Conversely, assume that z is a regular point of $U \cap V$. For any $v \in \mathbf{P}$, we define

$$g(y) = \begin{cases} v(y) & \text{if} \quad y \in \partial U \cap \overline{V}, \\ H_v^U(y) & \text{if} \quad y \in \partial V \cap U. \end{cases}$$

It is easy to see that g is resolutive for $U \cap V$ and $H_g^{U \cap V} = H_v^U | U \cap V$ (cf. [2, Lemma 4.2.4]). Since g is equal to v on a neighborhood of z and $0 \leqq g \leqq v$, we can easily show that $\lim_{x \to z} H_g^{U \cap V}(x) = g(z)$, and hence $\lim_{x \to z} H_v^U(x) = v(z)$. Thus by Lemma 4.1, z is a regular point of U.

A set $E \subset \Omega$ is said to be *thin* at a point $x \in E$, if

$$\inf_{V \in \mathfrak{B}_x} \hat{R}_1^{E \cap V}(x) < 1,$$

where $\mathfrak{V}_x$ is the set of all neighborhoods of x ([2, p. 107]). It is easy to see that E is thin at $x \in E$ if and only if there are $v \in P$ and $V \in \mathfrak{V}_x$ such that $\hat{R}_v^{E \cap V}(x) < v(x)$.

PROPOSITION 4.6. *Let U be an open set in Ω with $\partial U \neq \phi$ and $x_0 \in \partial U$. Then the following assertions are equivalent:*

(i) *x_0 is a regular point of U,*
(ii) *CU is not thin at x_0,*
(iii) $(\varepsilon_{x_0})^{CU} = \varepsilon_{x_0}$.

The proof of this proposition is similar to that of [2, Satz 4.3.1]. In fact, (i)⇒(ii) follows from Lemma 4.2 and Proposition 4.4, (c); (ii)⇒(iii) is immediate; and (iii)⇒(i) follows from Proposition 4.4, (c) and Lemma 4.1.

The following lemma is proved in the same way as [8, Lemma 3.1] by using Propositions 1.1 and 1.2, Lemma 3.1 and the previous proposition:

LEMMA 4.3. *If $z \in \partial U$, there exists a sequence $\{x_n\}$ in U converging to z for which the measure $\mu_{x_n}^U = (\varepsilon_{x_n})^{CU}$ converges to $\mu_x = (\varepsilon_z)^{CU}$ in the topology $\sigma(\mathfrak{M}_{\mathbf{P}}(\bar{U}), \mathbf{H}_{\mathbf{P}}(\bar{U}))$.*

COROLLARY 4.1 (cf. [2, Satz 3.4.3] and [7, Proposition 7.1.3]). *For each $z \in \bar{U}$, the balayaged measure $(\varepsilon_z)^{CU}$ is supported by dU.*

§4.4. The dilation given by balayaged measures

Let U be an open set in Ω with $\partial U \neq \phi$ and C be the set of all P-bounded continuous functions on $\bar{U}$ which are superharmonic in U. We know that C is a min-stable and linearly separating convex cone and $\mathbf{P}|\bar{U} \subset \mathbf{C} \subset \mathbf{H}_{\mathbf{P}}(\bar{U})$. By Proposition 4.1, dU is a C-determining set. Hence the Choquet boundary $\delta(\mathbf{C})$ of $\bar{U}$ is contained in dU (see §3.1). We write $B(x) = (\varepsilon_x)^{CU}$ for any $x \in \bar{U}$.

PROPOSITION 4.7. *The mapping: $x \mapsto B(x)$ from $\bar{U}$ into $\mathfrak{M}_{\mathbf{P}}^+(\bar{U})$ is a* C-*dilation and the set of all regular boundary points of U is just the set of all* B-*regular points.*

PROOF. For each $v \in \mathbf{P}$, the function: $x \mapsto B(x)(v) = \hat{R}_v^{CU}(x)$ is lower semicontinuous and hence Borel measurable. From Propositions 1.1 and 1.2, it follows that the mapping: $x \mapsto B(x)(f)$ is Borel measurable for each $f \in \mathbf{H}_{\mathbf{P}}(\bar{U})$ (cf. [7, Proposition 7.1.4]). Since $g \backslash U$ is an upper function of $g \backslash dU$ for each $g \in \mathbf{C}$,

$$B(x)(g) = H_g^U(x) \leqq g(x) \qquad \text{for any} \quad x \in U.$$

Hence $\varepsilon_x \prec_{\mathbf{C}} B(x)$ for $x \in U$. If $z \in \partial U$, then there exists a sequence $\{x_n\}$ in U

such that $B(x_n)$ converges to $B(z)$ in the topology $\sigma(\mathfrak{M}_P(\bar{U}), \mathbf{H}_P(\bar{U}))$ by Lemma 4.3. Hence

$$B(z)(g) = \lim B(x_n)(g) \leqq g(z) \qquad \text{for each} \quad g \in C.$$

Thus the mapping: $z \mapsto B(z)$ is a **C-dilation.** The last assertion of the proposition follows from Proposition 4.6.

By definition, the support of a superharmonic function s on Ω is the complement of the largest open set on which s is harmonic. We say that Ω satisfies *Axiom D* if for any locally bounded superharmonic function, the continuity of its restriction to its support implies the continuity on the whole Ω.

THEOREM 4.1 (cf. [9, Theorem 3.3]). *Suppose that Ω satisfies Axiom D. Then the balayage mapping: $x \mapsto B(x)$ is a weakly affine C-dilation.*

PROOF. Let $v \in P$. Then

$$B(x)v = \hat{R}_v^{CU}(x).$$

Since $\hat{R}_v^{CU}$ is a potential dominated by v, it follows from [7, Theorem 8.2.2] and Axiom D that there exists an increasing net $\{v_\alpha\}$ in P such that $\hat{R}_v^{CU} = \sup v_\alpha$ and each v_α is specifically smaller than R_v^{CU}, i. e., there is a potential w_α satisfying $\hat{R}_v^{CU} = v_\alpha + w_\alpha$ for each α. Since $\hat{R}_v^{CU}$ is harmonic on U, each v_α is harmonic on U. Hence $v_\alpha|\bar{U}$ is **C-affine** for each α. Since $\mathbf{P}|\bar{U}$ is **min-stable** and linearly separating, it follows that $x \mapsto B(x)$ is a weakly affine C-dilation.

COROLLARY 4.2. *Suppose that Ω satisfies Axiom D. Then the set of all regular boundary points is equal to the Choquet boundary $\delta(\mathbf{C})$ and $(\bar{U}, \mathrm{C})$ is a simplex. Further for each $x \in \bar{U}$, the balayaged measure $(\varepsilon_x)^{CU}$ is the unique extremal measure μ with $\varepsilon_x \prec \mu$.*

PROOF. This follows from Proposition 3.3, Proposition 4.7 and the above theorem.

THEOREM 4.2. *Suppose that Ω satisfies Axiom D. If the set S of all regular points of U is closed, then any P-bounded continuous function on S is uniquely extended to a continuous function on $\bar{U}$ which is harmonic in U.*

PROOF. Since $(\bar{U}, \mathrm{C})$ is a Bauer's simplex and $S = \delta(\mathbf{C})$, any P-bounded continuous **function** f on S is uniquely extended to a C-affine continuous function g on $\bar{U}$ by Theorem 3.1. By Corollary 2.2, we see that $Qg = g$ and $Q(-g) = -g$. It follows that g and $-g$ are superharmonic in U, and hence g is harmonic in U.

§4.5. The Dirichlet problem for the exterior of an open set

Let £ be a closed set in Ω with $\partial E\neq\phi$ and f an extended real-valued function on dE. We denote by $\overline{\mathfrak{K}}_f^E$ the set of all hyperharmonic functions v on an open set containing E which satisfy the following properties:

(i) $\liminf_{\substack{x\to y\\ x\in CE}} v(x)\geqq f(y)$ for any y e dE,

(ii) $v\geqq -p$ on E for some peP.

We define

$$\overline{K}_f^E = \inf\{v|E;\ v\in\overline{\mathfrak{K}}_f^E\}$$

and $\underline{K}_f^E=-\overline{K}_{-f}^E$. By Proposition 4.2, we see that $\underline{K}_f^E\leqq\overline{K}_f^E$. If $\underline{K}_f^E=\overline{K}_f^E$, we say that f is *resolutive* and write $K_f^E=\underline{K}_f^E=\overline{K}_f^E$.

