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Smooth S^3-Actions on n Manifolds for $n \leqq 4$

Tohl ASOH

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§1. Introduction

In this note, we say that M is an S^3 $(=SU(2))$-manifold, if M is a connected compact smooth manifold admitting a non-trivial smooth S^3-action $S^3 \times M \to M$. The purpose of this note is to classify such closed manifolds of dimension less than 5 by S^3-equivariant diffeomorphisms.

We notice the following results (cf. [1, Cor. 3.2] and [6, Th. 2.6.7]).

(1.1) Any closed proper subgroup of

$$S^3 = \{q \in H;\ |q| = 1\} \qquad (H \text{ is the quaternion field})$$

is conjugate to one of the following subgroups:

$S^1 = \{z \in C;\ |z| = 1\}$, the unit circle in the complex field C;
$NS^1 = \{z, zj;\ z \in S^1\}$, the normalizer of S^1 in S^3;
$Z_n = \{z \in S^1;\ z^n = 1\}$, the cyclic group of order n $(\geqq 1)$;
$D^*(4m) = \{z, zj;\ z \in Z_{2m}\} = \eta_2^{-1}(D(2m))$, the binary dihedral group of order $4m$ $(\geqq 8)$;
$T^* = \eta_2^{-1}(T)$, $O^* = \eta_2^{-1}(O)$ and $I^* = \eta_2^{-1}(I)$, the binary tetrahedral, octahedral and icosahedral groups of order 24, 48 and 120, respectively.

Here, η_2: $S^3 \to SO(3)$ is the double covering defined by

$$\eta_2(q)p = qpq^{-1} \qquad (q \in S^3,\ p \text{ is a pure quaternion}),$$

and $D(2m)$ is the dihedral group of order $2m$ and T, O and I are the tetrahedral, octahedral and icosahedral groups.

For an S^3-manifold M, we denote by (H) its type of principal isotropy subgroups, and consider the following two cases:

(a) *Every isotropy subgroup is principal.*

(b) *There exists a non-principal isotropy subgroup* $K \supsetneqq H$.

Unless otherwise stated, we consider S^3/H as the S^3-manifold with the action η_1, $\eta_1(q)[p] = [qp]$. Also, for any S^3-manifold M_1 and any manifold N, we consider $M_1 \times N$ as the S^3-manifold acting S^3 trivially on N.

Then, closed S^3-manifolds are classified up to equivariant diffeomorphisms by the following theorems.

THEOREM 1.2. *If* $\dim M=2$, *then only the case* (a) *holds, and* M *is determined uniquely by* $H=S^1$ *or* NS^1 *and* $M=S^3/H=S^2$ *or* $P_2(R)$, *respectively.*

THEOREM 1.3. *Assume that* $\dim M=3$.

The case (a). *When* H *is any subgroup of* (1.1) *except* S^1, M *is determined by* H, *and*

$$M=S^3/H \text{ if } H \text{ is finite}, \quad =P_2(R)\times S^1 \text{ if } H=NS^1.$$

When $H=S^1$, M *is determined by* H *and the orientability, and*

$$M=\begin{cases}(S^3/S^1)\times S^1 = S^2\times S^1 & \text{if } M \text{ is orientable,}\\ (S^2\times S^1)/((p,z)\equiv(-p,-z)) & \text{otherwise.}\end{cases}$$

The case (b). M *is determined by the principal isotropy subgroup* $H=S^1$ *and two non-principal ones* K_1 *and* K_2, *and*

$$M=\begin{cases}S^3 \ (S^3 \text{ acts on it via } \eta_2) & \text{if } K_1=K_2=S^3,\\ P_3(R)=S^3/(q\equiv -q)\ (S^3 \text{ acts on } S^3 \text{ via } \eta_2) & \\ & \text{if } K_1=S^3, K_2=NS^1,\\ (S^2\times S^1)/((p,z)\equiv(-p,\bar{z})) & \text{if } K_1=K_2=NS^1.\end{cases}$$

For the case that $\dim M=4$ and $H=S^1$, we take a small closed invariant tubular neighborhood U of the fixed point set $F(S^3, M)$ (cf. [3, VI, Th. 2.2]), and consider the S^3-submanifold $M'=M-\operatorname{Int} U$. ($U=\emptyset$ and $M'=M$ if $F(S^3, M)=\emptyset$.) Further we consider the fixed point set $F(S^1, M')$ of the restricted S^1-action. Then, we have

PROPOSITION 1.4. (i) $F(S^1, M')$ *admits the non-trivial* Z_2 $(=NS^1/S^1)$-*action induced from the given* S^3-*action, which is free on the boundary* $\partial F(S^1, M')$, *and* $F(S^1, M')/Z_2$ *is connected. Also* $F(S^1, M')$ *is a compact surface.*

(ii) *Let* D^3 *be the unit disk of dimension* 3, *admitting the* S^3-*action via* η_2. *Then we have an equivariant diffeomorphism*

$$M \approx \partial(D^3\times F(S^1, M'))/Z_2,$$

where Z_2 *acts on* D^3 *by the antipodal map and on* $F(S^1, M')$ *by* (i).

THEOREM 1.5. *Assume that* $\dim M=4$.

The case (a). (i) *If* $H=1$, Z_2, O^* *or* I^*, *then* M *is determined uniquely by* H, *and* $M=(S^3/H)\times S^1$.

(ii) *If* $H=Z_n$ $(n\geqq 3)$, $D^*(4m)$ $(m\geqq 2)$ *or* T^*, *then* M *is an* S^3/H-*bundle over* S^1 *with structure group* NH/H, *and* M *is determined by* H *and the first* *integral homology group $H_1(M)$, which is given by the following table:*

H	$H_1(M)$
Z_n $(n \geqq 3)$	Z_n+Z^*, $Z(n$: odd), Z_2+Z (n: even)
$D^*(4m)$ $(m \geqq 3)$	Z_4+Z (m: odd)*, Z_2+Z_2+Z (m: even)*, Z_2+Z
$D^*(8)$	$Z_2+Z_2+Z^*$, Z_2+Z, Z
T^*	Z_3+Z^*, Z

($M=(S^3/H)\times S^1$ *for the case indexed by* *).

(iii) *If $H=NS^1$, then $M=P_2(R)\times N$, where $P_2(R)$ is the S^3-manifold in Theorem* 1.2 *and N is any connected closed surface.*

If $H=S^1$, then $F(S^3, M)=\emptyset$ and M is determined by the above proposition, where the Z_2-surface $F(S^1, M)$ is a closed surface and the Z_2-action is free.

The case (b). (iv) *When H is finite, H is Z_n, $D^*(4m)$ $(m \geqq 2)$ or T^*, and M has two non-principal isotropy subgroups K_1 and K_2.*

If $H \neq D^*(8)$, M is determined uniquely by H, K_1 and K_2 of the following table:

H	K_l $(l=1, 2)$
Z_n (n: odd)	Z_{2n}, S^1, S^3 $(n=1)$
Z_n (n: even)	Z_{2n}, $D^*(2n)$, S^1, NS^1 $(n=4)$
$D^*(4m)$ $(m \geqq 3)$	$D^*(8m)$, NS^1
T^*	O^*

If $H=D^*(8)$, M is determined by H, K_1, K_2 and $H_1(M)$, which are given by the following table:

(K_1, K_2)	$H_1(M)$
$(D^*(16), D^*(16))$	$Z_2+Z_2+Z_2$, Z_2+Z_2
$(D^*(16), NS^1)$	Z_2+Z_2, Z_2
(NS^1, NS^1)	Z_2, 0

(v) *If* $\dim H \geqq 1$, *then $H=S^1$ and M is determined by the above proposition,* *where the Z_2-action on the surface $F(S^1, M')=F(S^1, M)$ is not free if $F(S^3, M)=\emptyset$.*

The results on the classification of Z_2-surfaces, which are used in (iii) and (v) of the above theorem, are given in § 7.

§ 2. Closed subgroups of S^3

In this section, we prepare some known results on closed subgroups of S^3 and their real representations.

LEMMA 2.1. *The binary octahedral group O^* in* (1.1) *is generated by $e=\exp(\pi i/4)$, $e'=(1+j)/\sqrt{2}$ and $e''=(1+k)/\sqrt{2}$.*

PROOF. We notice the following equalities for any $a, b, z \in C$, which are seen easily:

$$(2.2)\qquad \begin{aligned}(a+bj)z(\bar{a}-bj) &= (|a|^2 z+|b|^2\bar{z})+ab(-z+\bar{z})j,\\ (a+bj)j(\bar{a}-bj) &= (-\bar{a}b+a\bar{b})+(a^2+b^2)j.\end{aligned}$$

By considering the set $A=\{\pm i, \pm j, \pm k\}$ of vertices of the regular octahedron, we see that

$$O^* = \eta_2^{-1}(O) = \{q \in S^3;\ qAq^{-1} = A\}.$$

Therefore, we see easily by using (2.2) that O^* contains the subgroup O' of S^3 generated by e, e' and e''. Therefore $O^*=O'$ since these groups are of order 48. *q.e.d.*

LEMMA 2.3. *Let H be a finite subgroup of S^3. Then the normalizer NH of H in S^3, the factor group NH/H and $\#\hat{\pi}_0(NH/H)$ are given as follows:*

H	NH	NH/H	$\#\hat{\pi}_0(NH/H)$
Z_n $(n=1, 2)$	S^3	S^3 $(n=1)$, $SO(3)$ $(n=2)$	1
Z_n $(n\geqq 3)$	NS^1	NS^1 (n: odd), $O(2)$ (n: even)	2
$D^*(4m)$ $(m\geqq 3)$	$D^*(8m)$	Z_2	2
$D^*(8)$	O^*	$D(6)$	3
T^*	O^*	Z_2	2
O^*, I^*	O^*, I^*	1	1

In the above lemma, for a given topological group G,

$$\hat{\pi}_0(G) = \pi_0(G)/\sim \tag{2.4}$$

is the set of equivalence classes of elements of $\pi_0(G)$ under the inner automorphisms, and $\#\hat{\pi}_0(G)$ is its cardinal number.

PROOF. When H is 1 or Z_2, the results are clear.

Assume $H = Z_n$ $(n \geqq 3)$. By (2.2), it is easy to see that $a+bj \in NZ_n$ is equivalent to $ab=0$, and so $NZ_n = NS^1$. Further, there are isomorphisms $NS^1/Z_n \approx NS^1$ for odd n given by $z \to z^n$, $j \to j$ and $NS^1/Z_n \approx O(2)$ for even n given by $\exp(\theta i) \to \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$, $j \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Assume that $H = D^*(8) = \{\pm 1\} \cup A$, where $A = \{\pm i, \pm j, \pm k\}$. Then, $ND^*(8) = O^*$ by the proof of Lemma 2.1. Furthermore, $e^2 = i$, $b^3 = -1$ and $ebeb = -1$ $(b = ee')$ are in $D^*(8)$, and

$$D(6) = \{1, x, xy, xy^2, y, y^2\}, \qquad x^2 = y^3 = xyxy = 1,$$

is the dihedral group of order 6. Hence $O^*/D^*(8) = D(6)$.

For the case $H = D^*(4m)$ $(m \geqq 3)$, we see easily by using (2.2) that $ND^*(4m) = \{a+bj \in S^3;\, ab=0,\, a^2+b^2 \in Z_{2m}\}$, which is equal to $D^*(8m)$. It is clear that $D^*(8m)/D^*(4m) = Z_2$.

Finally we consider the case $H = T^*$, O^* or I^*. It is well known that $T = A_4$, $O = S_4$ and $I = A_5$, where S_i and A_i are the symmetric and alternating groups of i letters. Therefore, T^* is the normal subgroup of O^* and $T^* \subset I^*$, and also $O^* \not\subset I^*$ since 120 is not a multiple of 48. Since T^* has two non-commutative elements of order 6, T^*, O^* and I^* are not contained in any conjugate of $D^*(4m)$ or NS^1. Also, we see that $\{qz\bar{q};\, q \in S^3\} = S^3$ if $z - \bar{z} \neq 0$ by using (2.2), and so the proper normal subgroup of S^3 is 1 or Z_2. Therefore we see that $NO^* = O^*$, $NI^* = I^*$ and $NT^* = O^*$.

The results of $\#\hat{\pi}_0(NH/H)$ are obtained easily. *q.e.d.*

Now, we prepare some results on real representations of closed subgroups of S^3.

LEMMA 2.5. *Let K be a closed subgroup of S^3, $\rho: K \to O(k)$, $k \leqq 4 - \dim S^3/K$, be a non-trivial representation, and H be a principal isotropy subgroup of the K-action on the unit disk D^k via ρ. Then, these are given by the following table up to equivalence:*

K	k	ρ	H
S^3	3, 4	$\eta_2: \eta_2(q)p=qpq^{-1}$	S^1
	4	$\eta_1: \eta_1(q)p=qp$	1
NS^1	1, 2	$\nu: \nu(z)=1,\ \nu(j)=-1$	S^1
	2	$\gamma_{2n}\ (n\geqq 1): \gamma_{2n}(z)p=z^{2n}p,\ \gamma_{2n}(j)p=-\bar{p}$	$D^*(4n)$
S^1	2	$\delta_n\ (n\geqq 1): \delta_n\ (z)p=z^n p$	Z_n
Z_{2n}	1	$\lambda^{(n)}: \lambda^{(n)}(a_{2n})=-1$	Z_n
O^*	1	$\mu: \mu(e)=\mu(e')=\mu(e'')=-1$	T^*
$D^*(4m)$	1	$\chi_1: \chi_1(a_{2m})=1,\ \chi_1(j)=-1$	Z_{2m}
$D^*(8m)$	1	$\chi_2: \chi_2(a_{4m})=-1,\ \chi_2(j)=1$	$D^*(4m)$
	1	$\chi_3: \chi_3(a_{4m})=-1,\ \chi_3(j)=-1$	$a_{8m}D^*(4m)a_{8m}^{-1}$

$(a_n=\exp(2\pi i/n),\ e=a_8,\ e'=(1+j)/\sqrt{2},\ e''=(1+k)/\sqrt{2})$.

PROOF. When $K=S^3$, the results follow immediately from [1, