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A Note on Coalgebras and Rational Modules

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1. Introduction

Let C be a coalgebra and C* its dual algebra. Every right C-comodule is equipped functorially with a left C*-module structure. A left C*-module thus obtained is called rational. On the other hand every left C*-module M has a unique maximal rational submodule M^{rat} and the correspondence $M \mapsto M^{rat}$ is a functor from the category of left C*-modules to that of rational ones which form a full subcategory of the former. This functor is left exact.

In this note we study the relation between the structure of a coalgebra C and the functor $M \mapsto M^{rat}$.

In Section 3 we consider the exactness of the functor and show the following: When C is irreducible, the functor is exact if and only if C is of finite dimension. When C is cosemisimple, the functor is exact. When C is cocommutative, the functor is exact if and only if C is a direct sum of finite-dimensional subcoalgebras.

In [3] Radford has proved that if every open left ideal in C* is finitely generated, then the class of rational modules is closed under group extensions. And recently Lin [2] investigated as an application of the torsion theories the structure of a coalgebra for which the functor is a left exact radical, i.e., the class of rational modules is closed under group extensions. In Section 4 we study the extension problem and prove the converse of the Radford's result above when the coalgebra has a finite-dimensional coradical or when the coalgebra is cocommutative (Theorem 4.6 and Corollary 4.9). We don't use the torsion theories but some topological concepts in [3].

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2. Preliminaries.

Throughout this note, vector spaces, coalgebras and algebras we consider are all over a fixed commutative field k and all linear mappings are k-linear. We follow the terminology in [4] with a few exceptions.

(2.1) Let £ be a vector space and E^* its dual space. E^* has the weak-*topology. An open subspace of E^* with this topology is just a closed and cofinite subspace. Let / be a linear mapping of E into a vector space F. Then the transposed mapping $f^*: F^* \to E^*$ is continuous. If $\operatorname{Ker} f$ is finite-dimensional, then f^* sends an open subspace of F^* onto an open subspace of E*. In particular, it is the case if f is injective.

(2.2) The dual space C* of a coalgebra C is a topological algebra. Every finitely generated left (or right) ideal of C* is closed.

(2.3) Let C be a coalgebra and C* its dual algebra. A left C*-module M is called rational if M is mapped into $M \otimes C$ under the canonical injection M $\to \operatorname{Hom}(C^*, M)$ ([4], p. 37). Every left C*-module M has a unique maximal rational submodule which we denote by $\operatorname{Rat}(M)$ or $\operatorname{Rat}_C(M)$ rather than M^{rat} [4], Th. 2.1.3). For an element $m \in M$ the following conditions are equivalent ([3], p. 519):

(a) $m \in \operatorname{Rat}(M)$,

(b) $\operatorname{Ann}(m)$ is an open left ideal of C^*,

(c) $\operatorname{Ann}(m)$ contains an open (two-sided) ideal of C*.

Here $\operatorname{Ann}(m)$ is the annihilator of m, i.e., $\operatorname{Ann}(m) = \{c^* \in C^* | c^*m = 0\}$.

(2.4) Let C and D be coalgebras and let f: $C \to D$ be a coalgebra homomorphism. Then f^*: $D^* \to C^*$ is a continuous algebra homomorphism, which induces a functor from the category of C*-modules to the category of D^*-modules by change of rings.

If M is a rational left C*-module, then M is also rational as a D*-module. This implies that

$$\operatorname{Rat}_C(M) \subset \operatorname{Rat}_D(M)$$

as subsets of M. In fact, for any $m \in M$ we have

$$\operatorname{Ann}_{D^*}(m) = f^{*-1}(\operatorname{Ann}_{C^*}(m)).$$

In particular, if Ker/ is finite-dimensional, then by (2.1) as subsets of M we have

$$\operatorname{Rat}_C(M) = \operatorname{Rat}_D(M).$$

(2.5) C* is regarded as a left C*-module by multiplication. Then by (1.8) in [5] we have

$$\operatorname{Rat}(C^*) = \text{the sum of all finite-dimensional left ideals in } C^*.$$

Moreover, it is a two-sided ideal in C^*, and $\operatorname{Rat}(C^*) = C^*$ if and only if C is of finite dimension. In fact, for any $c^*, d^* \in C^*$ we have

$$\operatorname{Ann}(c^*d^*) \supset \operatorname{Ann}(c^*).$$

The second part is obvious.

3. Functor Rat.

(3.1) For a coalgebra C, $\mathbf{Rat}_C$ is a functor from the category of left C*-modules to that of rational ones which is a full subcategory of the former, where $\mathrm{Rat}(f) = f|_{\mathrm{Rat}(M)}$ the restriction for every C^*-homomorphism / of a C^*-module M. And for any submodule N of M we have

$$\mathrm{Rat}(N) = N \text{ n } \mathrm{Rat}(M).$$

It follows that the functor $\mathbf{Rat}_C$ is left exact. But in general it is not exact.

(3.2) If $\mathbf{Rat}_C$ is exact, then for every left C*-module M we have

$$\mathrm{Rat}(M/\mathrm{Rat}(M)) = 0.$$

In fact, from the exact sequence $0 \to \mathrm{Rat}(M) \to M \to M/\mathrm{Rat}(M) \to 0$ we have an exact sequence $0 \to \mathrm{Rat}(\mathrm{Rat}(M)) \to \mathrm{Rat}(M) \to \mathrm{Rat}(M/\mathrm{Rat}(M)) \to 0$. But (3.1) implies $\mathrm{Rat}(\mathrm{Rat}(M)) = \mathrm{Rat}(M)$, so we have the result.

REMARK. More generally, (3.2) holds if the rationality satisfies the condition that if L and M/L are rational then so is M.

PROPOSITION 3.3. *// C is a subcoalgebra of a coalgebra D and if* $\mathbf{Rat}_D$ *is exact, then* $\mathbf{Rat}_C$ is *exact.*

PROOF. This follows from (2.4).

LEMMA 3.4. *Let C be irreducible with coradical R. If* $\mathrm{Rat}_C(C^*) \neq C^*$, *then* $\mathrm{Rat}_C(C^*) \subset R^{\perp}$.

PROOF. Let / be a finite-dimensional left ideal. By (2.2) the right ideal IC^* in C* generated by / is closed, and it is a two-sided ideal. Therefore $IC^* \subset R^{\perp}$ or $IC^* = C^*$. But since $\mathrm{Rat}(C^*)$ is a proper ideal, $IC^* \subset \mathrm{Rat}(C^*) \subsetneqq C^*$.

THEOREM 3.5. *Let C be an irreducible coalgebra. Then* $\mathbf{Rat}_C$ is *exact if and only if C is of finite dimension.*

PROOF. If $\dim C < \infty$, then every left C*-module is rational. Therefore the functor $\mathbf{Rat}_C$ is identical.

Conversely, assume that $\mathbf{Rat}_C$ is exact. By (2.5) it suffices to prove that $\mathrm{Rat}_C(C^*) = C^*$. Suppose now $\mathrm{Rat}_C(C^*) \neq C^*$. Then $\mathrm{Rat}_C(C^*) \subset R^{\perp}$ by Lemma 3.4, where R is the coradical of C. Consider the exact sequence of left C*-modules and C*-homomorphisms

$$C^*/\mathrm{Rat}_C(C^*) \longrightarrow C^*/R^\perp \longrightarrow 0.$$

By (3.2) we have $\mathrm{Rat}_C(C^*/\mathrm{Rat}_C(C^*))=0$. This and the exactness of Rat_C imply that $\mathrm{Rat}_C(C^*/R^\perp)=0$. So we have $\mathrm{Rat}_C(R^*)=0$ since $C^*/R^\perp \simeq R^*$ as C*-modules. On the other hand, since $\dim R<\infty$, R^* is a rational R^*-module, that is, $\mathrm{Rat}_R(R^*)=R^*$. Using (2.4) we have $R^*=\mathrm{Rat}_C(R^*)=0$ which is a contradiction.

