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Meromorphic Mappings of a Covering Space over C^m into a Projective Variety and Defect Relations

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1. Introduction

Carlson-Griffiths [1] and Griffiths-King [4] studied the value distribution of holomorphic mappings from a smooth affine variety A into a smooth projective variety V. Among others, they established Nevanlinna's second main theorem and defect relation for holomorphic mappings from A into V. Recently, these results were generalized to the case of meromorphic mappings by Shiffman [13].

In the present paper we study the value distribution of meromorphic mappings from X into V, where X is the complex space of a finite analytic covering $X \xrightarrow{\pi} \boldsymbol{C}^m$ (see Definition 1 in section 2). The main purpose is to show Nevanlinna's second main theorem and defect relation of Griffiths-King's type for meromorphic mappings from X into V (see Theorems 1 and 2 in section 6 and cf. [4]).

The next section will be devoted to the notation and terminologies. In section 3 we shall prove two preparatory lemmas concerning positive currents on X. In section 4 we shall generalize the ramification estimate in Selberg [12] to the case of the finite analytic covering $X \xrightarrow{\pi} \boldsymbol{C}^m$ (Lemma 4.1). This estimate and the use of a singular volume form on V constructed by Carlson-Griffiths [1] and Griffiths-King [4] will play essential roles to obtain the second main theorem in section 6. In section 5 we shall investigate the proper domain of existence of a meromorphic mapping $f: X \to V$. This investigation will make the ramification estimate, obtained in section 4, possible to apply to the proof of the second main theorem. In the same section we shall prove that the characteristic function T(r, L) of a meromorphic mapping $f: X \to V$ with respect to a positive line bundle $L \to V$ (see section 3 for the definition) satisfies

$$T(r, L) = O(\log r)$$

if and only if the finite analytic covering $X \xrightarrow{\pi} \boldsymbol{C}^m$ is algebraic (see Definition 2 in section 2) and f is rational, provided that f separates the fibres of $X \xrightarrow{\pi} \boldsymbol{C}^m$ (see Definition 3 in section 2). This is a fundamental property of the growth of the characteristic function T(r, L).

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2. Notation

DEFINITION 1. We call $X \xrightarrow{\pi} \boldsymbol{C}^m$ a finite analytic covering over the m-dimensional complex affine space $\boldsymbol{C}^m$ or simply a finite analytic covering if

i) X is an irreducible normal complex space,

ii) $\pi: X \to \boldsymbol{C}^m$ is an onto proper holomorphic mapping with discrete fibres.

If the covering $X \xrightarrow{\pi} \boldsymbol{C}^m$ is fc-sheeted, we call it an analytic k-covering.

DEFINITION 2. A finite analytic covering $X \xrightarrow{\pi} \boldsymbol{C}^m$ is said to be algebraic if

i) X is biholomorphic to an affine variety,

ii) $\pi: X \to \boldsymbol{C}^m$ is a rational mapping.

In general, we write $S(Y)$ for the set of singularities of a complex space Y. Let $X \xrightarrow{\pi} \boldsymbol{C}^m$ be a finite analytic covering. Then the analytic set

$$C = \{z \in \boldsymbol{C}^m;\ \pi \textit{ is ramifiedat some point of } \pi^{-1}(z)\}$$

is called the critical set of the finite analytic covering $X \xrightarrow{\pi} \boldsymbol{C}^m$. Let $(z^1, \dots, z^m)$ be the natural coordinate system in $\boldsymbol{C}^m$ and set

$$\|z\|^2 = \sum_{\nu=1}^{m} z^\nu \bar{z}^\nu, \qquad \boldsymbol{C}^m(r) = \{\|z\| < r\},$$

$$X(r) = \pi^{-1}(\boldsymbol{C}^m(r)), \qquad d^c = \frac{i}{4\pi}(\bar{\partial} - \partial),$$

$$\phi = dd^c\|z\|^2, \qquad \psi = dd^c \log \|z\|^2,$$

$$\eta = d^c \log \|z\|^2 \wedge \psi^{m-1}\ ^{1)}.$$

For simplicity, we abbreviate $\pi^*\phi$, $\pi^*\psi$ and $\pi^*\eta$ (differential forms on X) to ϕ, ψ and η, respectively, if no confusion occurs.

A locally finite sum $\sum_{j=1}^{\infty} s_j D_j$ of irreducible analytic sets D_j of codimension 1 in X with integer coefficients $s_j \in \boldsymbol{Z}$ is called a divisor on X. We denote by $Supp(D)$ the support of the divisor D. If $D = \sum_{j=1}^{\infty} s_j D_j$ is a divisor with positive s_j, D is said to be effective. An effective divisor on X determines a d-closed positive current of type (1, 1) on X (for the notion of positive currents, see [7, 5]).

Let V be a projective variety and $L \to V$ a holomorphic line bundle over V. Then we denote by $|L|$ all of the effective divisors (σ) on V determined by $\sigma \in H^0(V, L)$.

1) ψ^{m-1} stands for $\psi \wedge \cdots \wedge \psi$ ($m-1$ times).

Let $f: X \to V$ be a meromorphic mapping with singularity $N(f)$. Then by Remmert [10], $\operatorname{codim} N(f) \geqq 2$ and there are a proper modification $\tau: (\tilde{X}, \tilde{N}) \to (X, N(f))$ and a holomorphic mapping $\tilde{f}: \tilde{X} \to V$ such that the diagram

$$\begin{array}{ccc} \tilde{X} - \tilde{N} & \xrightarrow{\ \tilde{f}\ } & \\ \downarrow \tau & & V \\ X - N(f) & \xrightarrow{\ f\ } & \end{array} \tag{2.1}$$

is commutative.

