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The Enumeration of Liftings in Fibrations and the Embedding Problem I

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Introduction

As for the enumeration problem of embeddings of manifolds, many results have been obtained up to the present (e.g. [2], [5], [6], [7], [20] and [21]) but they are small in number compared with those of the existence problem. In this paper, we try one approach to the enumeration problem of embeddings of n-dimensional differentiable manifolds into the real $(2n-1)$-space R^{2n-1}. As an application, we determine the cardinality of the set of isotopy classes of embeddings of the n-dimensional real projective space RP^n into R^{2n-1}.

Our plan is as follows. An embedding $f: M\to R^m$ of a space M into R^m induces a Z_2-equivariant map $F: M\times M-\Delta\to S^{m-1}$ by $F(x,y)=\frac{f(x)-f(y)}{\|f(x)-f(y)\|}$ for distinct points x, y of M, where Δ is the diagonal of M and the Z_2-actions on $M\times M-\Delta$ and S^{m-1} are the interchange of the factors and the antipodal action, respectively. Consider the correspondence which associates with an isotopy class of an embedding $f: M\to R^m$ the equivariant homotopy class of the map F made above. Then this correspondence is surjective if $2m\geq 3(n+1)$ and bijective if $2m>3(n+1)$ for any n-dimensional compact differentiable manifold M by the theorem of A. Haefliger [5, § 1]. On the other hand, there is a one-to-one correspondence between the set of the equivariant homotopy classes of equivariant maps of $M\times M-\Delta$ to S^{m-1} and the set of homotopy classes of cross sections of the sphere bundle $S^{m-1}\to(M\times M-\Delta)\times_{Z_2}S^{m-1}\to(M\times M-\Delta)/Z_2$, where the reduced symmetric product $M^*=(M\times M-\Delta)/Z_2$ of M has the homotopy type of a CW-complex X of dimension less than $2n$ ($n=\dim M$). Therefore, the enumeration problem of embeddings of an n-dimensional manifold M into R^m arrives at the enumeration problem of cross sections of an S^{m-1}-bundle ξ over a CW-complex X of dimension less than $2n$.

Now, consider the case that $m=2n-1$, and let p: $BO(m-1)\to BO(m)$ be the universal S^{m-1}-bundle. Then the enumeration of cross sections of an S^{m-1}-bundle ξ over X is equivalent to the enumeration of liftings of the classifying map $\xi: X\to BO(m)$ of ξ to $BO(m-1)$. We construct the third stage Postnikov factorization

(*)

$$\begin{array}{ccc} BO(m-1) & \xrightarrow{q_2} & T \\ & \searrow^{q_1} & \downarrow p_2 \\ & {}_{p}\searrow & E \\ & & \downarrow p_1 \\ & & BO(m) \end{array}$$

of p. Here p_1 is the twisted principal fibration, p_2 is the principal fibration and q_2 is an $(m+1)$-equivalence. Since the dimension of X is less than $m+1$, the enumeration of liftings of ξ to $BO(m-1)$ is equivalent to the enumeration of liftings to T by the theorem of I. M. James and E. Thomas [11, Theorem 3.2].

From the above considerations, this paper is divided into three chapters.

In Chapter I, we study the enumeration problem of liftings of a map into the base space of a certain fibration to the total space. In § 1, the twisted principal fibration is defined and the enumeration of liftings for this fibration is treated. Further, we are concerned with the composition of two twisted principal fibrations $T \xrightarrow{q} E \xrightarrow{p} D$ under the assumption that it is stable (see § 2). We describe the set of homotopy classes of liftings of a map $u: X \to D$ to the composition pq: $T \to D$ in Theorem A of § 2, which is a generalization of the theorem of I. M. James and E. Thomas [12, Theorem 2.2] for principal fibrations. After preparing several propositions for the composition pq in §§ 3–4 without assuming the stability, Theorem A is proved in § 5.

The purpose of Chapter II is to study the enumeration problem of cross sections of sphere bundles. In § 6, we notice the cohomology $H^*(X; \underline{Z})$ with coefficients in the local system defined by $\phi: \pi_1(X)$-Aut(Z). In § 7, the third stage Postnikov factorization (*) of p: $BO(n-1) \to BO(n)$ is constructed, and we show in § 8 that the composition of fibrations $p_1 p_2$: $T \to BO(n)$ is stable in the sense of § 2. From Theorem A and the fact that q_2: $BO(n-1) \to T$ is an $(n+1)$-equivalence, we have the following theorem in § 9.

THEOREM B. *Let ξ be a real n-plane bundle over a CW-complex X of dimension less than $n+1$ and let $n>4$. If ξ has a non-zero cross section, then the set cross (ξ) of homotopy classes of non-zero cross sections of ξ is given, as a set, by*

$$\text{cross}(\xi) = H^{n-1}(X; Z) \times \text{Coker}\, \Theta,$$

where the homomorphism

$$\Theta: H^{n-2}(X; \underline{Z}) \longrightarrow H^n(X; Z_2)$$

is *defined by*

$$\Theta(a) = (\rho_2 a) w_2(\xi) + Sq^2 \rho_2 a \qquad \text{for} \quad a \in H^{n-2}(X; \text{Z}),$$

ρ_2 is the mod 2 *reduction, Z is the local system on X associated with ξ and $w_2(\xi)$ is the second Stiefel-Whitney class of ξ.*

Chapter III is devoted to an application of A. Haefliger's theorem and Theorem B on the enumeration problem of embeddings of n-dimensional manifolds into R^{2n-1}. In § 10, the set $[M \subset R^{2n-1}]$ of isotopy classes of embeddings of n-dimensional closed differentiable manifolds M into R^{2n-1} is described with the cohomology of M*. As an application for the n-dimensional real projective space RP^n, we calculate the cohomology group $H^{2n-2}((RP^n)^*; Z)$ and the homomorphism $\Theta: H^{2n-3}((RP^n)^*; Z) \to H^{2n-1}((RP^n)^*; Z_2)$, and we have the following theorem in §§ 11–12.

THEOREM C. *Let $n \neq 2^r$ and $n > 6$. Then the n-dimensional real projective space RP^n is embedded in the real $(2n-1)$-space R^{2n-1}, and there are just four and two isotopy classes of embeddings of RP^n into R^{2n-1} for $n = 3(4)$ and $n \not\equiv 3(4)$, respectively.*

Chapter I. Enumeration of liftings in certain fibrations

§ 1. Twisted principal fibrations

Let Z be a given space. By a *Z-space* $X = (X, f)$, we mean a space X together with a (continuous) map $f: X \to Z$. For two Z-spaces $X = (X, f)$ and $Y = (Y, g)$, the pull back

$$X \times_Z Y = \{(x, y) \mid f(x) = g(y)\} \qquad (\subset X \times Y)$$

of f and g is a Z-space with $(f, g): X \times_Z Y \to Z$, $(f, g)(x, y) = f(x) = g(y)$. A map $h: X \to Y$ is called a *Z-map* if $gh = f$, and a homotopy $h_t: X \to Y$ is called a *Z-homotopy* if $gh_t = f$ for all t. In this case, we say that h_0 is *Z-homotopic* to h_1 and denote by $h_0 \simeq_Z h_1$ Further,

$$[X, Y]_Z$$

denotes the set of all Z-homotopy classes of Z-maps of X to Y.

Now, let B be a space (with base point *) and π be a discrete group, and assume that π acts on B preserving the base point by a homomorphism $\phi: \pi \to \text{Homeo}(B, *)$. Then, considering the Eilenberg-MacLane space $K = K(\pi, 1)$, the universal covering $\tilde{K} \to K$ and the usual action of π on $\tilde{K}$, we have the fiber bundle

$$(1.1) \quad B \longrightarrow L_\phi(B) = \tilde{K} \times_\pi B \xrightarrow{q} K = K(\pi, 1)$$

with structure group π. Since $\tilde{K} \times_\pi * = K$, we have the canonical cross section

$s: K \to \tilde{K} \times_{\pi} B$ such that $s(K) = K = \tilde{K} \times_{\pi} *$.

In this paper, we consider the following situation.

(1.2) *Let π act on an H-group*) B by ϕ satisfying the following assumptions: The multiplication μ: $B \times B \to B$ and the homotopy inverse* v: $B \to B$ *of B are π-equivariant and there are π-equivariant homotopies*

$$\mu(1_B, c) \simeq 1_B \simeq \mu(c, 1_B),\ \mu(\mu \times 1_B) \simeq \mu(1_B \times \mu)\ and\ \mu(\nu, 1_B) \simeq c \simeq \mu(1_B, \nu),$$

*where $c: B \to B$ is the constant map to *. Also, if B is homotopy abelian, we assume in addition that there is a π-equivariant homotopy $\mu t \simeq \mu$, where $t: B \times B \to B \times B$ is the map defined by $t(x, y) = (y,$* x).

Then, for the K-space $(L_{\phi}(B), q)$ of (1.1), we can define K-maps

(1.3) μ_{ϕ}: $L_{\phi}(B) \times_K L_{\phi}(B) \longrightarrow L_{\phi}(B)$, ν_{ϕ}: $L_{\phi}(B) \longrightarrow L_{\phi}(B)$

by

$$\mu_{\phi}([\tilde{x}, b], [\tilde{x}, b']) = [\tilde{x}, \mu(b, \text{fr}')],\quad \nu_{\phi}([\tilde{x}, \text{ft}]) = [\tilde{x}, \nu(b)],$$

and there exist the following relations:

$$\mu_{\phi}(1 x\, sq)\Delta \simeq_K 1 \simeq_K \mu_{\phi}(sq x\, 1)\Delta: L_{\phi}(B) \longrightarrow L_{\phi}(B),$$

$$\mu_{\phi}(\mu_{\phi} \times 1) \simeq_K \mu_{\phi}(1 \times \mu_{\phi}): L_{\phi}(B) \times_K L_{\phi}(B) \times_K L_{\phi}(B) \longrightarrow L_{\phi}(B),$$

$$\mu_{\phi}(\nu_{\phi} x\, i)\Delta \simeq_K sq \simeq_K \mu_{\phi}(1 x\, \nu_{\phi})\Delta: L_{\phi}(B) \longrightarrow L_{\phi}(B),$$

and

$$\mu_{\phi} t \simeq_K \mu_{\phi}: L_{\phi}(B) \times_K L_{\phi}(B) \longrightarrow L_{\phi}(B),$$

if B is homotopy abelian, where A is the diagonal map and t is the map defined by $t(x, y) = (y, x)$.

Therefore we have the following

LEMMA 1.4. *Let X be a K-space with a map $u: X \to K$. Then the homotopy set $[X, L_{\phi}(B)]_K$ of K-maps is a group with unit $[su]$ by the multiplication*

$$[/] \ [g] = [\mu_{\phi}(f \times g)\Delta] \textit{for } K\textit{-maps } f, g: X \longrightarrow L_{\phi}(B).$$

If, furthermore, B is homotopy abelian, then this group $[X, L_{\phi}(B)]_K$ is abelian.

