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On the Supports of the Transition Densities for Certain Stable Processes

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§ 1. Introduction

Let $\{X(t), t \geqq 0\}$ be a non-degenerate, drift free, d-dimensional stable process having exponent α, $0<\alpha<1$, and characteristic function $\psi(t, \xi)$ given by

$$(1)\quad \psi(t, \xi)=\exp\left\{-t|\xi|^{\alpha}\int_{S^{d-1}}[1-i\operatorname{sgn}(<\xi, \theta>)\tan(\pi\alpha/2)]|<\xi/|\xi|, \theta>|^{\alpha}\mu(d\theta)\right\}.$$

Here $t>0$, $\xi \in R^d$, S^{d-1} is the unit sphere with center origin in R^d, $\mu(d\theta)$ is a probability measure on S^{d-1} and $<\ ,\ >$ is the usual inner product in R^d. Then, $X(t)$ has the continuous transition density:

$$(2)\qquad p(t, x) = (2\pi)^{-d}\int_{R^d} e^{-i<x, \xi>}\psi(t, \xi)d\xi, \qquad t>0,\ x \in R^d.$$

The support of the transition density $p(t, x)$ was investigated by Taylor [4]. His result combined with Port's work [3] is that $p(t, x)>0$ for all $t>0$ and $x \in R^d$ if $p(1, 0)>0$. In the case $p(1, 0)=0$ and $d \geqq 2$, the properties of the support of $p(t, x)$ seems to be unknown except Taylor's remarks (see [4, p. 1233]) about what can be expected to hold. The purpose of this paper is to investigate the support of $p(t, x)$ in connection with that of μ for this case.

We denote by supp (μ) the smallest closed set with full μ-measure and put

$$M = \{\lambda\theta; \theta\ e\ \mathrm{supp}(\mu),\ \lambda \geqq 0\},$$

$$K_1 = \{x; p(t, x) > 0 \text{ for some } t > 0\},$$

$$K_2 = \{x; p(t, x) > 0 \text{ for any } t > 0\},$$

$$S(t) = \{x; p(t, x) > 0\}, \qquad t > 0.$$

Also we denote by A^-, $A^\wedge$ and A° the closure, the convex hull and the interior of a set A, respectively. Our theorem is stated as follows:

THEOREM. *Ifp*$(1, 0)=0$, *then* $K_1=((M^\wedge)^-)^\circ$. *//, in addition, the dimension* $l(M^\wedge)$ *of the largest subspace contained in* $M^\wedge$ *is zero, then* $K_1=K_2$—$(M^\wedge)^\circ$.

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§ 2. The Properties of $S(t)$

Making use of Kolmogorov-Chapman equation for $p(t, x)$ and the well known scaling relationship $p(t, x) = p(rt, r^{1/\alpha}x)r^{d/\alpha}$, we have

$$S(t_1)+S(t_2) = S(t_1+t_2), \tag{3}$$

$$S(rt) = r^{1/\alpha}S(t), \qquad r > 0. \tag{4}$$

If we set $S = S(1)$, (3) and (4) imply that

$$r^{1/\alpha}S+(1-r)^{1/\alpha}S = S, \qquad 0 \leqq r \leqq 1. \tag{5}$$

It is not obvious whether in general (5) implies

$$r^{1/\alpha}S^{-}+(1-r)^{1/\alpha}S^{-} = S^{-}, \qquad 0 \leqq r \leqq 1, \tag{6}$$

or not. However, under some condition we can prove (6), as will be seen below.

LEMMA 1. *If S is contained in a closed convex cone C with* 1(C) = 0, *then S~ has the property* (6).

PROOF. For each x e 5~, we choose a sequence $\{x_n\}$ in S converging to x as $n\to\infty$. Then, by (5) for each r, $0\leqq r\leqq 1$, there exist two sequences $\{y_n\}$ and $\{z_n\}$ in S such that

$$r^{1/\alpha}y_n+(1-r)^{1/\alpha}z_n = x_n. \tag{7}$$

We now prove that both $\{y_n\}$ and $\{z_n\}$ are bounded. For this purpose denote by L_n the straight line passing through y_n and z_n, and let w_n be the point on L_n which is nearest to the origin. Then, with a suitable unit vector e_n perpendicular to w_n, y_n and z_n can be expressed as $y_n = a_n e_n + w_n$ and $z_n = b_n e_n + w_n$, a_n, $b_n \in R^1$. From (7) we have $[r^{1/\alpha}+(1-r)^{1/\alpha}]^{-1}x_n = c_n e_n + w_n$ where $c_n = [r^{1/\alpha}a_n + (l-r)^{1/} b_n][r^{1/\alpha}+(1-r)^{1/\alpha}]^{-1}$, and hence $[r^{1/} + (1-r)^{1/}]^{-1}x_n \in L_n$. This implies that $\{w_n\}$ and $\{c_n\}$ are bounded. The definition of c_n then implies that both $\{a_n\}$ and $\{b_n\}$ are bounded or both $\{a_n\}$ and $\{b_n\}$ are unbounded. Suppose the latter case happens. Then we have $a_n\to\infty$ (or $-\infty$) and fo$_n$— › $-\infty$ (or ∞) as $n\to\infty$ via some subsequence. Since $\{w_n\}$ is bounded, the sequence of the line segments $\overline{y_n z_n}$ joining y_n to z_n converges to some full straight line L as $n\to\infty$ via some further subsequence. Since $y_n z_n$ is in the closed set C, L is also in C, and this contradicts 7(C) = 0. Thus we have proved that $\{y_n\}$ and $\{z_n\}$ are bounded. Therefore $\{y_n\}$ and $\{z_n\}$ converge to some $y \in S^{-}$ and $z \in S^{-}$, respectively, as tt-»oo via some (common) subsequence, and $r^{1/\alpha}y+(1-r)^{1/\alpha}z=$ x, which shows $r^{1/\alpha}S^{-}+(1-r)^{1/\alpha}S^{-}\subset S^{-}$. Finally the inverse relation $r^{1/\alpha}S^{-}+(1-r)^{1/\alpha}$ $S^{-}\subset S^{-}$ is easily shown by (5). This completes the proof.

LEMMA 2 ([1]). *If C is a closed convex* **cone,** *then* 7(C) = 0 *if and only if there exists a* **unit** *vector e such that* <**β**, ξ> > 0 *for any* ***non-zero*** *vector ξ in C.*

The proof of this lemma is found in Fenchel [1, pp. 10–11].

LEMMA 3. *Under the hypothesis of* Lemma 1 *and* $\alpha<1$, *S~ is a convex cone.*

PROOF. We first note that (6) and $\alpha<1$ imply that $\lambda S^{-}\subset S^{-}$ for $0\leqq\lambda\leqq 1$. Next we put $S_0=\{x\in S^{-};\lambda x\in S^{-}$ for any $\lambda\geqq 0\}$. Then, from this definition and (6), it follows that S_0 is a closed convex cone. We show by contradiction $S^{-}=S_0$. Suppose that there exists a point $x_0\in S^{-}\backslash S_0$ and denote by y_0 the point on S_0 which is nearest to x_0. Noting that S_0 is a closed convex cone, we can choose a supporting hyperplane P_0 of S_0 at y_0. Obviously P_0 passes the origin. Since $I(C)=0$, there exists a unit vector e such that <**e**, ξ> >0 for any non-zero vector ξ in C by Lemma 2. Let us set $H=\{x;$ <x, e> $=0\}$, denote by P the hyperplane determined by H n P_0 and $\frac{1}{2}x_0+\frac{1}{2}y_0$ and by P^{+} the closed half space bounded by P and not containing the interior of S_0. We now claim that there exists a supporting hyperplane β of S^{-} which is parallel to P. For the proof it is enough to see that P^{+} n S^{-} is bounded. If it is unbounded, then there exists a sequence $\{x_n\}$ in P^{+} n S~ such that $|x_n|\to\infty$ as $n\to\infty$. The sequence of rays $L_n^{+}=\{\lambda x_n;\lambda>0\}$ has a convergent subsequence; let L^{+} be the limiting ray. Since $\lambda S^{-}\subset S^{-}$ for $0\leqq\lambda\leqq 1$, L^{+} is contained in S^{-} and lies outside S_0. This contradicts the definition of S_0, and so P^{+} n S~ is bounded.

Choose a point x in S~ n β. By (6) there exist $y, z\in S^{-}$ such that $y+z=2^{1/\alpha}x$. Since $x\neq 0$ and $2^{\frac{1}{\alpha}-1}>1$, the point $2^{\frac{1}{\alpha}-1}$ x lies in Q^{+} (the region bounded by Q and not containing the origin) and hence so does $(y+$ z$)/2$. But then, one of y and z must lie in Q^{+}, which is a contradiction since $y, z\in S^{-}$. We thus finally proved that S~ $=S_0$. Since S_0 is a convex cone, the lemma is proved.