DEFINITION 3. We say that a meromorphic mapping $f: X \to V$ separates the fibres of $X \xrightarrow{\pi} C^m$ if there exists a point $z \in C^m - (C \cup \pi(N(f)))$ such that $f(x) \neq f(y)$ for any distinct points x, y of $\pi^{-1}(z)$.

3. Preparatory lemmas and the first main theorem

LEMMA 3.1. *Let ξ be a plurisubharmonic function in C^m and set $T = dd^c\xi$ (in the sense of currents). Then*

$$2\int_1^r \frac{n(t)}{t}\,dt = \int_{\partial C^m(r)} \xi\eta - \int_{\partial C^m(1)} \xi\eta, \tag{3.1}$$

where $n(t) = t^{2-2m} T|_{C^m(t)}(\phi^{m-1})$ (cf. [7]).

PROOF. When ξ is smooth, it is easy to prove (3.1) by using Stokes' theorem, Fubini's theorem and the formula

$$\psi^{m-1} = t^{2-2m}\phi^{m-1} \qquad \text{on} \quad \partial C^m(t).$$

For a general ξ, set $\xi_\varepsilon = \xi * \mu_\varepsilon$, where $\mu_\varepsilon(z)$ is a convolution kernel depending only on $\|z\|$. Then $\xi_\varepsilon \to \xi$ decreasingly as $\varepsilon \to 0$ (see [7]). Since ξ_ε is smooth, we have

$$2\int_1^r \frac{dt}{t^{2m-1}} \int_{C^m(t)} dd^c\xi_\varepsilon \wedge \phi^{m-1} = \int_{\partial C^m(r)} \xi_\varepsilon\eta - \int_{\partial C^m(1)} \xi_\varepsilon\eta.$$

Since $t^{2-2m}\int_{C^m(t)} dd^c\xi_\varepsilon \wedge \phi^{m-1} \to n(t)$ uniformly with respect to $t \in [1, r]$ and $\xi_\varepsilon \to \xi$ decreasingly as $\varepsilon \to 0$, we have (3.1). Q. E. D.

Let $X \xrightarrow{\pi} C^m$ be an analytic k-covering and D an effective divisor on X. Then the push-forward $\pi_* D$ is a d-closed positive current of type (1, 1) on C^m since π is proper.

LEMMA 3.2. *For an effective divisor D on X, there is a holomorphic function α in C^m satisfying*

$$\pi_* D = dd^c \log |\alpha|^2 \qquad (\textit{in the sense of currents}).$$

PROOF. Since C^m is a *Cousin*-II domain, by Poincaré's equation (see, e.g., [4, Proposition 1.1]) it suffices to show that $\pi_* D$ coincides with a current determined by an effective divisor on C^m. For $z \in C^m - \pi(S(X))$, set $\pi^{-1}(z) = \{x_1, \ldots, x_l\}$. Then we can take neighborhoods U of z and W_i of x_i so that $W_i \cap W_j = 0$ if $i \neq j$, $\pi|_{W_i}: W_i \to U$ is onto proper for each i and there is a holomorphic function β_i in each W_i satisfying $(\beta_i) = D \cap W_i$. Set

$$\alpha(z) = \prod_{i=1}^{l} \prod_{x \in \pi^{-1}(z) \cap W_i} \beta_i(x).$$

Then by Riemann's extension theorem $\alpha(z)$ is holomorphic in U. It is easy to show the current equation

$$\pi_* D = (\alpha) \qquad \text{in} \quad U.$$

Thus there exists a divisor Σ' on $C^m - \pi(S(X))$ satisfying $\pi_* D = \Sigma'$ in $C^m - \pi(S(X))$. Since $\operatorname{codim} \pi(S(X)) \geqq 2$, by Remmert-Stein's theorem Σ' has a unique extension Σ as a divisor on C^m. Since an analytic set of codimension greater than 1 is negligible for the currents $\pi_* D$ and Σ (cf. [6, 5]), the equality $\pi_* D = \Sigma$ holds in C^m. Q.E.D.

We define the counting functions for an effective divisor D on the analytic k-covering X (resp. C^m) by

$$n(t, D) = \frac{1}{kt^{2m-2}} \int_{D \cap X(t)} \phi^{m-1} \quad \left(\text{resp.} \frac{1}{t^{2m-2}} \int_{D \cap C^m(t)} \phi^{m-1}\right),$$

$$N(r, D) = \int_1^r \frac{n(t, D)}{t} dt.$$

Let V be a smooth projective variety, $L \to V$ a holomorphic line bundle over V with an hermitian metric $|\ |$ whose curvature form is ω, and $f: X \to V$ a meromorphic mapping. Then $f^*\omega$ may have singularities on $N(f)$ but it is a differential form with coefficients belonging to L^1_{loc}; this fact readily follows from (2.1). We define the characteristic function of f with respect to the line bundle L by

$$T_f(r, L) = T(r, L) = \frac{1}{k} \int_1^r \frac{dt}{t^{2m-1}} \int_{X(t)} f^*\omega \wedge \phi^{m-1}.$$

For $D \in |L|$ such that $Supp(D) \not\supset f(X)$, taking always a section $\sigma \in H^0(V, L)$ so that $(\sigma) = D$ and $|\sigma| \leqq 1$, we set

(3.2) $$m_f(r, D) = m(r, D) = \frac{1}{k} \int_{\partial X(r)} \log \frac{1}{f^*|\sigma|} \eta.$$

Since the integrand is non-negative, the integral is well defined. We shall show that $m(r, D)$ is finite for all $r>0$. Since V is a projective variety, we may assume that the line bundle L is positive, i.e., ω is positive definite. By Lemma 4.2 there is an entire function α on C^m such that $(\alpha)=\pi_*(f_*D)$. Setting

$$\xi(z) = -\sum_{x\in\pi^{-1}(z)} \log f^*|\sigma|^2(x),$$

we easily deduce by using the extension theorem of Grauert-Remmert [3] that

(3.3) $\xi_1=\xi+\log|\alpha|^2$ is plurisubharmonic in C^m,

(3.4) $dd^c\log\xi=\pi_*(f^*\omega)-\pi_*(f^*D)$ in the sense of currents.

We observe that $m(r, D)=\frac{1}{2k}\int_{\partial C^m(r)}\xi\eta$. It follows from (3.3) that the integral is finite.

Combining Lemma 3.1 with (3.4), we get the so-called first main theorem:

(3.5) $$T(r, L) = N(r,f^*D)+m(r, D)-m(1, D)$$

for divisors $D\in|L|$ such that $Supp(D)\not\supset f(X)$.

Let α be a meromorphic function on X. Then α is canonically identified with a meromorphic mapping from X into the 1-dimensional complex projective space P^1 (cf. [10]). We shall freely use this identification. We denote by $(\alpha)_\infty$ (resp. $(\alpha)_0$) the divisor of poles (resp. zeros) of the meromorphic function α. Set

$$m(r, \alpha) = \frac{1}{k}\int_{\partial X(r)}\log^+|\alpha|\eta^{1)},$$

$$T(r, \alpha) = N(r,(\alpha)_\infty)+m(r, \alpha).$$

From Poincaré's equation $dd^c\log|\alpha|^2=(\alpha)_0-(\alpha)_\infty$ (see [4, Proposition 1.1]) and Lemma 3.1, it follows that

(3.6) $$T(r,\alpha) = T\left(r, \frac{1}{\alpha}\right)+\frac{1}{k}\int_{\partial X(1)}\log|\alpha|\eta.$$

Let $L\to P^1$ be the hyperplane bundle over P^1. Then

(3.7) $$T(r, \alpha) = T_\alpha(r, L)+O(1).$$

4. Ramification estimate

In general, let $g: X\to Y$ be a meromorphic mapping from a normal com-

1) $\log^+ s=0$ for $s\leqq 1$ and $\log^+ s=\log s$ for $s\geqq 1$.

plex space X into a complex manifold Y with $\dim X = \dim Y$. Let R'_g be the divisor determined by the Jacobian of the holomorphic mapping $g|_{X-(N(g)\cup S(X))}$: $X-(N(g)\cup S(X))\rightarrow Y$ if the Jacobian does not identically vanish. Since codim $(N(g)\cup S(X))\geqq 2$, by Remmert-Stein's extension theorem R'_g has a unique extension R_g as a divisor on X, which is called the ramification divisor of the meromorphic mapping g.

Let $X\xrightarrow{\pi}C^m$ be a finite analytic covering and B the ramification divisor of $\pi: X\rightarrow C^m$. Then we call B the ramification divisor of the finite analytic covering $X\xrightarrow{\pi}C^m$. One notes that the analytic set *Supp* $(\pi_* B)$ (cf. Lemma 4.2) coincides with the critical set of $X\xrightarrow{\pi}C^m$.

LEMMA 4.1. *Let $X\xrightarrow{\pi}C^m$ be an analytic k-covering with ramification divisor B and α a meromorphic function on X which separates the fibres of $X\xrightarrow{\pi}C^m$. Then*

$$N(r, B) \leqq 2(k-1)T(r, \alpha)+O(1). \tag{4.1}$$

REMARK. In case $m=1$, this was proved by Selberg [12].

PROOF. Since $T(r, \alpha)=T(r, 1/(\alpha-c))+O(1)$ by (3.6), we may assume that

$$\begin{cases} \text{any irreducible component of } Supp((\alpha)_\infty) \text{ is not} \\ \text{contained in } \pi^{-1}(C), \end{cases} \tag{4.2}$$

where C is the critical set of $X\xrightarrow{\pi}C^m$. Represent the ramification divisor B as

$$B=\Sigma(k_\nu-1)B_\nu,$$

where B_ν are irreducible components of $Supp(B)$ and k_ν are integers greater than 1. Let A be the set of points $w\in C-(S(C)\cup \pi(S(X)\cup Supp((\alpha)_\infty)))$ such that

i) each point of $\pi^{-1}(w)$ belongs to $Supp(B)-S(Supp(B))$,
ii) there is a neighborhood U of each point of $\pi^{-1}(w)$ such that $d(\pi|_{Supp(B)\cap U})$ has maximal rank $m-1$.

We put

$$\Delta(z)=\prod_{i<j}(\alpha(x_i)-\alpha(x_j))^2$$

for $z\in C^m$, where $\pi^{-1}(z)=\{x_1,\ldots,x_k\}$ (counting multiplicities). Then $\Delta\not\equiv 0$ by the hypothesis. First we show

$$\pi_* B \leqq (\Delta)_0. \tag{4.3}$$

Since an analytic set of codimension greater than 1 is negligible for both the currents of (4.3), by virtue of (4.2) it suffices to prove (4.3) in a neighborhood of each point of A. Let $w_0 \in A$, $x_0 \in \pi^{-1}(w_0) \cap B_\nu$ and take local coordinate neighborhoods $W(w^1,\dots,w^m)$ of w_0 and $U(x^1,\dots,x^m)$ of x_0 so that $\pi|_U\colon U \to W$ is onto proper, $\pi^{-1}(w_0) \cap U = \{x_0\}$, $w_0 = (0,\dots,0)$, $x_0 = (0,\dots,0)$, $C \cap W = \{w^1 = 0\}$, *Supp (B)* $\cap U = B_\nu \cap U = \{x^1 = 0\}$ and

$$w^1 = \pi^1(x) = (x^1)^{k_\nu},$$

$$w^j = \pi^j(x) = x^j \qquad \text{for } j \geqq 2.$$

Thus $\pi|_U\colon U \to W$ is a k_ν-sheeted covering. Since α is holomorphic in U which is chosen smaller if necessary, α can be represented by a series of the form

$$\alpha = a_0 + \sum_{j \geqq 1} a_j(w^2,\dots, w^m)(\sqrt[k_\nu]{w^1})^j .$$

Therefore we have

$$\prod_{1 \leqq i < j \leqq k_\nu} (\alpha(x_i) - \alpha(x_j))^2 = (w^1)^{k_\nu - 1}\zeta(w),$$

where $(\pi|_U)^{-1}(w) = \{x_1,\dots, x_{k_\nu}\}$. Applyting this argument to each point of $\pi^{-1}(w_0)$, we see that (4.3) is valid in a neighborhood of w_0. Hence (4.3) is proved.

From (4.3) it immediately follows that

$$N(r, B) \leqq \frac{1}{k} N(r, (\Delta)_0) .$$

According to (3.6) this yields

$$N(r, B) \leqq \frac{1}{k} T(r, \Delta) + O(1) . \tag{4.4}$$

Referring to (4.2), we have

$$\frac{1}{k} N(r, (\Delta)_\infty) = 2(k-1)N(r, (\alpha)_\infty) . \tag{4.5}$$

Since $\log^+|s_1 s_2| \leqq \log^+|s_1| + \log^+|s_2|$ and $\log^+|s_1 + s_2| \leqq \log^+|s_1| + \log^+|s_2| + \log 2$,

$$\begin{aligned} \frac{1}{k} m(r, \Delta) &= \frac{1}{k} \int_{z \in \partial C^m(r)} \log^+ \Big| \prod_{i<j} (\alpha(x_i) - \alpha(x_j)) \Big|^2 \eta(z) \\ &\leqq \frac{2(k-1)}{k} \int_{\partial X(r)} \log^+ |\alpha| \eta + O(1) \qquad (4.6) \\ &= 2(k-1)m(r, \alpha) + O(1), \end{aligned}$$

where $\pi^{-1}(z) = \{x_1,\dots,x_k\}$. The inequalities (4.4), (4.5) and (4.6) imply (4.1).

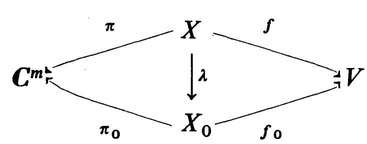
Q.E.D.

5. Meromorphic mappings into a projective variety and their rationality

As is well known, for any *k-valued* meromorphic function α on $\boldsymbol{C}$, there is a Riemann surface S which is a fc-sheeted covering over $\boldsymbol{C}$, such that α becomes a *1-valued* meromorphic function on S. Such Riemann surface S is called the proper domain of existence of α. Let τ: $\boldsymbol{S} \to \boldsymbol{C}$ denote the natural projection. Then the meromorphic function α separates the fibres of $\boldsymbol{S} \xrightarrow{\tau} \boldsymbol{C}$ in our sense.

Let $\boldsymbol{X} \xrightarrow{\pi} \boldsymbol{C}^m$ be a finite analytic covering, V a projective variety and /: $\boldsymbol{X} \to \boldsymbol{V}$ a meromorphic mapping. If f separates the fibres of $X \xrightarrow{\pi} \boldsymbol{C}^m$, we may say that X is the proper domain of existence of /. The following proposition asserts the existence of such a domain for an arbitrary meromorphic mapping from X into V.

PROPOSITION 1. *Let f: $\boldsymbol{X} \to V$ be a meromorphic mapping. Then there are a finite analytic covering $\boldsymbol{X}_0 \xrightarrow{\pi_0} \boldsymbol{C}^m$, and onto proper holomorphic mapping λ: $\boldsymbol{X} \to \boldsymbol{X}_0$ with discrete fibres and a meromorphic mapping f_0: $X_0 \to V$ which separates the fibres of $\boldsymbol{X}_0 \xrightarrow{\pi_0} \boldsymbol{C}^m$ and is **non-degenerate**[1)] if so is f, such that the diagram*



*is **commutative.***

PROOF. We may assume that $V \subsetneq \boldsymbol{P}^N$. Let $[w^0,\dots, w^N]$ be a homogeneous coordinate system in $\boldsymbol{P}^N$. Then / can be represented in a neighborhood of each point of X by

$$f = [f^0,\dots,f^N],$$

where f^j are holomorphic functions in the neighborhood. Letting C denote the critical set of $X \xrightarrow{\pi} \boldsymbol{C}^m$, we set $\mathring{X} = X - \pi^{-1}(C)$. Then $\pi|_{\mathring{X}}: \mathring{X} \to \boldsymbol{C}^m - C$ is unramified, i.e., locally biholomorphic. For $z \in \boldsymbol{C}^m - C$, set $\pi^{-1}(z) = \{x_1,\dots, x_k\}$ and take neighborhoods U of z and W_i of x_i so that $\pi|_{W_i}$: $W_i \to U$ are biholomorphic. We write

$$(5.1) \qquad f_i(z) = [f_i^0 \circ (\pi|_{W_i})^{-1}(z), \dots, f_i^N \circ (\pi|_{W_i})^{-1}(z)],$$

where $f = [f_i^0,\dots,f_i^N]$ in W_i and $\{f_i^0 = \cdot = f_i^N = 0\} = N(f) \cap W_i$. Let $\mathcal{O}$ denote