Let p: $E \to A$ be a fibration with fiber $F = p^{-1}(*)$, and assume that p admits a cross section s: $(A, *) \to (E, *)$. Then, we can consider the path spaces

*) The H-group is the homotopy associative H-space with a homotopy inverse.

$$P_AE = \{\lambda: I \longrightarrow E \mid \lambda(0) \in s(A), p\lambda(0) = p\lambda(t) \text{ for all } t \in I\},$$

$$\Omega_A E = \{\lambda \in P_A E \mid \lambda(0) = \lambda(1)\},$$

and we have the following well-known lemma.

LEMMA 1.5. *The projection*

$$r: P_A E \longrightarrow E, \qquad r(\lambda) = \lambda(1),$$

is a fibration with fiber ΩF. Furthermore,

$$pr: P_A E \longrightarrow A \text{ and } pr: \Omega_A E \longrightarrow A$$

are fibrations with fibers PF and ΩF, respectively, and they admit the canonical cross sections induced by s, where $PF = \{\lambda: I \to F \mid \lambda(0) = *\}$ and $\Omega F = \{\lambda \in PF \mid \lambda(0) = \lambda(1)\}$ are the ordinary path space and loop space of F.

By applying this lemma to the fibration $q: L_\phi(B) \to K$ of (1.1), we obtain the fibration

$$qr: \Omega_K L_\phi(B) \longrightarrow K, \ (qr)^{-1}(*) = \Omega B,$$

admitting the canonical cross section s. On the other hand, the given homomorphism $\phi: \pi \to \mathrm{Homeo}(B, *)$ induces the homomorphism

$$\phi': \pi \longrightarrow \mathrm{Homeo}\,(\Omega B, *), \ \phi'(g)(\lambda)(t) = \phi(g)(\lambda(t)).$$

This determines by (1.1) the fibration

$$q': L_{\phi'}(\Omega B) \longrightarrow X,$$

with fiber ΩB admitting the canonical cross section s', and we have the natural homeomorphism

$$\psi: L_{\phi'}(\Omega B) \xrightarrow{\approx} \Omega_K L_\phi(B), \qquad \psi([\tilde{x}, \lambda])(t) = [\mathrm{S}, \lambda(t)],$$

which satisfies $qr\psi = q'$. Also, the loop space ΩB is a homotopy abelian H-group by the join $\vee$ of loops:

$$(\lambda_1 \vee \lambda_2)(t) = \begin{cases} \lambda_1(2t) & 0 \le 2t \le 1 \\ \lambda_2(2t-1) & 1 \le 2t \le 2, \end{cases}$$

and the action of π on ΩB by ϕ' satisfies (1.2). Therefore, Lemma 1.4 shows that the homotopy set $[X, L_{\phi'}(\Omega B)]_K$ of K-maps is an abelian group by the multiplication induced by $\vee_{\phi'}$. Furthermore, the above natural homeomorphism ψ commutes with $\vee_{\phi'}$ and the K-map

$$\vee : \Omega_K L_\phi(B) \times_K \Omega_K L_\phi(B) \longrightarrow \Omega_K L_\phi(B)$$

given by the join of loops, and we have the following

LEMMA 1.6. *The natural K-homeomorphism $\psi: L_{\phi'}(\Omega B) \to \Omega_K L_\phi(B)$ induces an isomorphism*

$$\psi_*: [X, L_{\phi'}(\Omega B)]_K \xrightarrow{\approx} [X, \Omega_K L_\phi(B)]_K$$

for any K-space X, where the domain is the abelian group of Lemma 1.4 *and the multiplication in the range is induced by $\vee$ mentioned above.*

Also, applying Lemma 1.5 to $q: L_\phi(B) \to K$ of (1.1), we obtain the fibration

$$r: P_K L_\phi(B) \longrightarrow L_\phi(B) \quad \text{with fiber } \Omega B.$$

Now, let $\theta: D \to L_\phi(B)$ be a given map. Then, from this fibration, θ induces a fibration

$$p: E = D \times_L P_K L_\phi(B) \longrightarrow D \ (L = L_\phi(B)) \quad \text{with fiber } \Omega B,$$

which is called *the twisted principal fibration* with classifying map θ.

Let $u: X \to D$ be a given map and consider the diagram

$$\begin{array}{ccccccc} & & E & & P_K L_\phi(B) & & \Omega_K L_\phi(B) \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} & & \| \\ X & \xrightarrow{u} & D & \xrightarrow{\theta} & L_\phi(B) & \xrightarrow{q} & K. \end{array}$$

We define a D-map

(1.7) $$m: \Omega_K L_\phi(B) \times_K E \longrightarrow E$$

by the relation $m(\lambda_1, (x, \lambda_2)) = (x, \lambda_1 \vee \lambda_2)$, where $\vee$ is the join of paths, and the domain is the pull back of K-spaces $(\Omega_K L_\phi(B), qr)$ and $(E, q\theta p)$ and is understood as a D-space $(\Omega_K L_\phi(B) \times_K E, p\pi_2)$ (π_2 is the projection to the second factor in this paper). Hereafter, we often write $\lambda_1 \vee (x, \lambda_2)$ for $m(\lambda_1, (x, \lambda_2))$ simply. By considering a D-space $X = (X, u)$ as a K-space $(X, q\theta u)$, this map m induces a function

$$m_*: [X, \Omega_K L_\phi(B)]_K \times [X, E]_D \longrightarrow [X, E]_D.$$

PROPOSITION 1.8. *The function m_* mentioned above is an action of the abelian group $[X, \Omega_K L_\phi(B)]_K$ of* Lemma 1.6 *on the homotopy set $[X, E]_D$. If $u: X \to D$ has a lifting $v: X \to E$, that is, if there is a D-map $v: (X, u) \to (E, p)$, then the function $m_*(\ , [v]): [X, \Omega_K L_\phi(B)]_K \to [X, E]_D$ is a bijection.*

PROOF. This is a straightforward modification of the case that $p: E \to D$

is a usual principal fibration (cf. [12, Lemma 3.1]).

§ 2. The main result in Chapter I

Let B and C be H-groups with homomorphisms $\phi(B):\pi(B)\to\mathrm{Homeo}(B, *)$ and $\phi(C):\pi(C)\to\mathrm{Homeo}(C, *)$ such that they satisfy the assumption (1.2), and let

$$q_A: L(A)=L_{\phi(A)}(A)\longrightarrow K(A)=K(\pi(A), 1) \qquad (A = \mathrm{B},\ \mathrm{C})$$

be the fiber bundle of (1.1) with the canonical cross section s_A. Consider the following situation:

$$\begin{array}{ccccccc} & & T & & & & \\ & & \big\downarrow{\scriptstyle q} & & & & \\ & & E & \xrightarrow{\rho} & L(C) & \xrightarrow{q_C} & K(C) \\ & & \big\downarrow{\scriptstyle p} & & & \nearrow{\scriptstyle \bar\rho} & \\ X & \xrightarrow{u} & D & \xrightarrow{\theta} & L(B) & \xrightarrow{q_B} & K(B)\,. \end{array} \tag{2.1}$$

Here p is the twisted principal fibration with fiber ΩB induced from $P_{K(B)}L(B)\to L(B)$ by 0, q is the one with fiber ΩC induced from $P_{K(C)}L(C)\to L(C)$ by p, and it is assumed that

$$q_C\rho = \bar\rho p.^{*)}$$

For a given map $u:X\to D$, the homotopy set $[X, T]_D$ of D-maps of the D-space (X, u) to the D-space (T, pq) is the set of homotopy classes of liftings of u to T. The investigation of this set is our main purpose of Chapter I.

From now on, we assume that C is a topological group.**) For the simplicity,

$$n: L(C)\times_{K(C)} L(C)\longrightarrow L(C) \quad\text{and}\quad {}^{-1}: L(C)\longrightarrow L(C)$$

denote the $K(C)$-maps $\mu_{\phi(C)}$ and $\nu_{\phi(C)}$ of (1.3) induced from the multiplication and the inverse of C.

Let

$$m_B: \Omega_{K(B)}L(B)\times_{K(B)}E\longrightarrow E \tag{2.2}$$

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*) In our applications of the later chapters, we are concerned with the case where $K(C)=*$. For this case, $L(C)=C$ and q is a usual principal fibration and the existence of such a map p with $q_C\rho=\rho p$ is trivial.

**) This assumption gives neat formulas but essentially the same theory carries through in the case that C is an H-group.

be the D-map defined in (1.7), and consider the map

$$\rho_1 : \Omega_{K(B)}L(B)\times_{K(B)}E \longrightarrow L(C)$$

defined by

$$\rho_1(\lambda, y) = n(\rho m_B(\lambda, y), [\rho m_B(c_{\lambda(0)}, y)]^{-1}) \qquad \text{for} \quad \lambda \text{ e } \Omega_{K(B)}L(B), y \in E,$$

where c_x denotes the constant loop at x. Then, ρ_1 maps $E = s_B(K(B))\times_{K(B)} E$ to $K(C) = s_C(K(C))$, and ρ_1 is a $K(C)$-map, where $\Omega_{K(B)}L(B)\times_{K(B)}E$is considered as a $K(C)$-space by the composition $\rho p\pi_2 = q_C\rho\pi_2$($\pi_2$is the projection to the second factor). Therefore, we have $K(C)$-maps ρ_1 and $1 \times p$ in the diagram

$$(2.3) \qquad \begin{array}{ccc} (\Omega_{K(B)}L(B)\times_{K(B)}E,\ E) & \xrightarrow{\rho_1} & (L(C),\ K(C)) \\ \downarrow{\scriptstyle 1\times p} & & \| \\ (\Omega_{K(B)}L(B)\times_{K(B)}D,\ D) & \xrightarrow{d} & (L(C),\ K(C)), \end{array}$$

where $\Omega_{K(B)}L(B)\times_{K(B)}D$is also considered as a $K(C)$-space by the composition $\bar{\rho}\pi_2$.

Now, we say that the composition of fibrations $T \xrightarrow{q} E \xrightarrow{p} D$ in (2.1) is ***stable***, if there exists a $K(C)$-map d in (2.3) such that the diagram (2.3) is $K(C)$-homotopy commutative.

Suppose that the composition pq is stable by a $K(C)$-map d. From the fibration $\Omega_{K(B)}L(B) \rightarrow K(B)$,we obtain the fibration

$$\text{fli}_{(B)}\text{JXB}) = \Omega_{K(B)}(\Omega_{K(B)}L(B)) \text{— } K(B)$$

with the canonical cross section, by Lemma 1.5. Then, the map d induces a $K(C)$-map

$$(2.4) \qquad d' : (\Omega^2_{K(B)}L(B)\times_{K(B)}D,\ D) \longrightarrow (\Omega_{K(C)}L(C), K(C))$$

by the equation

$$d'(\lambda, x)(0 = d(\lambda(t), x) \qquad \text{for} \quad \lambda \text{ e } \Omega^2_{K(B)}L(B),\ \text{x e } D \quad \text{and} \quad t \text{ e } /.$$

For a given D-space $X = (X, u)$, these $K(C)$-maps d and rf induce two functions

$$(2.5) \qquad \begin{array}{l} \Theta_u : [X, \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X, L(C)]_{K(C)}, \\ \Theta'_u : [X, \Omega^2_{K(B)}L(B)]_{K(B)} \longrightarrow [X, \Omega_{K(C)}L(C)]_{K(C)}, \end{array}$$

given by $\Theta_u([a]) = [d(a, u)]$ and $\Theta'_u([b]) = [d'(b, u)]$, where X is considered as a $K(B)$-space $(X, q_B\theta u)$ and $K(C)$-space $(X, \bar{\rho}u)$. Here Θ'_u is a homomorphism of groups by the definition of d' and so Coker Θ'_u is defined. Set Ker $\Theta_u = \Theta_u^{-1}$ $([s_C\bar{\rho}u])$. Then we have the following main theorem in this chapter, which is a generalization of [12, Theorem 2.2].