COROLLARY 3.6. *Let C be any coalgebra. If* Rat_C *is exact, then any irreducible subcoalgebra of C is of finite dimension.*

(3.7) Let $C=\bigoplus_\alpha C_\alpha$ be a direct sum of coalgebras. Then $C^*=\prod_\alpha C^*_\alpha$ is a direct product of algebras and each C* may be regarded as an ideal of C*. If M is a left C*-module, then the submodule $C^*_\alpha M$ of M is simultaneously a left C^*_α-module. By (2.4) we have

$$\mathrm{Rat}_C(C^*_\alpha M) = \mathrm{Rat}_{C_\alpha}(C^*_\alpha M).$$

PROPOSITION 3.8. *Let* $C=\bigoplus_\alpha C_\alpha$ *be a direct sum of coalgedras. Then* Rat_C is *exact if and only if* Rat_{C_α} is *exact for all* α.

PROOF. "Only if" part follows from Proposition 3.3. Assume now that Rat_{C_α} is exact for all α. Let $M \xrightarrow{f} N \to 0$ be an exact sequence of C*-modules. It suffices to prove that $\mathrm{Rat}_C(f)$ is surjective. Let $n \in \mathrm{Rat}_C(N)$ be any element. Since for any submodule M' of M we have $\mathrm{Rat}(M') \subset \mathrm{Rat}(M)$, we may consider $f^{-1}(C^*n)$ instead of M. Moreover, since $\dim C^*n<\infty$ ([4], Th. 2.1.3, *b*)), we may assume N to be a finite-dimensional rational C*-module, so that there exists an open ideal I of C* such that $IN=0$ by (2.3). Since $I^\perp$ is a finite-dimensional subcoalgebra of C, $I^\perp \subset \bigoplus_{i=1}^d C_{\alpha_i}$ for some finitely many indices $\alpha_1, \dots, \alpha_d$. Therefore $I \supset (\bigoplus_{i=1}^d C_{\alpha_i})^\perp = \prod_{\alpha\neq\alpha_i} C^*$. Note that $C^*=(\bigoplus_{i=1}^d C^*_{\alpha_i}) \oplus (\prod_{\alpha\neq\alpha_i} C^*_\alpha)$. It follows that $N=C^*N=(\bigoplus_{i=1}^d C^*_{\alpha_i})N=\sum_{i=1}^d C^*_{\alpha_i}N$. This implies that $n=\sum_{i=1}^d n_i$, where $n_i \in C^*_{\alpha_i}N$ for each i. By (3.7) $C^*_{\alpha_i}N$ is a rational $C^*_{\alpha_i}$-module, and the restriction of f: $C^*_{\alpha_i}M \to C^*_{\alpha_i}N$ is a surjective $C^*_{\alpha_i}$-homomorphism. Thus by the assumption for each i there exists an $m_i \in \mathrm{Rat}_{C_{\alpha_i}}(C^*_{\alpha_i}M)=\mathrm{Rat}_C(C^*_{\alpha_i}M) \subset \mathrm{Rat}_C(M)$ such that $n_i=f(m_i)$ Then $m=\sum_{i=1}^d m_i \in \mathrm{Rat}_C(M)$ and $f(m)=n$ which prove the proposition.

COROLLARY 3.9. *If C is cosemisimple, then* Rat_C is *exact.*

THEOREM 3.10. *Let C be a cocommutative coalgebra. Then the following conditions are equivalent:*

(1) Rat_C is *exact.*

(2) *C is a direct sum of finite-dimensional subcoalgebras.*

(3) *Every irreducible subcoalgebra of C is of finite dimension.*

PROOF. (1)⇒(3) and (2)⇒(1) follow from Theorem 3.5 and Proposition 3.8 respectively. Since C is cocommutative, it is a direct sum of its irreducible components ([4], Th. 8.0.5). It follows that (3) implies (2).

4. Extensions of rational modules

DEFINITION 4.1. We say that a coalgebra C has the property (E) when for any exact sequence $0 \to L \to M \to N \to 0$ of left C^*-modules and C^*-homomorphisms, if L and N are rational, then so is M.

(4.2) (Radford [3]). If every open left ideal of C^* is finitely generated, then C has the property (E).

PROPOSITION 4.3. *Let C have the property* (E). *// / and J are open ideals of* C^*, *then so is the product ideal* IJ.

PROOF. Consider the following exact sequence of C^*-modules

$$0 \longrightarrow J/IJ \longrightarrow C^*/IJ \longrightarrow C^*/J \longrightarrow 0.$$

Since Ann(J/IJ) (resp. Ann(C^*/J)) contains an open ideal / (resp. J), both C^*-modules J/IJ and C^*/J are rational. Hence C^*/IJ is also rational by the property (E) of C. But since C^*/IJ is a cyclic module, $IJ = \text{Ann}(C^*/IJ)$ is an open ideal.

LEMMA 4.4. *Let E be a vector space and* E^* *its dual space. If* $\{A_n\}_{n=1}^{\infty}$ *is a decreasing chain of closed subspaces in* E^* *such that* $\bigcap_n A_n = 0$, *then the linear topology on* E^* *defined by* $\{A_n\}$ *is complete.*

PROOF. It is clear that the topology is Hausdorff. Let $E_n = A_n^{\perp}$, $n = 1, 2, \ldots$. Then $E_n^{\perp} = A_n$ and $\{E_n\}$ is an increasing chain of subspaces in E. Since, in general, for a family of subspaces $\{E_\lambda\}$ in E we have $(\sum_\lambda E_\lambda)^{\perp} = \bigcap_\lambda E_\lambda^{\perp}$, ft follows that $\bigcup_n E_n = E$.

Now let $\{x_n^*\}$ be any Cauchy sequence. We may assume that $x_n^* - x_{n+1}^* \in A_n$ for all n. Define a linear form x^* on E as follows:

$$x^* = x_n^* \quad \text{on} \quad E_n, \; n = 1, 2, \ldots.$$

Then it is easy to see that x^* is well defined and $\lim x_n^* = x^*$.

(4.5) A coalgebra is called almost irreducible if its coradical is finite-dimensional.

THEOREM 4.6. *Let C be an almost irreducible coalgebra with coradical R and let $J=R^{\perp}$. Then the following conditions are equivalent:*

(1) *C has the property* (E).

(2) *J is finitely generated as a left ideal.*

(3) *The coradical filtration of C consists of finite-dimensional subcoalgebras.*

(4) *Any open left ideal in* C^* *is finitely generated.*

PROOF. (1)$\Rightarrow$(2). Since J is open, Proposition 4.3 implies that every power J^n is also open, and thus it is closed. It follows from Lemma 4.4 that C^* is complete with respect to J-adic topology. On the other hand since J^2 is open, the left C^*-module J/J^2 is finite-dimensional as well as finitely generated. The completeness of J-adic topology shows that J itself is finitely generated.

(2)$\Rightarrow$(4). Let / be any open left ideal. Then $I\supset J^n$ for some n. Since J is finitely generated, so is J^n, and this is cofinite ([1], 1.3.9). Therefore / is finitely generated.

(4)$\Rightarrow$(1). This is just (4.2).

(1)$\Rightarrow$(3). Recall that $C_n=(J^{n+1})^{\perp}$ ([4], p. 185). By Proposition 4.3 every J^{n+1} is open, so that it is cofinite.

(3)$\Rightarrow$(2). Let I_n be the closure of the ideal J^n. Then $\{I_n\}$ is a decreasing chain of ideals and $\bigcap_n I_n=0$. It follows from Lemma 4.4 that the topology on C^* defined by $\{I_n\}$ is complete, which we call $\{I_n\}$-topology. Since J^2 is cofinite, we have

$$J = C^*c_1^* + \cdots + C^*c_r^* + J^2$$

for some $c_1^*,\ldots, c_r^*$ in J, so that for every $n>1$ we have

$$J^n = J^{n-1}c_1^* + \cdots + J^{n-1}c_r^* + J^{n+1},$$

where J^0 means C^*. We now prove that J is actually generated by $c_1^*,\ldots,$ c*. Take any element x^* in J. Define a sequence $\{x_n^*\}$ inductively as follows:

$$x_1^* = x^*,$$

$$x_n^* = a_n^{1*}c_1^* + \cdots + a_n^{r*}c_r^* + x_{n+1}^*,$$

where $a_n^{i*}\in J^{n-1}$, $i=1,\ldots, r$, and $x_{n+1}^*\in J^{n+1}$. Then for any n we have $x^* = \sum_{i=1}^{r}(\sum_{j=1}^{n} a_j^{i*})c_i^* + x_{n+1}^*$. As $n\rightarrow\infty$, $x_{n+1}^*\rightarrow 0$ and $\sum_{j=1}^{n} a_j^{i*}$ converges to some a^{i*} for $i=1,\ldots,$ r in $\{I_n\}$-topology. Therefore $x^*=\sum_{i=1}^{r} a^{i*}c_i^*$ This completes the proof.

PROPOSITION 4.7 (Lin [2]). *Let C be a subcoalgebra of a coalgebra D.*

If D has the property (E), *then also does C.*

PROOF. It is clear from (2.4).

PROPOSITION 4.8 (Lin [2]). *Let* $C=\bigoplus_{\alpha} C_{\alpha}$ *be a direct sum of subcoalgebras. Then C has the property* (E) *if and only if also does each* C_{α}.

PROOF. Because of Proposition 4.7, it suffices to show "if" part. Let each C_{α} have the property (E). Let $0 \to L \to M \xrightarrow{f} N \to 0$ be an exact sequence of C*-modules with L and N rational. Let $m \in M$ be any fixed element and put $n=f(m)$. Then Ann(n) is open. Therefore,

$$\mathrm{Ann}(n) \supset (\bigoplus_{i=1}^{d} C_{\alpha_i})^{\perp} = \prod_{\alpha \neq \alpha_i} C^*$$

for some finitely many indices $\alpha_1, \ldots, \alpha_d$. Since

$$M = (\bigoplus_{i=1}^{d} C^*_{\alpha_i} M) \oplus (\prod_{\alpha \neq \alpha_i} C^*_{\alpha}) M,$$

m can be written as $m=\sum_{i=1}^{d} m_i + m'$, $m_i \in C^*_{\alpha_i} M$, $m' \in (\prod_{\alpha \neq \alpha_i} C^*_{\alpha}) M$. Here we have $m' \in L$. In fact, let e be the identity of the algebra $\prod_{\alpha \neq \alpha_i} C^*$. Multiply e on both sides of $n=\Sigma f(m_i)+f(m')$, and we have $f(m')=ef(m')=0$ because $en=0$, $ef(m_i)=f(em_i)=0$ and e acts identically on $(\prod_{\alpha \neq \alpha_i} C^*_{\alpha})N$.

By (3.7) C^*_{α}-modules $C^*_{\alpha}L$ and $C^*_{\alpha}N$ are rational! and the sequence of C^*_{α}-modules $0 \to C^*_{\alpha}L \to C^*_{\alpha}M \to C^*_{\alpha}N \to 0$ is exact. It follows from the assumption that $C^*_{\alpha}M$ is also rational as a C^*_{α}-module for every α. Therefore we have

$$m_i \in \mathrm{Rat}_{C_{\alpha_i}}(C^*_{\alpha_i} M) = \mathrm{Rat}_C(C^*_{\alpha_i} M) \subset \mathrm{Rat}_C(M).$$

Thus $m \in \mathrm{Rat}_C(M)$ which implies that M is rational.

COROLLARY 4.9. *When* C can *be expressed as a direct sum of almost irreducible* **subcoalgebras**, *in* **particular**, **when** *C is* **cocommutative**, *C has the property* (E) *if and only if every open left ideal of* C* is *finitely generated.*

PROOF. It suffices to prove "only if" part. Let $C=\bigoplus_{\alpha} C_{\alpha}$ and let I be any open left ideal in $C^*=\prod_{\alpha} C^*_{\alpha}$ Then there exist finitely many indices $\alpha_1, \ldots, \alpha_d$ such that $I \supset \prod_{\alpha \neq \alpha_i} C^*_{\alpha}$. Let $\bar{I}$ be the image of I under the canonical homomorphism $C^* \to C^*/\prod_{\alpha \neq \alpha_i} C^*_{\alpha} \simeq \bigoplus_{i=1}^{d} C^*_{\alpha_i} = (\bigoplus_{i=1}^{d} C_{\alpha_i})^*$. Since C has the property (E), also does the subcoalgebra $\bigoplus_{i=1}^{d} C_{\alpha_i}$ which is almost irreducible. By (2.1) $\bar{I}$ is open in $(\bigoplus_{i=1}^{d} C_{\alpha_i})^*$. Therefore by Theorem 3.6 $\bar{I}$ is a finitely generated ideal, so that I is a finitely genrated left C^*-module. Since the ideal $\prod_{\alpha \neq \alpha_i} C^*_\alpha$ is finitely generated, so is I. This completes the proof.

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