1) A meromorphic mapping f: $X \to V$ is said to be *non-degenerate* if df has maximal rank in a non-empty open set where f is holomorphic.

the sheaf of holomorphic functions over $\boldsymbol{C}^m$, $\mathcal{O}_z$ its stalk at z and $\mathcal{O}_z^*$ the set of invertible elements of $\mathcal{O}_z$. For each $z \in \boldsymbol{C}^m - C$, $f_i(z)$ $(i=1,\dots,$ fc) determine elements $\underline{f_i}(z) \in (\mathcal{O}_z^{N+1})/\mathcal{O}_z^*$,each of which is independent of the representation (5.1). One should note that $\underline{f_i}(z)$ and $\underline{f_j}(z)$ may be the same elements of $(\mathcal{O}_z^{N+1})/\mathcal{O}_z^*$ though $i \neq j$. Set

$$\mathscr{F} = \text{fe}/\underline{\text{Xz}})); \; z \in \boldsymbol{C}^m - C, \; 1 \leq i \leqq k\}.$$

Then in the same manner as Weyl's "*AnalytischeGebilde*", we can construct an unramified finite analytic covering $X' \xrightarrow{\pi'} \boldsymbol{C}^m - C$ over $\boldsymbol{C}^m - C$ associated with $\mathscr{F}$. In the natural way we get a holomorphic mapping $\lambda': \mathring{X} \to X'$ which is onto proper and whose fibres are discrete, and a meromorphic mapping $f': X' \to V$, which satisfy the commutative diagram

$$\begin{array}{ccccc} & \overset{\pi}{\swarrow} & \mathring{X} & \overset{f}{\searrow} & \\ \boldsymbol{C}^m - C & & \downarrow{\scriptstyle \lambda'} & & V. \\ & \underset{\pi'}{\nwarrow} & X' & \underset{f'}{\nearrow} & \end{array}$$

Since C is a thin analytic set in $\boldsymbol{C}^m$, by Grauert-Remmert [3] the finite analytic covering $X' \xrightarrow{\pi'} \boldsymbol{C}^m - C$ over $\boldsymbol{C}^m - C$ can be uniquely extended over all $\boldsymbol{C}^m$. Let us denote it by $X_0 \xrightarrow{\pi'} \boldsymbol{C}^m$. It is easy to see that λ' has a unique holomorphic extension $\lambda: X \to X_0$ which is onto proper and whose fibres are discrete. Let Γ and Γ' be the graphs of the meromorphic mappings / and /'. Then setting $\tau = \lambda \times id_V$,we get

$$\begin{array}{lcl} \Gamma \subset & \longrightarrow & X \times V \\ & & \downarrow \tau \\ \Gamma' \subsetneq X' \times V \subsetneq & & X_0 \times V. \end{array}$$

From the construction, $\Gamma' = \tau(\Gamma \cap (\mathring{X} \times V)) \subset \tau(\Gamma) \subset X_0 \times V$. Since τ is proper, τ(Γ) is an analytic set. Since the closure $\overline{\Gamma'}$ of Γ' coincides with the analytic set τ(Γ), we obtain a meromorphic mapping $f_0: X_0 \to V$which is an extension of f'. It is clear that $X_0 \xrightarrow{\pi_0} \boldsymbol{C}^m$, $\lambda: X \to X_0$ and $f_0: X_0 \to V$ satisfy our requirements. Q.E.D.

REMARK. Let us assume that V is smooth. Let $T_f(r, \mathrm{L})$ and $T_{f_0}(r, \mathrm{L})$ be the characteristic functions of the meromorphic mappings / and f_0 in Proposition 1 with respect to a line bundle $L \to V$. Then

$$(5.2) \qquad \begin{cases} T_f(r, L) = T_{f_0}(r, L), \\ m_f(r, D) = m_{f_0}(r, D), \\ N(r, f^*D) = N(r, f_0^*D) \end{cases}$$

for divisors $D\in|L|$ such that $Supp(D)\not\supset f(X)$. Indeed, let k, k_0 and l be the numbers of sheets of the coverings $X\xrightarrow{\pi}C^m$, $X_0\xrightarrow{\pi_0}C^m$ and $X\xrightarrow{\lambda}X_0$, respectively. Then $k=k_0 l$. The equalities (5.2) immediately follow from this fact (for the last equality, refer to the proof of Lemma 4.1).

PROPOSITION 2. *Let f: $X\to V$ be a meromorphic mapping which separates the fibres of $X\xrightarrow{\pi}C^m$ and let $L\to V$ be a positive line bundle over V.*

Then $X\xrightarrow{\pi}C^m$ is an algebraic covering and f is rational if and only if

$$T(r, L) = O(\log r).$$

First we show

LEMMA 5.1. *Let $X\xrightarrow{\pi}C^m$ be a finite analytic covering with critical set C. Then $X\xrightarrow{\pi}C^m$ is algebraic if and only if C is an algebraic set.*