THEOREM A. *Suppose that the composition of the fibrations*

$$T \xrightarrow{q} E \xrightarrow{p} D$$

in the diagram (2.1) *is stable by the map d in* (2.3). *Let X be a CW-complex and* $u: X \to D$ *admit a lifting* $X \to T$. *Then the set*

$$[X, T]_D$$

of homotopy classes of liftings of u to T is equivalent to the product

$$\text{Ker } \Theta_u \times \text{Coker } \Theta'_u,$$

where Θ_u *and* Θ'_u *are the functions of* (2.5).

§3. Correlations

Consider the diagram (2.1) and let $v: X \to E$ be a lifting of $u: X \to D$. We say that two maps $h, h': X \to T$ are *v-related* if (1) $qh = qh' = v$ and (2) h is D-homotopic to h'. The relation "*v-related*" is an equivalence relation, and if v is D-homotopic to v', then the set of v-relation classes is equivalent to the set of v'-relation classes.

For $\eta = [v] \in [X, E]_D$, let $N(\eta)$ denote the set of v-relation classes of D-maps of X to T. Then

$$N(\eta) = q_*^{-1}(\eta) \quad \text{and} \quad [X, T]_D = \cup \{q_*^{-1}(\eta) \mid \eta \in [X, E]_D\},$$

where $q_*: [X, T]_D \to [X, E]_D$. Thus we have the following

LEMMA 3.1 [12, Theorem 3.2]. *The set* $[X, T]_D$ *is equivalent to the disjoint union of the set* $N(\eta)$, *where* η *runs through the elements of* $[X, E]_D$.

Since the set $[X, E]_D$ is equivalent to the group $[X, \Omega_{K(B)}L(B)]_{K(B)}$ by Proposition 1.8, we study the set $N(\eta)$ for each $\eta \in [X, E]_D$ in the rest of this section.

As is constructed in (1.7), there is a D-map

$$m_C: \Omega_{K(C)}L(C) \times_{K(C)} T \longrightarrow T.$$

This D-map m_C induces an action of the group $[X, \Omega_{K(C)}L(C)]_{K(C)}$ on $[X, T]_D$ by the same way as Proposition 1.8. It is easily seen that (1) if $h: X \to T$ is a D-map and if $k, k': X \to \Omega_{K(C)}L(C)$ are $K(C)$-homotopic, then $m_C(k, h)$ and $m_C(k', h)$ are ι-related, where $v = qh$, and (2) if $k: X \to \Omega_{K(C)}L(C)$ is a $K(C)$-map and if h, h': $X \to T$ are ι-related, then $m_C(k, h)$ and $m_C(k, h')$ are ι-related. Hence, using

Proposition 1.8, we see that the above action of $[X, \Omega_{K(C)}L(C)]_{K(C)}$ is transmitted to a transitive action on $N(\eta)$. We, therefore, have the following

LEMMA 3.2. *Let η be the element in the image of $q_*: [X, T]_D \to [X, E]_D$. The set $N(\eta)$ is equivalent to the quotient of $[X, \Omega_{K(C)}L(C)]_{K(C)}$ by the stabilizer of an element of $N(\eta)$.*

Let $p: E \to A$ be a fibration with fiber F and let

$$\Omega_A^* E = \{\lambda: I \longrightarrow E \mid p\lambda(t) = p\lambda(0) \text{ for all } t \in I,\ \lambda(0) = \lambda(1)\},$$

$$\Omega^* F = \{\lambda: I \longrightarrow F \mid \lambda(0) = \lambda(1)\}.$$

Then the following results are known and will be used later on.

LEMMA 3.3. *Let $r: \Omega_A^* E \to E$ be a map defined by $r(\lambda) = \lambda(1)$. Then $r: \Omega_A^* E \to E$ is a fibration with fiber ΩF and $pr: \Omega_A^* E \to A$ is also a fibration with fiber $\Omega^* F$.*

The map $p: E \to L(C)$ in (2.1) induces a map

$$\rho': \Omega_D^* E \longrightarrow \Omega_{K(C)}^* L(C),$$

which is given by $\rho'(\lambda)(t) = \rho(\lambda(t))$, and there follows a commutative diagram below,

$$\begin{array}{ccccc} \Omega_D^* E & \xrightarrow{r} & E & \xrightarrow{p} & D \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \bar{\rho}} \\ \Omega_{K(C)}^* L(C) & \xrightarrow{r} & L(C) & \longrightarrow & K(C). \end{array}$$

Therefore we have a commutative diagram

$$\begin{array}{ccccc} & & [X, \Omega_D^* E]_D & \xrightarrow{r_*} & [X, E]_D \\ & & \downarrow{\scriptstyle \rho'_*} & & \downarrow{\scriptstyle \rho_*} \\ [X, \Omega_{K(C)}L(C)]_{K(C)} & \xrightarrow{i_*} & [X, \Omega_{K(C)}^* L(C)]_{K(C)} & \xrightarrow{r_*} & [X, L(C)]_{K(C)}, \end{array}$$

where $i: \Omega_{K(C)}L(C) \to \Omega_{K(C)}^* L(C)$ is the natural inclusion. We say that an element $\gamma \in [X, \Omega_{K(C)}L(C)]_{K(C)}$ is *ρ-correlated to* $\eta \in [X, E]_D$ if there is an element $\chi \in [X, \Omega_D^* E]_D$ such that $r_*(\chi) = \eta$ and $\rho'_*(\chi) = i_*(\gamma)$

LEMMA 3.4. *Let $h: X \to T$ be a D-map and let $v = qh$. Suppose that $k \vee h = m_C(k, h)$ is v-related to h for a $K(C)$-map $k: X \to \Omega_{K(C)}L(C)$. Then the class of k in $[X, \Omega_{K(C)}L(C)]_{K(C)}$ is ρ-correlated to the D-homotopy class of $v: X \to E$.*

LEMMA 3.5. *For a $K(C)$-map $k: X \to \Omega_{K(C)}L(C)$, suppose that the class of*

k in $[X, \Omega_{K(C)}L(C)]_{K(C)}$ is ρ-correlated to the D homotopy class of $v\colon X\to E$. Then $k\vee h$ is v-related to h for any lifting $h\colon X\to T$ of v.

Combining Lemma 3.2 and Lemmas 3.4-5, we have the following

PROPOSITION 3.6. *If $\eta\in[X,E]_D$ lies in the image of $q_*\colon[X,T]_D\to[X,E]_D$, then the set $N(\eta)=q_*^{-1}(\eta)$ is equivalent to the factor group of $[X,\Omega_{K(C)}L(C)]_{K(C)}$ by the subgroup of elements which are ρ-correlated to η.*

PROOF OF LEMMA 3.4. Let $g_t\colon X\to T$ be a D-homotopy such that $g_0=h$ and $g_1=k\vee h$ and let $g\colon X\to\Omega_D^*E$ be a D-map given by $g(x)(t)=qg_t(x)$ for any $x\in X$ and $t\in I$. Then $rg(x)=g(x)(1)=qg_1(x)=v(x)$. Hence it is sufficient to show that $i_*([k])=\rho'_*([g])$ in $[X,\Omega_{K(C)}^*L(C)]_{K(C)}$. Let $\tilde\rho\colon T\to P_{K(C)}L(C)$ be the map induced by ρ, which makes the following diagram commutative:

$$\begin{array}{ccc} T & \xrightarrow{\tilde\rho} & P_{K(C)}L(C) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\rho} & L(C)\,. \end{array}$$

Then there is a homotopy $l_s\colon X\to\Omega_{K(C)}^*L(C)$ ($s\in I$) given by

$$l_s(x)(t)=\begin{cases}\tilde\rho g_{1+2st-2s}(x)(t/2) & 0\le 2s\le 1\\ \tilde\rho g_t(x)(2s+t-st-1) & 1<2s\le 2\end{cases}$$

which is a $K(C)$-homotopy between ik and $\rho'g$. q.e.d.

PROOF OF LEMMA 3.5. Let $g\colon X\to\Omega_D^*E$ be a D-map such that $rg\simeq_D v$ and $\rho'g\simeq_{K(C)}ik$. Since $\Omega_D^*E\to E$ is a fibration by Lemma 3.3, we may assume that $rg=v$. Let $\tau\colon\Omega_{K(C)}L(C)\to\Omega_{K(C)}L(C)$ be a $K(C)$-map given by $\tau(\lambda)(t)=\lambda(1-t)$ for all $t\in I$. Let $k'\colon X\to\Omega_{K(C)}L(C)$ be a $K(C)$-map defined by $k'=\tilde\rho h\vee\rho'g\vee\tau(\tilde\rho h)$. Then ik' is $K(C)$-homotopic to $l_0\colon X\to\Omega_{K(C)}^*L(C)$ defined by

$$l_0(x)(t)=\begin{cases}\tilde\rho h(x)(3t) & 0\le 3t\le 1\\ \rho'g(x)(3t-1) & 1\le 3t\le 2\\ \tilde\rho h(x)(3-3t) & 2\le 3t\le 3.\end{cases}$$

Let $l_s\colon X\to\Omega_{K(C)}^*L(C)$ be a $K(C)$-homotopy which is defined by

$$l_s(x)(t)=\begin{cases}l_0(x)(t+s/3) & 0\le 3t\le 1-s\\ l_0(x)((t+s)/(1+2s)) & 1-s\le 3t\le 2+s\\ l_0(x)(t-s/3) & 2+s\le 3t\le 3.\end{cases}$$

Then $l_1(x)(t)=l_0(x)((1+t)/3)=\rho' g(x)(t)$ and so $i_*([k'])=\rho'_*([g])$. Therefore, there follows $ik\simeq_{K(C)} ik'$ because $i_*([k])=\rho'_*([g])$ by the assumption. Let $f_t\colon X\to\Omega^*_{K(C)}L(C)$ be a $K(C)$-homotopy between ik' and ik, and let $f\colon X\to\Omega_{K(C)}L(C)$ be a $K(C)$-map given by $f(x)(t)=f_t(x)(0)$. Then it is easily seen that $k'\vee f\simeq_{K(C)} f\vee k$, i.e., $[k'\vee f]=[f\vee k]$ in $[X,\Omega_{K(C)}L(C)]_{K(C)}$. Because $[X,\Omega_{K(C)}L(C)]_{K(C)}$ is an abelian group by Lemma 1.6, it follows that $[k]=[k']$. Therefore, we have

$$k\vee\tilde{\rho}h\simeq_{K(C)}k'\vee\tilde{\rho}h\simeq_{K(C)}(\rho h\vee\rho' g\vee\tau(\tilde{\rho}h))\vee\tilde{\rho}h\simeq_{K(C)}\tilde{\rho}h\vee\rho' g.$$

Let $w\colon X\to T$ be the map defined by $w(x)=(v(x),(\tilde{\rho}h\vee\rho' g)(x))$. Then w is a lifting of v and w is D-homotopic to $(v,k\vee\tilde{\rho}h)=k\vee h$, i.e., w is v-related to $k\vee h$. On the other hand, let $w_s\colon X\to T$ be a homotopy which is given by

$$w_s(x)=(g(x)(1-s),\ l'_s(x)),$$

$$l'_s(x)(t)=\begin{cases}\tilde{\rho}h(x)(2t/(1+s)) & 0\le 2t\le 1+s\\ \rho' g(x)(2t-1-s) & 1+s\le 2t\le 2.\end{cases}$$

Then w_s is a D-homotopy between w and h. Therefore, w is ι-related to h and so $k\vee h$ is ι-related to h. q. e. d.

§4. Compositions of twisted principal fibrations

Let $p\colon E\to D$ be the twisted principal fibration with fiber $F\,(=\Omega B)$ in the diagram (2.1) and let

$$m_B\colon(\Omega_{K(B)}L(B)\times_{K(B)}E,\ \Omega_{K(B)}L(B)\times_{K(B)}F)\longrightarrow(E,F)$$

be the map of (2.2). Obviously, $\Omega_{K(B)}L(B)\times_{K(B)}F=F\times F$ and $m_B\colon F\times F\to F$ is the ordinary multiplication of $F=\Omega B$. Consider the map

$$m'_B\colon(\Omega^2_{K(B)}L(B)\times_{K(B)}E,\ \Omega^2_{K(B)}L(B)\times_{K(B)}F)\longrightarrow(\Omega^*_D E,\ \Omega^* F),$$

which is given by

$$m'_B(\lambda,x)(t)=m_B(\lambda(t),x)\quad\text{for}\quad \lambda\in\Omega^2_{K(B)}L(B),\ \ x\in E\ \ \text{and}\ \ t\in I.$$

It is easily seen that $\Omega^2_{K(B)}L(B)\times_{K(B)}F=\Omega F\times F$ and $m'_B\colon\Omega F\times F\to\Omega^*F$ coincides with the map defined in [10, Theorem 2.7]. Now, $pr\colon\Omega^*_D E\to D$ is a fibration with fiber Ω^*F by Lemma 3.3 on the one hand and on the other hand $p\pi_2\colon\Omega^2_{K(B)}L(B)\times_{K(B)}E\to D$ (π_2 is the projection to the second factor) is a fibration with fiber $\Omega F\times F$, and m'_B makes the following diagram of fibrations commutative:

$$\begin{array}{ccccc} \Omega F\times F & \xrightarrow{\subset} & \Omega^2_{K(B)}L(B)\times_{K(B)}E & \xrightarrow{p\pi_2} & D \\ \downarrow{\scriptstyle m'_B} & & \downarrow{\scriptstyle m'_B} & & \downarrow{\scriptstyle 1} \\ \Omega^*F & \xrightarrow{\Xi} & \Omega^*_D E & \xrightarrow{pr} & D. \end{array}$$

The map $m'_B\colon \Omega F\times F\to\Omega^*F$ is a weak homotopy equivalence by [10, Theorem 2.7] and so is the map $m'_B\colon \Omega^2_{K(B)}L(B)\times_{K(B)}E\to\Omega^*_D E$, which is seen immediately by using the homotopy exact sequences of fibrations and the five lemma. Therefore the function

$$m'_{B*}\colon [X, \Omega^2_{K(B)}L(B)]_{K(B)}\times[X,E]_D \longrightarrow [X, \Omega^*_D E]_D$$

is a bijection for all CW-complex X, by [11, Theorem 3.2].

The $K(C)$-map ρ_1 in (2.3) induces a $K(C)$-map

$$\rho'_1 : (\Omega^2_{K(B)}L(B)\times_{K(B)}E,\ E) \longrightarrow (\Omega_{K(C)}L(C), K(C)),$$

which is defined by

$$\rho'_1(\lambda, x)(t) = \rho_1(\lambda(t), x).$$

If $v\colon X\to E$ is a D-map and $a, b\colon X\to\Omega^2_{K(B)}L(B)$ are $K(B)$-maps, then the relation

$$\rho'_1(a\vee b, v) = \rho'_1(a, v)\vee\rho'_1(b, v)$$

holds. Therefore the function

(4.1) $$\Delta(\rho, \mathrm{M})\colon\ [X, \Omega^2_{K(B)}L(B)]_{K(B)} - [X, \Omega_{K(C)}L(C)]_{K(C)},$$

defined by

$$\Delta(\rho, [v])([a]) = [\rho'_1(a, v)],$$

is a homomorphism of groups. We consider also a $K(C)$-map

$$n'\colon \Omega_{K(C)}L(C)\times_{K(C)}L(C) \longrightarrow \Omega^*_{K(C)}L(C),$$

defined by the relation

$$n'(\lambda, x)(t) = n(\lambda(t), x) \quad \text{for} \quad \lambda\in\Omega_{K(C)}L(C),\quad x\in L(C) \quad \text{and} \quad t\in I,$$

where $n=\mu_{\phi(C)}\colon L(C)\times_{K(C)}L(C)\to L(C)$ is the induced multiplication of (1.3). Because C is a topological group, the map n' is a $K(C)$-homeomorphism. Therefore the induced function

$$n'_*\colon [X, \Omega_{K(C)}L(C)]_{K(C)}\times[X, L(C)]_{K(C)} \longrightarrow [X, \Omega^*_{K(C)}L(C)]_{K(C)}$$

is a bijection for any space X. By the direct calculations, we obtain

$$n'(\rho'_1, \rho r m_B)\Delta = \rho' m'_B\colon \Omega^2_{K(B)}L(B)\times_{K(B)}E \longrightarrow \Omega^*_{K(C)}L(C),$$

where Δ is the diagonal map. This implies the following lemma.

LEMMA 4.2. *There are the following relations:*

(1) $r_* m'_{B*}(\beta, \eta) = \eta$,

(2) ρ roi $(\beta, \eta) = n'_*(\Delta(\rho, \eta)(\beta), \rho_*\eta)$,

(3) $n'_*(\gamma, [s_C\bar{\rho}u]) = i_*(\gamma)$.

Using the above lemma, we can prove the following

PROPOSITION 4.3. *Under the situation of* (2.1), *the conditions* (i) *and* (ii) *are equivalent.*
(i) *The element* $\eta \in [X, E]_D$ *is contained in the image of* $q_*\colon [X, T]_D \to [XE]_D$ *and* $\gamma \in [X, \Omega_{K(C)}L(C)]_{K(C)}$ *is* ρ*-correlated to* η.
(ii) *The element* η $E[X,E]_D$ *is contained in* $\rho_*^{-1}([s_C\bar{\rho}u])$ *and* γ *lies in the image of* $\Delta(\rho, \eta)\colon [X, \Omega^2_{K(B)}L(B)]_{K(B)} \to [X, \Omega_{K(C)}L(C)]_{K(C)}$.

From Lemma 3.1, Proposition 3.6 and Proposition 4.3, we have the following

THEOREM 4.4. *Under the situation of* (2.1), *the set* $[X, T]_D$ is *equivalent to the disjoint union of* $\mathrm{Coker}\,\Delta(\rho, \eta)$ *of the homomorphism* $\Delta(\rho, \eta)$ *of* (4.1), *as* η *runs through* $\rho_*^{-1}([s_C\bar{\rho}u])$, *where* $\rho_*\colon [X, E]_D \to [X, L(C)]_{K(C)}$.

§ 5. Proof of Theorem A in § 2

Assume that the composition of fibrations $T \xrightarrow{q} E \xrightarrow{p} D$ in the diagram (2.1) is stable by a $K(C)$-map $d\colon (\Omega_{K(B)}L(B)\times_{K(B)}D, D) \to (L(C), K(C))$, i.e., the following diagram is $K(C)$-homotopy commutative:

$$\begin{array}{ccc}
(\Omega_{K(B)}L(B)\times_{K(B)}E,\ E) & \xrightarrow{\rho_1} & (L(C),\ K(C)) \\
\downarrow{\scriptstyle 1\times p} & & \| \\
(\Omega_{K(B)}L(B)\times_{K(B)}D,\ D) & \xrightarrow{d} & (L(C),\ K(C)),
\end{array}$$

where ρ_1 is the map defined in (2.3). Let

$$d'\colon (\Omega^2_{K(B)}L(B)\times_{K(B)}D, D) \longrightarrow (\Omega_{K(C)}L(C), K(C))$$

be the map induced from the map d by $d'(\lambda, x)(t) = d(\lambda(t), x)$. Then the diagram below is $K(C)$-homotopy commutative:

$$\begin{array}{ccc}(\Omega^2_{K(B)}L(B)\times_{K(B)}E,\ E) & \xrightarrow{\rho'_1} & (\Omega_{K(C)}L(C),\ K(C)) \\ \downarrow{\scriptstyle 1\times p} & & \| \\ (\Omega^2_{K(B)}L(B)\times_{K(B)}D,\ D) & \xrightarrow{d'} & (\Omega_{K(C)}L(C),\ K(C)).\end{array}$$

For any map $u: X\to D$, there are two functions

$$\Theta_u: [X, \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X, L(C)]_{K(C)},$$

$$\Theta'_u: [X, \Omega^2_{K(B)}L(B)]_{K(B)} \longrightarrow [X, \Omega_{K(C)}L(C)]_{K(C)},$$

which are defined by

$$\Theta_u([a]) = [d(a, u)], \qquad \Theta'_u([b]) = [d'(b, u)].$$

If $u: X\to D$ has a lifting to E, then the homomorphism Θ'_u is equal to the homomorphism $\Delta(\rho, \eta)$ of (4.1) for any $\eta \in [X, E]_D$ by the definition of $\Delta(\rho, \eta)$ and the above commutative diagram. Therefore

$$\text{Coker}\,\Theta'_u = \text{Coker}\,\Delta(\rho, \eta) \qquad \text{for any} \quad \eta \in [X,E]_D.$$

Let $\eta=[v] \in [X, E]_D$. Then

$$\Theta_u([a]) = [d(a, u)] = [\rho_1(a, v)] = [n(\rho m_B(a,v), \rho m_B(c_{a(0)}, v)^{-1})]$$

by definition. If $v: X\to E$ has a lifting to T, then $[\rho m_B(c_{a(0)}, v)]$ is equal to the unit $[s_C\bar{\rho}u]$. Thus the function

$$\rho_* m_{B*}(\ ,\eta): [X, \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X, E]_D \longrightarrow [X, L(C)]_{K(C)}$$

is equal to Θ_u, if u has a lifting to T. Since $m_{B*}(\ ,\eta)$ is a bijection by Proposition 1.8, we see that $\rho_*^{-1}([s_C\bar{\rho}u])$ is equivalent to $\text{Ker}\,\Theta_u = \Theta_u^{-1}([s_C\bar{\rho}u])$.