§ 3. Proof of the theorem

First we note that $\alpha<1$ and supp(μ) is contained in some hemisphere by the assumption $p(1, 0)=0$ (see [4]). Thus M^ is a convex cone **which** is not the whole space R^d, and hence $(M\hat{})^{-}$ is equal to the intersection of all closed half spaces which contain M^, and for each closed half space which contains M^ there is a unit vector e such that $\{x;<e,x>\geqq 0\}$ is equal to this half space.

Now we set $X^e(t) = <X(t), e>$. Then $X^e(t)$ is a one-dimensional stable process with the exponent α and characteristic function

$$\exp\{-\lambda t|\eta|^\alpha[1 - i\,\mathrm{sgn}(\eta)\tan(\pi\alpha/2)]\}, \tag{8}$$

where $\lambda = \int |<e, \theta>|^\alpha \mu(d\theta)$. Thus, by the result in the case $n=1$ (see for a summary [4]), the transition density $p^e(t, y)$ of $X^e(t)$ is zero on the interval $(-\infty, 0]$ for all $t>0$. Since $p^e(t, y)$ can be obtained by

$$p^e(t, y) = \int_{\{<z,e>=0\}} p(t, ye+z)dv(z), \tag{9}$$

where v is the volume element on the hyperplane $\{z; <z, e>=0\}$, we have $p(t, x)=0$ in $\{x; <x, e> \leqq 0\}$ for all $t>0$. Noting once more that $(M^\wedge)^-$ is equal to the intersection of all closed half spaces which contain $M^\wedge$, it follows that $p(t, x)=0$ in the complement of $(M^\wedge)^-$ for all $t>0$, and hence $K_1 \subset (M^\wedge)^-$. Next we prove that $K_1 = ((M^\wedge)^-)^\circ$. For this we will use the following facts (i), (ii) and (iii):

(i) $\lim_{t\downarrow 0}\int_{R^d} f(x)p(t, x)/t\,dx = \int_{R^d} f(x)r^{-\alpha-d}dr\mu(d\theta), \quad f \in C_0(R^d\backslash\{0\})$,

where $r=|x|$, $\theta=x/r$ and $C_0(R^d\backslash\{0\})$ is the space of cotinuous functions with compact supports in $R^d\backslash\{0\}$.

(ii) $p(t+s, x) = \int_{R^d} p(t, x-y)p(s, y)dy$.

(iii) K_1 is an open convex set (see Taylor [4]).

In fact, (i) implies $K_1^- \supset M$, (ii) implies $K_1^- + K_1^- \subset K_1^-$ and hence

$$K_1^- \supset K_1^- + \cdots + K_1^- \supset M + \cdots + M \equiv M_n.$$

Because M is a cone, $M^\wedge = \cup_n M_n$ and so $K_1^- \supset (M^\wedge)^-$. Since $K_1^- \subset (M^\wedge)^-$ is already known, we have $K_1^- = (M^\wedge)^-$. On the other hand (iii) implies that $K_1 = (K_1^-)^\circ$ (see P]) and so $K_1 = ((M^\wedge)^-)^\circ$ as was to be proved.

We now proceed to the proof of the latter half of the theorem. For $x \in S$ and $0 \leqq r < 1$, $(1-r)^{1/\alpha}(S-x)$ is an open neighborhood of 0. Therefore, by (5) $r^{1/\alpha}S^- = (r^{1/\alpha}S)^- \subset r^{1/\alpha}S + (1-r)^{1/\alpha}(S-x) = r^{1/\alpha}S + (1-r)^{1/\alpha}S - (1-r)^{1/\alpha}x = S - (1-r)^{1/\alpha}x$, and hence $r^{1/\alpha}S^- + (1-r)^{1/\alpha}S \subset S$. From this result we can prove that $(S^-)^\circ \subset S$. Hence $S = (S^-)^\circ$. As a consequence of Lemma 3, we see that 5 is an open convex cone. From this fact and (4) it follows that $S = S(t)$ for all $t>0$, that is, $K_1 = K_2$. Finally $K_1 = (M^\wedge)^\circ$ follows from the fact: If C is a closed cone and $I(C^\wedge)=0$, then $(C^\wedge)^- = C^\wedge$ (see [1]). The proof of our theorem is completed.

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