PROOF. It is clear that C is algebraic if so is $X\xrightarrow{\pi}C^m$. Suppose that C is algebraic. Then the closure $\bar{C}$ in $P^m(\supset C^m=P^m-H_0$, where H_0 is a hyperplane in $P^m)$ is an analytic set. By Grauert-Remmert [3] there is a unique extension $\bar{X}\xrightarrow{\bar{\pi}}P^m$ of the finite analytic covering $X\xrightarrow{\pi}C^m$. Let $F\to P^m$ be the hyperplane bundle over P^m. Then $F^{-1}\to P^m$ is weakly negative in the sense of Grauert [2]. Since π: $\bar{X}\to P^m$ is proper and its fibres are discrete, it is easy to check that the zero section of $\bar{\pi}^*F^{-1}$ is exceptional, that is, $\bar{\pi}^*F^{-1}\to\bar{X}$ is weakly negative. Thus for an integer ν large enough, $\bar{\pi}^*F^{\nu}\to\bar{X}$ is very ample. The basis $\{\bar{\pi}^*\sigma_0^{\nu}, \bar{\pi}^*\sigma_1^{\nu},\dots, \bar{\pi}^*\sigma_m^{\nu},\dots\}$ of $H^0(\bar{X},\bar{\pi}^*F^{\nu})$ over C gives an embedding $\bar{X}\subsetneq P^N$ $(N=\dim H^0(\bar{X},\bar{\pi}^*F^{\nu})-1)$, where $\sigma_i\in H^0(P^m, F)$, $i=1,\dots, m$ such that $\{\sigma_0=0\}=H_0$ and $z^i=\sigma_i/\sigma_0$ for $i\geqq 1$. By this embedding, we regard $\bar{X}$ as a subvariety in P^N. Since $X=\bar{X}-\{\bar{\pi}^*\sigma_0^{\nu}=0\}$ and $\pi=\bar{\pi}|_X$, X is an affine variety and π is a rational mapping. Q. E. D.

REMARK. Let $X\xrightarrow{\pi}C^m$ be an algebraic fc-covering. A meromorphic function α on X is rational if and only if there are polynomials $P_i(z^1,\dots, z^m)$, $i=0, 1,\dots, k$ such that

$$P_0(\pi(x))\alpha^k(x)+\cdots+P_k(\pi(x))\equiv 0.$$

Moreover this is equivalent to $T(r,\alpha)=O(\log r)$.

PROOF OF PROPOSITION 2. We may assume that $L\to V$ is very ample. If $X\xrightarrow{\pi}C^m$ is algebraic and f is rational, it readily follows that $T(r, L)=O(\log r)$ (see [4]). Suppose that $T(r, L)=O(\log r)$. Since f separates the fibres of $X\xrightarrow{\pi}C^m$, there is a pair of sections σ_0, $\sigma_1\in H^0(V, L)$ such that the meromorphic function $\alpha=f^*(\sigma_1/\sigma_0)$ separates the fibres of $X\xrightarrow{\pi}C^m$. By definition, $N(r, (\alpha)_\infty)\leqq N(r, f^*D_0)$ with $D_0=(\sigma_0)$ and

$$m(r, \alpha) = \frac{1}{k}\int_{\partial X(r)} \log^+|\alpha|\eta = \frac{1}{k}\int_{\partial X(r)} \log^+ \frac{f^*|\sigma_1|}{f^*|\sigma_0|}\eta$$

$$\leqq \frac{1}{k}\int_{\partial X(r)} \log^+ \frac{1}{f^*|\sigma_0|}\eta + \frac{1}{k}\int_{\partial X(r)} \log^+ f^*|\sigma_1|\eta$$

$$= \frac{1}{k}\int_{\partial X(r)} \log \frac{1}{f^*|\sigma_0|}\eta = m(r, D_0)\,,$$

since $f^*|\sigma_0| \leqq 1$ and $f^*|\sigma_1| \leqq 1$. Thus we have by using (3.5)

$$T(r, \alpha) \leqq T(r, L) + O(1)\,. \tag{5.3}$$

This yields that $T(r, \alpha) = O(\log r)$. By Lemma 4.1, $N(r, B) = O(\log r)$. Letting C denote the critical set of $X \xrightarrow{\pi} C^m$, we deduce that $N(r, C) = O(\log r)$. Therefore C is algebraic (see, e.g., [14, 4]). By Lemma 5.1 the finite analytic covering $X \xrightarrow{\pi} C^m$ is algebraic. To conclude the rationality of f, by the remark of Lemma 5.1, it suffices to see that $T(r, f^*(\sigma/\sigma')) = O(\log r)$ for an arbitrary pair of sections $\sigma, \sigma' \in H^0(V, L)$ such that $\{\sigma' = 0\} \not\supset f(X)$ This follows from (5.3). Q.E.D.

6. The second main theorm and defect relation

Let $X \xrightarrow{\pi} C^m$ be an analytic k-covering, F a smooth projective variety of dimension $n \leqq m$, $L \to$ Fa positive line bundle over Fand $K_V \to$ Fdenote the canonical bundle over F. Carlson-Griffiths [1] and Griffiths-King [4] showed the following:

For divisors $D_i = (\sigma_i) \in |L|$, $i = 1, \dots, q$ such that ΣD_i has simple normal crossings and $qc(L) + c(K_V) > 0$[1]*, there exista volumeform Ω on Vand an hermitian metric \ \ in L such that the singular volumeform*

$$\Psi = \frac{\Omega}{\prod_{i=1}^{q} (\log|\sigma_i|^2)^2 |\sigma_i|^2} \tag{6.1}$$

satisfies

$$\mathrm{Ric}\,\Psi > 0, \quad (\mathrm{Ric}\,\Psi)^n \geqq \Psi, \quad \int_{V - Supp(\Sigma D_i)} (\mathrm{Ric}\,\Psi)^n < \infty.$$

In this section we assume that a meromorphic mapping /: $X \to V$ is nondegenerate.