The above argument and Theorem 4.4 complete the proof of Theorem A.

REMARK. We see easily that the function Θ_u is also a homomorphism.

Chapter II. Enumeration of cross sections of sphere bundles

§ 6. Some remarks on the cohomology with local coefficients

The non-trivial homomorphism $\phi: Z_2\to \text{Aut}(Z)$, where $\text{Aut}(Z)$ is the group of automorphisms of the infinite cyclic group Z, induces a homomorphism $\phi: Z_2 \to \text{Homeo}(K(Z, n))$ $(n>1)$. As indicated in (1.1), there is a fibration

$$K(Z, n) \xrightarrow{i} L_\phi(Z, n) \xrightarrow{q} K = K(Z_2, 1), \quad L_\phi(Z, n) = L_\phi(K(Z,n)),$$

with a canonical cross section s. A map $u: X\to K$ determines a local system on

X which is given by $\phi u_*: \pi_1(X)\to\pi_1(K)=Z_2\to\mathrm{Aut}(Z)$. We denote the cohomology with coefficients in the above local system by $H^*(X; Z_{u^*\phi})$ *or* $H^*(X; Z)$ simply. Notice that the following results.

PROPOSITION 6.1 [13, §1 and §3]. *There is a unique element* $\lambda\in H^n(L_\phi(Z, n), K; Z_{q^*\phi})$ *such that* $\bar{i}^*\lambda=\iota_n\in H^n(K(Z,n); Z)$, *the fundamental class of* $K(Z, n)$, *where* $\bar{i}: K(Z,n)\to(L_\phi(Z,n), K)$ *is the natural inclusion, and there is a natural isomorphism*

$$\Phi: [X, A; L_\phi(Z, n), K]_K \xrightarrow{\approx} H^n(X,A; Z_{u^*\phi})$$

for any pair of regular cell complex (X, A) *and for any map* $u: X\to K$ *which is defined by*

$$\Phi([a]) = a^*(\lambda).$$

If A *is empty, this is the isomorphism*

$$\Phi: [X, L_\phi(Z, n)]_K \longrightarrow H^n(X; Z_{u^*\phi}), \qquad \Phi([a]) = a^*j^*\lambda,$$

where $j: L_\phi(Z, n)\to(L_\phi(Z, n), K)$ *is the natural inclusion.*

We say that the elements λ and $j^*\lambda$ are the fundamental classes of the fibration $q: L_\phi(Z, n)\to K$ and we denote λ, $j^*\lambda$ and their mod 2 reductions by the same symbol λ, whenever no confusion can arise.

For a map $u: X\to K$, consider the pull back of $q: L_\phi(Z, n)\to K$ by u,

$$\begin{array}{ccccc} K(Z,n) & \xrightarrow{i} & L_\phi(Z,n)\times_K X & \xrightarrow{\pi_1} & L_\phi(Z,n) \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle q} \\ & & X & \xrightarrow{\quad u \quad} & K, \end{array}$$

(π_i is the projection to the i-th factor). Then $i^*\pi_1^*\lambda=\iota_n$ follows immediately from the relation $\bar{i}^*\lambda=\iota_n$. Therefore, we see easily the following

LEMMA 6.2. *Let* $\nu: H^*(K(Z,n); Z_2)\to H^*(L_\phi(Z, n)\times_K X; Z_2)$ *be the homomorphism of* Z_2*-algebras given by* $\nu(Sq^I\iota_n)=Sq^I\lambda_X$, *where* ι_n *is the image of the* mod 2 *reduction of the fundamental class* ι_n *of* $K(Z,n)$ *and* $\lambda_X=\pi_1^*\lambda\in H^n(L_\phi(Z, n)\times_K X; Z_2)$. *Then*

$$\nu\otimes\pi_2^*: H^*(K(Z,n); Z_2)\otimes H^*(X;Z_2) \longrightarrow H^*(L_\phi(Z,n)\times_K X;Z_2)$$

is an isomorphism of Z_2*-algebras and so any element* x *in* $H^*(L_\phi(Z, n)\times_K X; Z_2)$ *is described uniquely in the form*

$$x = \sum_i Sq^{I_i}\lambda_X\pi_2^*a_i, \qquad a_i\in H(X;Z_2).$$

§ 7. The third stage **Postnikov** factorization of $BO(n-1)\rightarrow BO(n)$

Let $p\colon BO(n-1)\rightarrow BO(n)$ be the universal S^{n-1}-bundle ($n\geq 4$). Our purpose in this section is the construction of the third stage Postnikov factorization of this bundle using the methods of J. F. McClendon [13] and E. Thomas [19].

Let $\phi\colon \pi_1(BO(n))=Z_2\rightarrow \mathrm{Aut}(\pi_{n-1}(S^{n-1}))=\mathrm{Aut}(Z)$ be the local system on $BO(n)$ associated with $p\colon BO(n-1)\rightarrow BO(n)$, and let s_{n-1} be the generator of $H^{n-1}(S^{n-1};Z)=Z$. Then, by [13, Theorem 4.1 and §§ 2–3], there is a map $W\colon BO(n)\rightarrow L_\phi(Z,n)$ such that $[W]\in[BO(n),L_\phi(Z,n)]_K=H^n(BO(n);\underline{Z})$ is the transgression image of s_{n-1}, and we have a commutative diagram

$$\begin{array}{ccccc}
S^{n-1} & \longrightarrow & BO(n-1) & & \\
\downarrow{\scriptstyle s_{n-1}} & & \downarrow{\scriptstyle q_1} & & \\
\Omega K(Z,n) & \longrightarrow & E & \longrightarrow & P_K L_\phi(Z,n) \\
 & & \downarrow{\scriptstyle p_1} & & \downarrow \\
 & & BO(n) & \xrightarrow{W} & L_\phi(Z,n) \xrightarrow{q} K,
\end{array}$$

where $p_1q_1=p$ and p_1 is the twisted principal fibration induced by W. By using the homotopy exact sequences of fibrations, we see easily that both maps s_{n-1} and q_1 are homotopically equivalent to the fibrations $F\xrightarrow{\subset}S^{n-1}\xrightarrow{s_{n-1}}\Omega K(Z,n)$ and $F\xrightarrow{\subset}BO(n-1)\xrightarrow{q_1}E$ (cf. [19, § 1]) and

$$\pi_i(F)=\begin{cases} 0 & \text{for } i\leq n-1 \\ \pi_i(S^{n-1}) & \text{for } i>n. \end{cases}$$

Therefore $q_1\colon BO(n-1)\rightarrow E$ is an n-equivalence.*) Since the generator of $H^n(F;Z_2)=Z_2$ is transgressive for the fibration $q_1\colon BO(n-1)\rightarrow E$, its transgression image is a non-zero element p in $H^{n+1}(E;Z_2)$ and there is a commutative diagram

$$\begin{array}{ccc}
F & \longrightarrow & BO(n-1) \\
\downarrow & & \downarrow{\scriptstyle q_2} \\
K(Z_2,n) & \longrightarrow & \Gamma \\
 & & \downarrow{\scriptstyle p_2} \\
 & & E \xrightarrow{p} K(Z_2,n+1).
\end{array}$$

Here $p_2q_2=q_1$, p_2 is the principal fibration with the classifying map p and it is easily seen that q_2 is an $(n+1)$-equivalence and $q_2|F$ represents the generator of

*) A map $g\colon X\rightarrow Y$ (X, Y are connected) is called an n-equivalence if $g_*\colon \pi_i(X)\rightarrow\pi_i(Y)$ is isomorphic for $i<n$ and epimorphic for $i=n$.

$H^n(F; Z_2)$.

In the rest of this section, we concentrate ourselves on the characterization of the map $\rho: E \to K(Z_2, n+1)$. Let

$$m: \Omega_K L_\phi(Z,n) \times_K E \longrightarrow E$$

be the action defined in (1.7) and set

(7.1) $\mu = m(1 \times q_1): \Omega_K L_\phi(Z, n) \times_K BO(n-1) \longrightarrow E.$

The map μ makes the following diagram commutative:

$$\begin{array}{ccc} \Omega_K L_\phi(Z,n) \times_K BO(n-1) & \xrightarrow{\mu} & E \\ \downarrow{\pi_2} & & \downarrow{p_1} \\ BO(n-1) & \xrightarrow{p} & BO(n). \end{array}$$

The projection π_2 to the second factor admits a cross section s defined by s(x) $=(c_{qWp(x)}, x)$, where c_y is the constant loop at y, and the relation

(7-2). $$\mu s \simeq_{BO(n)} q_1$$

holds obviously. The local system $\pi_1(BO(n)) = Z_2 \to \mathrm{Aut}(H^i(K(Z, n-1); Z_2))$ on $BO(n)$, which is associated with $p_1: E \to BO(n)$, is trivial for $i = n-1$ and hence so for all i. Also $H^i(K(Z, n-1); Z_2) = 0$ for $0 < i < n-1$ and $H^i(BO(n), BO(n-1); Z_2) = 0$ for $i < n$. Therefore, by the similar proof to [19, Property 4], we see that the sequence

$$\cdots \longrightarrow H^i(\Omega_K L_\phi(Z, n) \times_K BO(n-1); Z_2) \xrightarrow{\tau_0} H^{i+1}(BO(n), BO(n-1); Z_2)$$
$$\xrightarrow{p^* j^*} H^{i+1}(E; Z_2) \xrightarrow{\mu^*} H^{i+1}(\Omega_K L_\phi(Zn) \times_K BO(n-1); Z_2) \longrightarrow$$
$$\cdots \longrightarrow H^{2n-2}(E; Z_2)$$

is exact, where $j: BO(n) \to (BO(n), BO(n-1))$ is the natural inclusion, and τ_0 is the relative transgression. On the other hand, $p^*: H^i(BO(n); Z_2) \to H^i(BO(n-1); Z_2)$ is epimorphic for all i. Also $\mathrm{Ker}\, p^*$ is the ideal generated by the universal n-th Stiefel- Whitney class w_n. Since w_n is the transgression image of s_{n-1} of $p: BO(n-1) \to BO(n)$, we have $w_n = \tau(\iota_{n-1}) \in \mathrm{Ker}\, p_1^*$, where τ is the transgression of $K(Z, n-1) \xrightarrow{\subset} E \xrightarrow{p_1} BO(n)$. Thus we see that $\mathrm{Ker}\, p^* = \mathrm{Ker}\, p_1^*$. Therefore, the same argument as in [19, Property 5] provides the exact sequence

(7.3) $$0 \longrightarrow H^t(E; Z_2) \xrightarrow{\mu^*} H^t(\Omega_K L_\phi(Z,n) \times_K BO(n-1); Z_2)$$
$$\xrightarrow{\tau_1} H^{t+1}(BO(n); Z_2)$$

for $t<2n-2$, where $\tau_1=j^*\tau_0$. (7.2) and (7.3) imply that

(7.4) $\mu^*:\ \mathrm{Ker}\ q_1^* \longrightarrow \mathrm{Ker}\ s^* \cap \mathrm{Ker}\ \tau_1$

is isomorphic in dimension less than $2n-2$.