PROPOSITION 4.8. (a) */// and g are resolutive functions on dE, then f+g (when it has a meaning everywhere on dE) and* λf (λ:real) *are resolutive and*

$$K_{f+g}^E = K_f^E+K_f^E, \qquad K_{\lambda f}^E = \lambda K_f^E.$$

(b) *If* $f\leqq g$ *on* ∂E, *then* $\overline{K}_f^E\leqq\overline{K}_g^E$ *and* $\underline{K}_f^E\leqq\underline{K}_g^E$.

(c) *For any* $v\in\mathbf{P}$, *its restriction to dE is resolutive and*

$$K_v^E=\hat{R}_v^{CE}=R_v^{CE}=\sup\{H_v^\omega;\ \omega\text{: open}\supset E\} \quad \textit{on} \quad E.$$

PROOF (cf. [13, p. 386]). In general, $\overline{K}_{f+g}^E\leqq\overline{K}_f^E+\overline{K}_g^E$ and $\overline{K}_{\lambda f}^E=\lambda\overline{K}_f^E$ for $\lambda\geqq 0$, from which (a) follows. (b) is immediate. To prove (c), let $v\in\mathbf{P}$. By [2, Satz 2.2.1 and Satz 3.2.7],

$$\text{(4.2)} \qquad \hat{R}_v^{CE}=R_v^{CE}=\sup\{R_v^{C\omega};\ \omega\text{: open}\supset E\}.$$

Since $p=\hat{R}_v^{CE}$ is a potential and $p=v$ on CE, $p\in\overline{\mathfrak{K}}_v^E$ and hence $p\geqq\overline{K}_v^E$ on E. On the other hand, for any open set $\omega\supset E$, $H_v^\omega=R_v^{C\omega}|\omega\in-\overline{\mathfrak{K}}_{-v}^E$, since $H_v^\omega\leqq v$ on ω. Hence

$$\text{(4.3)} \qquad H_v^\omega=R_v^{C\omega}\leqq\underline{K}_v^E\leqq\overline{K}_v^E\leqq\hat{R}_v^{CE} \quad \text{on } E.$$

By (4.2) and (4.3), we obtain (c).

PROPOSITION 4.9 (cf. [13, Théorème 2]). *Let E be a closed set with* $\partial E\neq\phi$. *Then any* $\varphi\in\mathbf{H}_\mathbf{P}(\partial E)$ *is resolutive and for any decreasing net* $\{\omega_i\}_{i\in I}$ *of open sets satisfying* $E=\bigcap_{i\in I}\omega_i$ *and a P-bounded continuous extension* Φ *of* φ,

$$\text{(4.4)} \qquad K_\varphi^E=\lim_{i\in I}H_\Phi^{\omega_i}.$$

PROOF. By Proposition 4.8, if $f \in \mathbf{P}-\mathbf{P}$, then $f|\partial E$ is resolutive and $K_f^E = \lim_{i \in I} H_f^{\omega_i}$. Hence by using Propositions 1.1 and 1.2, we see that for $\Phi \in \mathbf{H_P}$, $\varphi = \Phi|\partial E$ is resolutive and (4.4) holds.

A point $x_0 \in \partial E$ is called a *stable point* of E if $K_f^E(x_0) = f(x_0)$ holds for any $f \in \mathbf{H_P}(\partial E)$.

By Proposition 4.8, (c), we can easily show

PROPOSITION 4.10. *$x_0 \in \partial E$ is a stable point of E if and only if* $(\varepsilon_{x_0})^{CE} = \varepsilon_{x_0}$.

For each $x \in E$, the mapping: $f \mapsto K_f^E(x)$ on $\mathbf{H_P}(\partial E)$ defines a measure $K(x) \in \mathfrak{M}_{\mathbf{P}}^+$ on ∂E. We denote by C the set of all P-bounded continuous functions on E each of which is the restriction of a superharmonic function in an open set containing E. Then we have the following theorem.