Let $f: X \to V$ be a meromorphic mapping which separates the fibres of $X \xrightarrow{\pi} C^m$. Then $f^*\Psi \wedge \phi^{m-n} \not\equiv 0$. Thus we may assume that

1) $c(L)$ and $c(K_V)$denote the first Chern classes of the line bundles L and K_V in the de Rham cohomology group $H^*(V, R)$.

$$f^*\Psi \wedge \pi^* \left(\bigvee_{\nu=1} \frac{i}{2\pi} dz^\nu \wedge \text{rfz}\wedge_J \not\equiv 0 \,. \tag{6.2}$$

Set

$$f^*\Psi \wedge \pi^* \left(\bigwedge_{\nu=1}^{m-n} \frac{i}{2\pi} dz^\nu \wedge d\bar{z}^\nu \right) = \xi \phi^m \,.$$

Let R_f denote the ramification divisor of

$$f \times (\pi^1, \ldots, \pi^{m-n}) \colon X \longrightarrow V \times \boldsymbol{C}^{m-n}. \tag{6.3}$$

Let B be the ramification divisor of $X \xrightarrow{\pi} \boldsymbol{C}^m$. From Griffiths-King [4, Lemma 6.18] we obtain a current equation in $X - (S(X) \cup N(f))$

$$dd^c \log \xi = f^* \text{Ric}\, \Psi + R_f - B - \sum_{i=1}^q f^* D_i \,.$$

Setting $\zeta(z) = \prod_{x \in \pi^{-1}(z)} \xi(x)$, we have

$$dd^c \log \zeta = \pi_*(f^* \text{Ric}\, \Psi) + \pi_* R_f - \pi_* B - \sum_{i=1}^q \pi_*(f^* D_i) \tag{6.4}$$

in $\boldsymbol{C}^m - \pi(S(X) \cup N(f))$. By Lemma 3.2 there is an entire function α such that $(\alpha) = \pi_* B + \sum_{i=1}^q \pi_*(f^* D_i)$. Setting

$$\zeta_1 = \log \zeta + \log |\alpha|^2,$$

we have

$$dd^c \zeta_1 = \pi_*(f^* \text{Ric}\, \Psi) + \pi_* R_f \tag{6.5}$$

in $\boldsymbol{C}^m - \pi(S(X) \cup N(f))$.Since the right hand side is a positive current, ζ_1 is plurisubharmonic in $\boldsymbol{C}^m - \pi(S(X) N(f))$. Since $\operatorname{codim} \pi(S(X) \cup N(f)) \geqq 2$, by Grauert-Remmert [3] ζ_1 can be uniquely extended as a plurisubharmonic function in all $\boldsymbol{C}^m$. Therefore the current equation (6.5) is valid in all $\boldsymbol{C}^m$ and so is (6.4). We have

$$\log \zeta = \zeta_1 - \zeta_2, \tag{6.6}$$

where ζ_1 and $\zeta_2 = \log |\alpha|^2$ are plurisubharmonic in $\boldsymbol{C}^m$. Set

$$\begin{aligned} T^*(r) &= \frac{1}{k} \int_1^r \frac{1}{t^{2m-1}} \int_{X(t)} f^* \text{Ric}\, \Psi \wedge \phi^{m-1} \\ &= \frac{1}{k} \int_1^r \frac{1}{t^{2m-1}} \int_{C^m(t)} \pi_*(f^* \text{Ric}\ \Psi) \wedge \phi^{m-1} \,, \end{aligned}$$

$$\mu(r) = \frac{1}{k} \int_{\partial X(r)} \log \xi \eta = \frac{1}{k} \int_{\partial C^m(r)} \log \zeta \eta \,.$$

Combining (6.4), (6.6) with Lemma 3.1, we get

$$T^{\#}(r)+N(r,\ R_f)=N(r,B)+\sum_{i=1}^{q}N(r,f^*D_i)$$
$$+\mu(r)-\mu(1).$$

Once this equality is shown, in the same way as in Griffiths-King [4, sections 7 and 8 (c)] we obtain

$$qT(r,\ L)+T(r,\ K_V)\leqq\sum_{i=1}^{q}N(r,f^*D_i)+N(r,B)$$
$$-N(r,\ R_f)+\mu(r)+O(1), \tag{6.7}$$

where the remainder term $\mu(r)$ satisfies

$$\mu(r)=O(\log T(r,\ L))+(\theta-1)O(\log r) \tag{6.8}$$

outside an exceptional set I of r satisfying $\int_I dr^{\theta}<\infty$ for any $\theta>1$; moreover if the order of T(r, L) is finite,

$$\mu(r)=O(\log r)\qquad \text{for all}\quad r. \tag{6.9}$$

In what follows, we denote by $S(r)$ a function of r satisfying the properties (6.8) and (6.9) with respect to $T(r,$ L).

Since f separates the fibres of $X\xrightarrow{\pi}C^m$ and the line bundle $L\to V$ is positive, there is an integer l such that the following holds:

$$A(f,L^l)\left\{\begin{array}{l}\text{There exists a pair of sections } \sigma_0,\ \sigma_1\in H^0(V,L^l)\\ \text{such that the meromorphic function } f^*(\sigma_0/\sigma_1)\\ \text{separates the fibres of } X\xrightarrow{\pi}C^m.\end{array}\right.$$

Let l_0 be the least integer l for which $A(f,L^l)$ holds. From Lemma 4.1 and (5.3) we see that

$$N(r,B)\leqq 2(k-1)T(r,\ L^{l_0})+O(1)$$
$$=2(k-1)l_0T(r,\ L)+O(1).$$

From this and (6.7) it follows that

$$\{q-2(k-1)l_0\}T(r,\ L)+T(r,K_V)$$
$$\leqq\sum_{i=1}^{q}N(r,f^*D_i)-N(r,\ R_f)+S(r). \tag{6.10}$$

Now we let $f\colon X\to V$ be a meromorphic mapping which does not necessarily separates the fibres of $X\xrightarrow{\pi} \boldsymbol{C}^m$. Then by Proposition 1 there are an analytic fco-covering $X_0\xrightarrow{\pi_0}\boldsymbol{C}^m$, where k_0 is a divisor offc, and a meromorphic mapping $f_0\colon X_0\to V$which separates the fibres of $X_0\xrightarrow{\pi_0}\boldsymbol{C}^m$. The inequality (6.10) is valid for the meromorphic mapping f_0. By the remark of Proposition 1 we have

THEOREM 1 (The second main Theorem). *Let $f\colon X\to V$ be a meromorphic mapping and $L\to V$ a positive line bundle. Then for divisors $D_i\in|L|$, $i=1,\dots,q$ such that ΣD_i has simple normal crossings and $qc(L)+c(K_V)>0$,*

$$\{q-2(k_0-1)l_0\}T(rL)+T(r,K_V) \tag{6.11}$$

$$\leqq \sum_{i=1}^{q} N(r,f^*D_i)-N(r,R_{f_0})+S(r),$$

where $f_0\colon X_0\to V$ is the meromorphic mapping given by Proposition 1, k_0 is the number of sheets of $X_0\xrightarrow{\pi_0}\boldsymbol{C}^m$ and l_0 is the least integer $l\in Z$ for which $A(f_0, L^l)$ holds.

Let $f\colon X\to V$ be a meromorphic mapping. Then for a divisor $D\in \backslash L\backslash$ we set

$$\bar{n}(t,f^*D)=\frac{1}{kt^{2m-2}}\int_{Supp(f^*D)\cap X(t)}\phi^{m-1},$$

$$\bar{N}(r,f^*D)=\int_1^r\frac{\bar{n}(t,f^*D)}{t}dt,$$

$$\delta(D)=1-\varlimsup_{r\to\infty}\frac{N(r,f^*D)}{T(r,L)},$$

$$\Theta(D)=1-\varlimsup_{r\to\infty}\frac{\bar{N}(r,f^*D)}{T(r,L)}.$$

By definition, $\delta(D)\leqq\Theta(D)$.

THEOREM 2 (Defect Relation). *Under the same assumptions as in Theorem 1, we have*

$$\sum_{i=1}^{q}\delta(D_i)\leqq\left[\frac{c(K_V^{-1})}{c(L)}\right]+2(k_0-1)l_0, \tag{6.12}$$

where $[c(K_V^{-1})/c(L)]=\inf\{\theta\in\boldsymbol{R};\ \theta c(L)+c(K_V)>0\}$ and the integers k_0 and l_0 are the same as in (6.11). Moreover if f separates the fibres of the analytic k-covering $X\xrightarrow{\pi}\boldsymbol{C}^m$, then

$$\sum_{i=1}^{q}\Theta(D_i)\leqq\left[\frac{c(K_V^{-1})}{c(L)}\right]+2(k-1)l_0. \tag{6.13}$$

PROOF. The defect relation (6.12) readily follows from (6.11). Suppose that f separates the fibres of $X \xrightarrow{\pi} \boldsymbol{C}^m$. Let us recall the definition of R_f: R_f is the divisor of

$$f\times(\pi^1,\dots,\pi^{m-n})\colon X \longrightarrow V\times \boldsymbol{C}^{m-n} \quad \text{(cf. (6.2), (6.3))}.$$

The method of the proof of Sakai [11, Proposition 3] can apply to the holomorphic mapping $(f\times(\pi^1,\dots,\pi^{m-n}))|_{X-(S(X)\cup N(f))}\colon X-(S(X)\cup N(f))\to V\times \boldsymbol{C}^{m-n}$. So we have

$$(6.14) \qquad \sum_{i=1}^{q} f^*D_i - \sum_{i=1}^{q} Supp(f^*D_i) \leqq R_f$$

in $X-(S(X)\cup N(f))$.Since $\operatorname{codim}(S(X)\cup N(f))\geqq 2$, (6.14) holds in all X. The defect relation (6.13) follows from (6.10) and (6.14). Q. E. D.

For a detailed argument in case $\dim X=\dim V=1$, we refer to [8].

Let $X=\boldsymbol{C}^2$. In this case, the defect relation (6.12) was proved by Shiffman [13]. Let V be a non-singular hypersurface of degree d in $\mathbf{P}^3$ and L the restriction of the hyperplane bundle over $\boldsymbol{P}^3$ on V. Suppose that there exists a non-degenerate meromorphic mapping of C^2 into V. Then by Theorem 2, $d\leqq 4$. We give an example for $d=4$. Let V be a Fermat quartic surface:

$$(w^0)^4+(w^1)^4+(w^2)^4+(w^3)^4=0,$$

where $[w^0, w^1, w^2, w^3]$ is a homogeneous coordinate system in $\boldsymbol{P}^3$. Then by [9, Section 8] V is a Kummer surface $Km(A)$ associated with an abelian surface A, that is, V is a non-singular model (obtained by one σ-process at each fixed point of 0) of the factored space $A/<\theta>$, where θ denotes the involution $A\ni x\mapsto -x \in A$. By composing the covering mappings of $\boldsymbol{C}^2$ onto A and of A onto $A/<\theta>$ and the birational mapping of $A/<\theta>$ to V, we get a non-degenerate meromorphic mapping $f\colon \boldsymbol{C}^2\to V$.

Let X be an analytic 2-covering over $\boldsymbol{C}^2$, V and L as above. Then $d\leqq 6$. It is hoped to find examples for $d=6$.

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