By considering $\Omega_K L_\phi(Z,n)=L_\phi(Z,n-1)$ by the natural K-homeomorphism ψ of Lemma 1.6, there is an element $\lambda_{BO(n-1)}$ in $H^{n-1}(\Omega_K L_\phi(Z,n)\times_K BO(n-1); Z_2)$ by Lemma 6.2 for the fibration $\Omega_K L_\phi(Z,n)\times_K BO(n-1)\to BO(n-1)$ such that $i^*\lambda_{BO(n-1)}=\iota_{n-1}$, the mod 2 reduction of the fundamental class of $K(Z,n-1)$. Here the diagram

$$\begin{array}{ccccc} \Omega K(Z,n) & \xrightarrow{i} & \Omega_K L_\phi(Z,n)\times_K BO(n-1) & \xrightarrow{\pi_2} & BO(n-1) \\ \Big\| & & \downarrow & & \downarrow p \\ \Omega K(Z,n) & \longrightarrow & E & \xrightarrow{p_1} & BO(n) \end{array}$$

implies that $\tau_1(\lambda_{BO(n-1)})=j^*\tau_0(\lambda_{BO(n-1)})=\tau i^*(\lambda_{BO(n-1)})=\tau(\iota_n$ Any element x in $H^{n+1}(\Omega_K L_\phi(Z,n)\times_K BO(n-1);Z_2)$ is described in the form

$$x=\pi_2^*b+\varepsilon_1\lambda_{BO(n-1)}\pi_2^*w_1^2+\varepsilon_2\lambda_{BO(n-1)}\pi_2^*w_2+\varepsilon_3 Sq^2\lambda_{BO(n-1)},$$

where $\varepsilon_i=0$ or 1 for $i=1, 2, 3$ by Lemma 6.2. If $x\in \mathrm{Ker}\,s^* \cap \mathrm{Ker}\,\tau_1$, then $0=s^*x=b$. Because τ_1 is an $H^*(BO(n);Z_2)$-homomorphism and $\tau_1 Sq^i=Sq^i\tau_1$ by [19, § 3], it follows that

$$\tau_1(\lambda_{BO(n-1)}\pi_2^*w_1^2)=w_nw_1^2,\qquad \tau_1(\lambda_{BO(n-1)}\pi_2^*w_2)=w_nw_2,$$

$$\tau_1(Sq^2\lambda_{BO(n-1)})=Sq^2w_n=w_nw_2.$$

Hence $\mathrm{Ker}\,s^*\cap\mathrm{Ker}\,\tau_1=Z_2$ generated by $\lambda_{BO(n-1)}\pi_2^*w_2+Sq^2\lambda_{BO(n-1)}$ and so the map ρ: $E\to K(Z_2, n+1)$ is characterized by the relation

(7.5) $$\mu^*\rho=\lambda_{BO(n-1)}\pi_2^*w_2+Sq^2\lambda_{BO(n-1)}.$$

Summing up the above arguments, we have

THEOREM 7.6. *The third stage Postnikov factorization of* p: $BO(n-1)\to BO(n)$ *is given as follows:*

(7.7) $$\begin{array}{ccccc} BO(n-1) & \xrightarrow{q_2} & T & & \\ & \searrow^{q_1} & \downarrow p_2 & & \\ & {}_{p}\searrow & E & \xrightarrow{\rho} & K(Z_2, n+1) \\ & & \downarrow p_1 & & \\ & & BO(n) & \xrightarrow{w} & L_\phi(Z,n), \end{array}$$

where $\phi: \pi_1(K(Z_2,1)) = Z_2 \to \mathrm{Aut}(Z)$ *is the non-trivial local system on* $K(Z_2, 1)$, $p_1 : E \to BO(n)$ *is the twisted principal fibration induced by the map* $W, p_2: T \to E$ *is the principal fibration with classifying map* p, $q_1: BO(n-1) \to E$ *is an* n-*equivalence*, $q_2: BO(n-1) \to T$ *is an* $(n+1)$-*equivalence and the map* p *is characterized by the relation* (7.5).

§8. The stability of the third stage Postnikov factorization of p: $BO(n-1) \to BO(n)$

There is a map

(8.1) d: $(\Omega_K L_\phi(Z, n) \times_K BO(n), BO(n)) \longrightarrow (K(Z_2, n+1), *)$,

which represents the element $\lambda_{BO(n)}\pi_2^* w_2 + Sq^2\lambda_{BO(n)}$ in $H^{n+1}(\Omega_K L_\phi(Z,n) \times_K BO(n), BO(n); Z_2)$, i.e., $d^*(\iota) = \lambda_{BO(n)}\pi_2^* w_2 + Sq^2\lambda_{BO(n)}$, where ι is the fundamental class of $K(Z_2, n+1)$. The relation

$$(8.2) \qquad (1 \times p_1)^* d^*(\iota) = \lambda_E \pi_2^* p_1^* w_2 + Sq^2 \lambda_E \in H^{n+1}(\Omega_K L_\phi(Zn) \times_K E, E; Z_2)$$

follows easily. Let

$$\rho_1: (\Omega_K L_\phi(Z,n) \times_K E, E) \longrightarrow (K(Z_2, n+1), *)$$

be the map given by the relation $\rho_1(k, y) = \rho m(k, y) \cdot [\rho m(c_{k(0)}, y)]^{-1}$ (cf. (2.3)). Then the following relation holds:

$$(8.3) \qquad \rho_1^*(\iota) = m^*\rho^*(\iota) - \pi_2^*\rho^*(\iota) \in H^{n+1}(\Omega_K L_\phi(Zn) \times_K E, E; Z_2).$$

To see that the composition of fibrations $T \xrightarrow{p_2} E \xrightarrow{p_1} BO(n)$ in the diagram (7.7) is stable by the map d in the sense of § 2, it is sufficient to show that

$$(8.4) \qquad (m^* - \pi_2^*)\rho^*(\iota) = \quad {}_{E}\,{}_{2}\,{}_{1}\,{}_{2} \qquad {}_{E},$$

by (8.2) and (8.3). Now, consider the map μ of (7.1). Then the diagram

$$\begin{array}{ccc} H^{n+1}(E; Z_2) & \xrightarrow{m^* - \pi_2^*} & H^{n+1}(\Omega_K L_\phi(Z,n) \times_K E; Z_2) \\ \cup & & \downarrow (1 \times q_1)^* \\ \mathrm{Ker}\, q_1^* & \xrightarrow{\mu^*} & H^{n+1}(\Omega_K L_\phi(Z, n) \times_K BO(n-1) Z_2) \end{array}$$

is commutative because $(1 \times q_1)^*(m^* - \pi_2^*)(x) = (1 \times q_1)^* m^*(x) - (1 \times q_1)^* \pi_2^*(x) = \mu^*(x)$ for any x in $\mathrm{Ker}\, q_1^*$. Therefore we have

$$\begin{aligned} (1 \times q_1)^*(m^* - \pi_2^*)\rho^*(\iota) &= \mu^*\rho^*(\iota) \quad \text{by} \quad \rho^*(\iota) \in \mathrm{Ker}\, q_1^* \\ &= \lambda_{BO(n-1)}\pi_2^* p^* w_2 + Sq^2\lambda_{BO(n-1)} \text{ by (7.5)} \end{aligned}$$

$$= (1 \times q_1)^*(\lambda_E \pi_2^* p_1^* w_2 + Sq^2 \lambda_E).$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
H^{n+1}(\Omega_K L_\phi(Z,n) \times_K E;\ Z_2) & \xleftarrow{\nu \otimes \pi_2^*} & \sum_{i=0}^{2} H^{n-i}(K(Z,n-1);\ Z_2) \otimes H^i(E;Z_2) \\
\downarrow{\scriptstyle (1 \times q_1)^*} & & \downarrow{\scriptstyle 1 \otimes q_1^*} \\
H^{n+1}(\Omega_K L_\phi(Z,n) \times_K BO(n-1);Z_2) & \xleftarrow{\nu \otimes \pi_2^*} & \sum_{i=0}^{2} H^{n\ i}(K(Z,n-1);\ Z_2) \otimes \\
 & & H^i(BO(n-1);\ Z_2).
\end{array}$$

The horizontal maps are monomorphisms by Lemma 6.2. Further $q_1^*\colon H^i(E;Z_2) \to H^i(BO(n-1); Z_2)$ is monomorphic for $i<2$ because q_1 is an n-equivalence, and so the vertical map in the right hand side is a monomorphism. This result and the above equality imply (8.4), and we have the following

PROPOSITION 8.5. *The composition of the fibrations* $T \xrightarrow{p_2} E \xrightarrow{p_1} BO(n)$ *in the diagram* (7.7) *is stable by the map d in* (8.1).

§ 9. Enumeration of cross sections of sphere bundles

Let ξ be a real n-plane bundle over a CW-complex X. If ξ has a non-zero cross section, *cross* (ξ) denotes the set of (free) homotopy classes of non-zero cross sections of ξ. The space X is a $BO(n)$-space with the classifying map $\xi\colon X \to BO(n)$ of ξ. Then the relation

$$cross\,(\xi) = [X, BO(n-1)]_{BO(n)}$$

follows from [11, Lemma 2.2]. If the dimension of X is less than $n+1$ and $n>4$, then

$$[X,\ BO(n-1)]_{BO(n)} = [X,\ T]_{BO(n)}$$

follows from [11, Theorem 3.2], because $q_2\colon BO(n-1) \to T$ is an $(n+1)$-equivalence. On the other hand, it follows from Theorem A of § 2 that

$$[X,\ T]_{BO(n)} = \operatorname{Ker} \Theta_\xi \times \operatorname{Coker} \Theta'_\xi.$$

Here

$$\Theta_\xi\colon [X, \Omega_K L_\phi(Z,n)]_K \longrightarrow [X, K(Z_2,\ n+1)] = H^{n+1}(X\ Z_2) = 0,$$

$$\Theta'_\xi\colon [X, \Omega_K^2 L_\phi(Z,n)]_K \longrightarrow [X, \Omega K(Z_2, n+1)] = \text{fl«}(X; Z_2),$$

and $\Theta'_\xi([a]) = [d'(a,\xi)]$, where $d'\colon (\Omega_K^2 L_\phi(Z,n) \times_K BO(n), BO(n)) \to (\Omega K(Z_2, n+1), *)$ is the map given by $d'(a, x)(t) = d(a(t), x)$ (cf. (2.4)). Also,

$$[X, \Omega_K L_\phi(Z, rij]_x = H^{n-1}(X; Z), \qquad [X, \Omega_K^2 L_\phi(Z, n)]_K = H^{n-2}(X; Z)$$

by Proposition 6.1, where Z is the local system on X associated with ξ given by the composition

$$\pi_1(X) \xrightarrow{\xi_*} \pi_1(BO(n)) \xrightarrow{q_* W_*} \pi_1(K) = Z_2 \xrightarrow{\phi} \text{Aut}(Z), \qquad (K = K(Z_2, 1)).$$