THEOREM 4.3 (cf. [8, Theorem 4.1]). *The mapping: $x \mapsto K(x)$ is a weakly affine C dilation on E and the set of K-regular points on ∂E coincides with the set of stable points of E.*

PROOF. Since $K_v^E(x) = \hat{R}_v^{CE}(x)$ for $v \in \mathbf{P}$, the function: $x \mapsto K_v^E(x)$ is lower semicontinuous and hence Borel measurable for any $v \in$ P. Using Propositions 1.1 and 1.2, we can see that the function: $x \mapsto K_f^E(x) = K(x)(f)$ is Borel measurable for each $f \in \mathbf{H_P}(E)$. Since every $g \in$ C is the restriction of a function belonging to $\overline{\mathfrak{R}}_{g|\partial E}^E$, we have

$$K(x)(g) = K_g^E(x) \leqq g(x),$$

whence $\varepsilon_x \prec_{\mathbf{C}} K(x)$ for any $x \in \Omega$. Therefore K is a **C-dilation.** Let $v \in \mathbf{P}$. Since $H_v^\omega|E$ is **C-affine** for any open set $\omega \supset E$, we see that K is a weakly affine C-dilation by Proposition 4.8, (c). By definition, $x_0 \in \partial E$ is a K-**regular** point if and only if it is a stable point of E.

COROLLARY 4.3 (cf. [8, Corollary 4.2]). *The pair $(E$, C$)$ is a simplex and $\delta(\mathbf{C})$ is the set of all stable points of E.*

PROOF. Since the mapping: $x \mapsto K(x)$ is a weakly affine C-dilation, $(E$, C$)$ is a simplex and $\delta(\mathbf{C})$ coincides with the set of all K-**regular** points by Proposition 3.3. By Proposition 4.2, ∂E is a C-determining set and hence $\delta(\mathbf{C}) \subset \partial E$. From the above theorem it follows that $\delta(\mathbf{C})$ is the set of all stable points of E.

References

[1] H. Bauer, Šilovscher Rand und Dirichletsches Problem, Ann. Inst. Fourier 11 (1961), 89-136.
[2] H. Bauer, Harmonische Räume und ihre Potentialtheorie, Lecture Notes in Mathematics 22, Springer-Verlag, Berlin, 1966.
[3] N. Boboc and A. Cornea, Convex cones of lower semicontinuous functions on compact spaces, Rev. Roumaine Math. Pures Appl. 12 (1967), 471-525.
[4] N. Bourbaki, Integration, Ch. 1-4, 2^{e} éd., Hermann, Paris, 1965.
[5] G. Choquet, Lectures on analysis, 1 and 2, Benjamin, New York, 1969.
[6] G. Choquet and P. A. Meyer, Existence et unicité des representations integrates dans les convexes compacts quelconques, Ann. Inst. Fourier 13 (1963), 139-154.
[7] C. Constantinescu and A. Cornea, Potential theory on harmonic spaces, Springer-Verlag, Berlin, 1972.
[8] E. G. Effros and J. L. Kazdan, Applications of Choquet simplexes to elliptic and parabolic boundary value problems, J. Differential Equations 8 (1970), 95-134.
[9] G. Mokobodzki and D. Sibony, Cones adaptés de fonctions continues et theorie du potentiel, Séminaire Choquet, 6e annee, 1966/1967, no. 5.
[10] G. Mokobodzki and D. Sibony, Cones de fonctions continues, C. R. Acad. Sci. Paris, Sér. A 264 (1967), 15-18.
[11] G. Mokobodzki and D. Sibony, Cones de fonctions et theorie du potentiel II, Seminaire Brelot-Choquet-Deny, 11e annee, 1966/67, no. 9.
[12] R. R. Phelps, Lectures on Choquet's theorem, Van Norstrand Mathematical Studies 7, Princeton, 1966.
[13] A. de La Pradelle, Approximation et caractère de quasianalyticité dans la theorie axiomatique des fonctions harmoniques, Ann. Inst. Fourier, 17, 1 (1967), 383-399.
[14] A. de La Pradelle, A propos du mémoire de G. F. Vincent-Smith sur l'approximation des fonctions harmoniques, Ann. Inst. Fourier 19, 2 (1969), 355-370.
[15] H. Watanabe, Simplexes on a locally compact space, Natur. Sci. Rep. Ochanomizu Univ. 23 (1972), 35-42.
[16] H. Watanabe, On balayaged measures and simplexes in harmonic spaces, Natur. Sci. Rep. Ochanomizu Univ. 23 (1972), 61-68.

Department of Mathematics,
Faculty of Science,
Ochanomizu University