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Integrability of Nonoscillatory Solutions of a Delay Differential Equation

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The purpose of this note is to extend Dahiya and Singh's result [1, Theorem 4] to n-th order equations and give a straightforward proof of the theorem. Examples for $n=2$ which are not covered by Theorem 4 [1] will also be given.

We are concerned with the n-th order delay equation

$$(1)\qquad (r(t)y^{(n-1)}(t))'+(b(t)y(t))'+a(t)y(g(t))=f(t),\qquad n>2,$$

where the functions $r(t)$, fo(0>ⁿ(0> $g(t)$ and/(O arecontinuous on the whole real line.

A nontrivial solution of (1) which exists on $[t_0, \infty)$ is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is said to be *nonoscillatory.*

THEOREM. *Assume that*

(i) $a(t)\geq M$*on* $[t_0, \infty)$ *for some constant* $M>0$,

(ii) fe(0>0<m $[t_0,$ oo),

(iii) $g(t)\to\infty$ *as* $t\to\infty$ *and* $0\leq g'(t)\leq 1$ *on* $[t_0, \infty)$,

(iv) $r(t)>0$ *on* $[t_0,$ oo) *and* $\int_{t_0}^{\infty}\frac{1}{r(t)}dt=\infty$,

and

(v) $\int_{t_0}^{\infty}|f(t)|dt<\infty$.

Then every nonoscillatory solution of (1) *on* $[t_0,$ oo) *is integrable.*

PROOF. Let $y(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that there is a $T_0>t_0$ such that $y(t)>0$ for $t>T_0$. Since $g(t)\to\infty$ as $t\to\infty$, there is a $T_1>T_0$ such that $y(g(t))>0$ for $t>T_1$.

Integrating (1) from T_1 to $t>T_1$, we obtain

$$(2)\qquad r(t)y^{(n-1)}(t)-r(T_1)y^{(n-1)}(T_1)+b(t)y(t)-b(T_1)y(T_1)+\int_{T_1}^{t}a(s)y(g(s))ds$$
$$\leq\int_{T_1}^{t}|f(s)|ds.$$

The lefthand side of (2) remains bounded as $t\to\infty$.

Suppose $\int_{T_1}^{\infty}y(g(t))dt=\infty$. Then $r(t)y^{(n-1)}(t)\to-\infty$ as $t\to\infty$. Condition

(iv) implies that $y^{(n-2)}(t)\to-\infty$ as $t\to\infty$. Then $y(t)$ would be negative eventually, a contradiction. Therefore we have

$$\int_{T_1}^{\infty} y(g(s))ds < \infty.$$

By (iii) and change of variable, we get

$$\int_{T_1}^{\infty} y(g(s))ds > \int_{T_1}^{\infty} g'(s)y(g(s))ds = \int_{g(T_1)}^{\infty} y(t)dt.$$

This completes the proof.

Remark. For the case $n=2$, $r(t)=1$ and $b(t)=0$, the equation (1) is studied by Dahiya and Singh in [1]. In the assumptions of the theorem, we do not require the condition $a(t)a''(t)\le 2(a'(t))^2$ and the boundedness of the delay $\tau(t)=t-g(t)$ which are assumed in Theorem 4 [1].

EXAMPLE 1. Let us consider the equation

$$y''(t)+\left(\frac{t^3}{6}+t^2+\left(\frac{t^2}{4}-\frac{1}{8}\right)\sin 2t+\frac{t}{4}\cos 2t\right)y(t) \tag{3}$$

$$=\left(\frac{t^3}{6}+t^2+1+\left(\frac{t^2}{4}-\frac{1}{8}\right)\sin 2t+\frac{t}{4}\cos 2t\right)e^{-t}.$$

The functions $a(t)=\frac{t^3}{6}+t^2+\left(\frac{t^2}{4}-\frac{1}{8}\right)\sin 2t+\frac{t}{4}\cos 2t$, $g(t)=t$ and $f(t)=\left(\frac{t^3}{6}+t^2+1+\left(\frac{t^2}{4}-\frac{1}{8}\right)\sin 2t+\frac{t}{4}\cos 2t\right)e^{-t}$ satisfy the assumptions of the theorem. Then all nonoscillatory solutions of (3) are integrable. For instance, $y(t)=e^{-t}$ is a nonoscillatory solution of (3) which is integrable. We also observe that the inequality $a(t)a''(t)\le 2(a'(t))^2$ does not always hold. In fact, for each $\alpha>0$, there is a positive integer n such that $\beta=\left(n+\frac{3}{4}\right)\pi>\alpha$ and $a(\beta)a''(\beta)>2(a'(\beta))^2$. Dahiya and Singh's Theorem 4 [1] thus does not cover this example.

EXAMPLE 2. If we consider the delay equation

$$y''(t)+t^k y\left(\frac{t}{2}\right)=e^{-t}+t^k e^{-\frac{t}{2}}, \qquad k\ge 0, \tag{4}$$

then all conditions of the theorem are satisfied with an unbounded delay $\tau(t)=t-g(t)=\frac{t}{2}$. This equation has a nonoscillatory solution $y(t)=e^{-t}$ which is integrable on $[t_0,\infty)$.

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Reference

[1] R. S. Dahiya and B. Singh, *Certain results on nonoscillatory and asymptotic nature of delay* ***equations,*** Hiroshima Math. J. 5 (1975), 7-15.

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