Now, we show that the homomorphism $\Theta'_\xi: H^{n-2}(X; Z) \to H^n(X; Z_2)$ is given by

$$\Theta'_\xi(a) = (\rho_2 a) w_2(\xi) + Sq^2 \rho_2 a, \qquad \text{for any,} \quad a \in H^{n-2}(X; Z), \tag{9.1}$$

where ρ_2 is the mod 2 reduction and $w_2(\xi)$ is the second Stiefel-Whitney class of ξ

Let $\iota' \in H^n(K(Z_2, n); Z_2)$ be the fundamental class of $K(Z_2, n)$. Then

$$\Theta'_\xi([a]) = (a, \xi)^* d'^*(\iota') \tag{9.2}$$

for any K-map $\alpha: X \to \Omega_K^2 L_\phi(Z, n)$. Consider the two commutative diagrams of the mod 2 cohomology groups

$$\begin{array}{ccccc}
H^i(K', *) & \xrightarrow[\sim]{r^*} & H^i(PK', \Omega K') & \xleftarrow[\sim]{\delta} & H^{i-1}(\Omega K', *) \\
\downarrow d^* & & \downarrow d'^* & & \downarrow d'^* \\
H^i(\Omega' \times_K B, B) & \xrightarrow{r^*} & H^i(P_K \Omega' \times_K B, \Omega_K \Omega' \times_K B) & \xleftarrow[\approx]{\delta} & H^{i-1}(\Omega_K \Omega' \times_K B, B),
\end{array}$$

$$\begin{array}{ccccc}
H^{n-1}(\Omega', K) & \xrightarrow{r^*} & H^{n-1}(P_K \Omega', \Omega_K \Omega') & \xleftarrow[\sim]{\delta} & H^{n-2}(\Omega_K \Omega', K) \\
\downarrow \bar{\imath}^* & & \downarrow \bar{\imath}^* & & \downarrow \bar{\imath}^* \\
H^{n-1}(K'') & \xrightarrow[\approx]{r^*} & H^{n-1}(PK'', \Omega K'') & \xleftarrow[\approx]{\delta} & H^{n-2}(\Omega K''),
\end{array}$$

where $K' = K(Z_2, n+1)$, $\Omega' = \Omega_K L_\phi(Z, n)$, $B = BO(n)$, $K'' = \Omega K(Z, n)$ and d': $P_K \Omega' \times_K B \to PK'$ is the map defined by the same equation $d'(b, x)(t) = d(b(t), x)$ as (2.4). Since $\delta^{-1} r^*(\iota_{n-1}) = \iota_{n-2}$, we have

$$\delta^{-1} r^* \lambda = \lambda', \qquad \delta^{-1} r^* \lambda_B = \lambda'_B,$$

where $\lambda \in H^{n-1}(\Omega', K)$ and $\lambda' \in H^{n-2}(\Omega_K \Omega', K)$ are the fundamental classes of the fibrations $\Omega' \to K$ and $\Omega_K \Omega' \to K$ of Proposition 6.1 and $\lambda_B = \pi_1^* \lambda \in H^{n-1}(\Omega' \times_K B, B)$, $\lambda'_B = \pi_1^* \lambda' \in H^{n-2}(\Omega_K \Omega' \times_K B, B)$. Therefore, by the equation $d^*(\iota) = \lambda_B \pi_2^* w_2 + Sq^2 \lambda_B$ by (8.1) and $\delta^{-1} r^*(\iota) = \iota'$, we have $d'^*(\iota') = \delta^{-1} r^* d^*(\iota) = \lambda'_B \pi_2^* w_2 + Sq^2 \lambda'_B = (\pi_1^* \lambda')(\pi_2^* w_2) + Sq^2 \pi_1^* \lambda'$. This equality and (9.2) yield

$$\begin{aligned}
\Theta'_\xi([a]) &= (a, \xi)^*((\pi_1^* \lambda') \cdot (\pi_2^* w_2) + Sq^2 \pi_1^* \lambda') \\
&= (a^* \lambda')(\xi^* w_2) + Sq^2 a^* \lambda'.
\end{aligned}$$

Therefore, the homomorphism $\Theta'_\xi: H^{n-2}(X;\underline{Z})\to H^n(X;Z_2)$ is given by

$$\Theta'_\xi(a) = (\rho_2 a)w_2(\xi) + Sq^2\rho_2 a$$

by Proposition 6.1, where $w_2(\xi)$ is the second Stiefel-Whitney class of ξ and ρ_2 is the mod 2 reduction.

From the consideration made above, we obtain the following

THEOREM B. *Let ξ be a real n-plane bundle over a CW-complex X of dimension less than $n+1$ and let $n>4$. If ξ admits a non-zero cross section, then the set* cross (ξ) *of homotopy classes of non-zero cross sections of ξ is, as a set, given by*

$$\text{cross}(\xi) = H^{n-1}(X \quad Z) \times \text{Coker}\,\Theta,$$

where Θ: $H^{n-2}(X;\underline{Z})\to H^n(X;Z_2)$ is defined by

$$\Theta(a) = (\rho_2 a)w_2(\xi) + Sq^2\rho_2 a, \qquad \textit{for} \quad a\in H^{n-2}(X;Z),$$

ρ_2 is the mod 2 *reduction and Z is the local system on X associated with ξ.*

Chapter III. Enumeration of embeddings

§ 10. Enumeration of embeddings of manifolds

Let M be an n-dimensional differentiable closed manifold. Let M^* be the reduced symmetric product of M obtained from $M\times M-\Delta$ (Δ is the diagonal of M) by identifying (x, y) and (y, x) and let η be the real line bundle over M^* associated with the double covering $M\times M-\Delta\to M^*$. Then the set $[M\subset R^{2n-1}]$ of isotopy classes of embeddings of M into the real $(2n-1)$-space R^{2n-1} for $n\geq 6$ is equivalent to the set of homotopy classes of cross sections of the S^{2n-2}-bundle $(M\times M-\Delta)\times_{Z_2} S^{2n-2}\to M^*$ by the theorem of A. Haefliger [5, § 1]. Because this bundle is the associated S^{2n-2}-bundle of $(2n-1)\eta$, we have

$$[M\subset R^{2n-1}] = \text{cross}((2n-1)\eta).$$

Since M^* is an open $2n$-dimensional manifold, there is a proper Morse function on M^* with no critical point of index $2n$ by [15, Lemma 1.1], and so M^* has the homotopy type of a CW-complex of dimension less than $2n$ by [14, Theorem 3.5]. Therefore we have the following proposition from Theorem B of §9 and the fact

$$w_2((2n-1)\eta) = \binom{2n-1}{2} w_1(\eta)^2.$$

PROPOSITION 10.1. *Let $n>6$ and let M be an n-dimensional differentiable closed manifold which is embedded in R^{2n-1}. Then the set $[M\subset R^{2n-1}]$ of isotopy classes of embeddings of M into R^{2n-1} is, as a set, given by*

$$[M\subset R^{2n-1}]=H^{2n-2}(M^*;\underline{Z})\times \operatorname{Coker}\Theta,$$

where the homomorphism

$$\Theta: H^{2n-3}(M^*;\underline{Z})\longrightarrow H^{2n-1}(M^*;Z_2)$$

is given by

$$\Theta(a)=\binom{2n-1}{2}w_1(\eta)^2\rho_2 a+Sq^2\rho_2 a,$$

$w_1(\eta)$ is the first Stiefel-Whitney class of the double covering $M\times M-\Delta\rightarrow M^*$ and $\underline{Z}$ is the local system on M^* defined from this double covering.

COROLLARY 10.2. *In addition to the conditions of the above proposition, we assume that $H_1(M;Z_2)=0$. Then we have*

$$[M\subset R^{2n-1}]=H^{2n-2}(M^*;\underline{Z}).$$

PROOF. Because $H_1(M;Z_2)=0$, we have $H_1(M\times M,\Delta;Z_2)=0$ by the exact sequence of the pair $(M\times M,\Delta)$. The Thom-Gysin exact sequence

$$\longrightarrow H^{2n-1}(M\times M-\Delta;Z_2)\longrightarrow H^{2n-1}(M^*;Z_2)\longrightarrow H^{2n}(M^*;Z_2)\ (=0)$$

and the Poincaré duality $H^{2n-1}(M\times M-\Delta;Z_2)=H_1(M\times M,\Delta;Z_2)\ (=0)$ yield $H^{2n-1}(M^*;Z_2)=0$, which implies that $\operatorname{Coker}\Theta=0$.

REMARK. There is a description in [6, 1.3, e, Théorème] that

$$[M\subset R^{2n-1}]=H^{2n-2}(M^*;\underline{Z})=\begin{cases}H^{n-2}(M;Z) & \text{if } n-1 \text{ is odd}\\ H^{n-2}(M;Z_2) & \text{if } n-1 \text{ is even,}\end{cases}$$

under the assumption $H_1(M;Z)=0$.

§ 11. Enumeration of embeddings of real projective spaces RP^n

Our purpose in this section is to prove the following

THEOREM C. *Let $n\neq 2^r$ and let $n>6$. Then the n-dimensional real projective space RP^n is embedded into the real $(2n-1)$-space R^{2n-1}. Furthermore, the cardinality $\#[RP^n\subset R^{2n-1}]$ of the set $[RP^n\subset R^{2n-1}]$ of isotopy classes of embeddings of RP^n into R^{2n-1} is given by*

$$\#[RP^n \subset R^{2n-1}] = \begin{cases} 4 & n \equiv 3(4) \\ 2 & \textit{otherwise.} \end{cases}$$

The first half of this theorem is shown in [1, Theorem 1] for even n and in [9, Theorem 1.1] for odd n. Thus we concentrate ourselves on the study of the set $[RP^n \subset R^{2n-1}]$. Let η be the real line bundle associated with the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$. Then the set $[RP^n \subset R^{2n-1}]$ is equivalent to the set *cross* $((2n-1)\eta)$(cf. § 10).

In [8, (2.5–6)],

(11.1) *there is a commutative diagram of the double coverings*

$$\begin{array}{ccccc} V_{n+1,2}/(Z_2+Z_2) & = & Z_{n+1,2} & \xrightarrow{f'} & RP^n \times RP^n - \Delta \\ & & \downarrow & & \downarrow \\ V_{n+1,2}/D_4 & = & SZ_{n+1,2} & \xrightarrow{f} & (RP^n)^*, \end{array}$$

where $V_{n+1,2}$ is the Stiefel manifold of 2-*frames in* R^{n+1}, D_4 *is the dihedral group of order* 8, *both maps f and f' are homotopy equivalences and both spaces $Z_{n+1,2}$ and $SZ_{n+1,2}$ are* ($2n$ — 1)-*dimensional manifolds.*

The mod 2 cohomology of $(RP^n)^*$(and so $SZ_{n+1,2}$) is calculated by S. Feder [2], [3] and D. Handel [8] and is given as follows:

(11.2) *Let $G_{n+1,2}$ be the Grassmann manifold of* 2-*planes in the real* $(n+1)$-*space* R^{n+1}. *Then the* mod 2 *cohomology of $G_{n+1,2}$ is given by*

$$H^*(G_{n+1,2}; Z_2) = Z_2[x, y]/(a_n, a_{n+1}),$$

where $\deg x = 1$, $\deg y = 2$ *and* $a_r = \sum_i \binom{r-i}{i} x^{r-2i} y^i$ ($r = n, n+1$), *and there is a relation*

$$x^{2i} y^{n-i-1} \neq 0 \quad \textit{if and only if} \quad i = 2^t - 1 \quad \textit{for some} \quad t.$$

$H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as a basis of an $H^*(G_{n+1,2}; Z_2)$-module, where $v \in H^1((RP^n)^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ and there are the relations

$$v^2 = vx, \ Sq^1 y = xy \quad \textit{and} \quad x^{2^{r+1}-1} = 0 \quad \textit{for} \quad n = 2^r + s, \ 0 < s < 2^r.$$

By the Poincaré duality and (11.1–2),

(11.3) *$H^t((RP^n)^*; Z_2)$ $(n = 2^r + s, \ 0 < s < 2^r)$ for $2n-3 \leq t \leq 2n-1$ are given as follows* [20], [21]:

t	$H^t((RP^n)^*; Z_2)$	basis
$2n-1$	Z_2	$vx^{2^{r+1}-2}y^s$
$2n-2$	Z_2+Z_2	$vx^{2^{r+1}-3}y^s,\ x^{2^{r+1}-2}y^s$
$2n-3$	$Z_2+Z_2+Z_2$	$vx^{2^{r+1}-4}y^s,\ x^{2^{r+1}-3}y^s,\ vx^{2^{r+1}-2}y^{s-1}$

To apply Proposition 10.1, we must study the cohomology groups $H^i((RP^n)^*$ $\underline{Z})$ $(i=2n-2, 2n-3)$ with coefficients in the local system associated with the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$.

Let ρ_2: $H^i((RP^n)^*;\underline{Z}) \to H^i((RP^n)^*;Z_2)$ be the mod 2 reduction.

LEMMA 11.4. *Let* $n=0(2)$. *Then* $H^{2n-2}((RP^n)^*;Z)=Z_2$ *and* $\rho_2 H^{2n-3}((RP^n)^*;\underline{Z})=Z_2+Z_2$ *generated by* $\{vx^{2^{r+1}-4}y^s, vx^{2^{r+1}-2}y^{s-1}\}$.

LEMMA 11.5. *Let* $n=1(2)$. *Then* $H^{2n-2}((RP^n)^*;\underline{Z})=Z_2$ *and* $\rho_2 H^{2n-3}((RP^n)^*;\underline{Z})=Z_2+Z_2$ *generated by* $\{vx^{2^{r+1}-4}y^s+x^{2^{r+1}-3}y^s,\ vx^{2^{r+1}-2}y^{s-1}\}$.

The proofs of Lemmas 11.4–5 will be made in the next section and we go on proving Theorem C. By Proposition 10.1,

$$[RP^n \subset R^{2n-1}] = H^{2n-2}((RP^n)^*;\underline{Z}) \times \mathrm{Coker}\,\Theta,$$

where

$$\Theta: H^{2n-3}((RP^n)^*;\ \underline{Z}) \longrightarrow H^{2n-1}((RP^n)^*;\ Z_2),\ \Theta(a) = Sq^2\rho_2 a + \binom{2n-1}{2} v^2 \rho_2 a.$$

Now, there are relations

$$Sq^2(vx^{2^{r+1}-2}y^{s-1}) = (s-1)vx^{2^{r+1}-2}y^s,$$

$$Sq^2(vx^{2^{r+1}-4}y^s) = \left(s+\binom{s}{2}\right)vx^{2^{r+1}-2}y^s,$$

$$Sq^2(x^{2^{r+1}-2}y^s) = 0,$$

which are easily seen by using (11.2) and the fact $Sq^2(y^t)=ty^{t+1}+\binom{t}{2}x^2y^t$. Therefore we have

$$\left(Sq^2+\binom{2n-1}{2}v^2\right)(vx^{2^{r+1}-2}y^{s-1}) = \begin{cases} vx^{2^{r+1}-2}y^s & n\equiv 0(2) \\ 0 & n\equiv 1(2), \end{cases}$$

$$\left(Sq^2+\binom{2n-1}{2}v^2\right)(vx^{2^{r+1}-4}y^s+x^{2^{r+1}-3}y^s) = \begin{cases} vx^{2^{r+1}-2}y^s & n\equiv 1(4) \\ 0 & n\equiv 3(4). \end{cases}$$

From Lemmas 11.4–5 and (11.3), these relations show that

$$\operatorname{Coker}\Theta = \begin{cases} Z_2 & n = 3(4) \\ 0 & \text{elsewhere.} \end{cases}$$

Since $H^{2n-2}((RP^n)^*;Z) = Z_2$ by Lemmas 11.4–5, we have Theorem C.

§ 12. Proofs of Lemmas 11.4-5

There are two exact sequences of cohomology groups associated with the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ (cf. [17, pp. 282–283]), which is called the Thom-Gysin exact sequence:

(12.1) $$\cdots \to H^{i-1}(M^*;\, Z) \to H^i(M^*;\, Z) \to H^i(M \times M - \Delta;\, Z) \to H^i(M^*;\, Z) \to \cdots,$$

$$\cdots \to H^{i-1}(M^*;\, Z) \to H^i(M^*;\, Z) \to H^i(M \times M - \Delta;\, Z) \to H^i(M^*;\, Z) \to \cdots,$$

where $M = RP^n$. Moreover, there is the Bockstein exact sequence [18]

(12.2) ---- > tf'-KM*; $Z_2) \xrightarrow{\tilde\beta_2} H^i(M^*;\, Z) \xrightarrow{\times 2} H^i(M^*;\, Z)$

$$\xrightarrow{\rho_2} H^i(M^*;\, Z_2) \xrightarrow{\tilde\beta_2} \cdots, \qquad (M = RP^n),$$

associated with the short exact sequence $0 \longrightarrow Z \xrightarrow{\times 2} Z \xrightarrow{\rho_2} Z_2 \longrightarrow 0$. The homomorphism $\tilde\beta_2$ is called the twisted Bockstein operator, and by [4] and [16], the homomorphism $\rho_2\tilde\beta_2\colon H^{i-1}((RP^n)^*;\, Z_2) \to H^i((RP^n)^*;\, Z_2)$ is given by

(12.3) $$\rho_2\tilde\beta_2(a) = Sq^1 a + va \qquad \text{for} \quad a \in H^{i-1}((RP^n)^*;Z_2),$$

where v is the first Stiefel-Whitneyclass of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$.

From now on, set $n = 2^r + s$, $0 < s < 2^r$.

PROOF OF LEMMA 11.4. Since n is even, the space $SZ_{n+1,2}$ is an orientable $(2n-1)$-dimensional manifold by [2, § 3] and so it follows that

$$H^{2n-1}(SZ_{n+1,2};\, Z) = Z,$$

$$H^{2n-2}(SZ_{n+1,2};\, Z) = H_1(SZ_{n+1,2}Z) = D_4/[D_4,\, D_4] = Z_2 + Z_2.$$

Since the total space $Z_{n+1,2}$ is also orientable and $\pi_1(Z_{n+1,2}) = Z_2 + Z_2$, the following relations hold:

$$H^{2n-1}(Z_{n+1,2};\, Z) = Z, \quad H^{2n-2}(Z_{n+1,2};\, Z) = Z_2 + Z_2.$$

Hence (11.1) and the Thom-Gysin exact sequence (12.1) give rise to the two exact sequences

$$Z_2+Z_2\to H^{2n-1}((RP^n)^*;\ \underline{Z})\to Z\to Z\to 0,$$

$$Z_2+Z_2\to H^{2n-2}((RP^n)^*;\ \underline{Z})\to Z\to Z\to H^{2n-1}((RP^n)^*;\ \underline{Z})\to 0.$$

A simple calculation yields

(12.4) $$H^{2n-2}((RP^n)^*;\underline{Z}) = Z_2 \quad \text{or} \quad Z_2+Z_2 \quad \text{or} \quad 0.$$

On the other **hand,** there are relations

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-2}y^s) = vx^{2^{r+1}-2}y^s,$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-3}y^s) = x^{2^{r+1}-2}y^s+vx^{2^{r+1}-3}y^s,$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-4}y^s) = vx^{2^{r+1}-4}y^s,\quad \rho_2\tilde{\beta}_2(x^{2^{r+1}-2}y^{s-1}) = vx^{2^{r+1}-2}y^{s-1},$$

by **(11.2)** and (12.3) since n is even. Consider the Bockstein exact sequence (12.2)

$$\cdots\cdots \to H^{2n-3}((RP^n)^*;\underline{Z}) \xrightarrow{\rho_2} H^{2n-3}((RP^n)^*;Z_2) \xrightarrow{\tilde{\beta}_2} H^{2n-2}((RP^n)^*;\underline{Z})$$
$$\xrightarrow{\times 2} H^{2n-2}((RP^n)^*;Z) \xrightarrow{\rho_2} H^{2n-2}((RP^n)^*Z_2) \longrightarrow \cdots.$$

The last three relations of the above and (11.3) show the last half of Lemma 11.4. Also, the first two relations of the above show that the image $\rho_2H^{2n-2}((RP^n)^*;\underline{Z})=Z_2$ generated by $x^{2^{r+1}-3}y^s+vx^{2^{r+1}-3}y^s$. Therefore we have the first half of Lemma **11.4** by the above Bockstein exact sequence, **(11.3)** and (12.4).

PROOF OF LEMMA 11.5. Consider the Bockstein exact sequence (12.2)

$$H^{2n-3}((RP^n)^*;Z) \xrightarrow{\rho_2} H^{2n-3}((RP^n)^*;Z_2) \xrightarrow{\tilde{\beta}_2} H^{2n-2}((RP^n)^*\ \underline{Z})$$
$$\xrightarrow{\times 2} H^{2n-2}((RP^n)^*\ Z) \xrightarrow{\rho_2} H^{2n-2}((RP^n)^*\ Z_2)\ .$$

Since n is odd, there are relations

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-2}y^s) = vx^{2^{r+1}-2}y^s,$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-3}y^s) = vx^{2^{r+1}-3}y^s,$$

$$\rho_2\tilde{\beta}_2(vx^{2^{r+1}-3}y^{s-1}) = vx^{2^{r+1}-2}y^{s-1},$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-4}y^s) = vx^{2^{r+1}-4}y^s+x^{2^{r+1}-3}y^s,$$

by (11.2) and (12.3). Therefore, the lemma can be proved in the same way as the proof of Lemma 11.4, by using the Bockstein exact sequence (12.2) and (11.3).

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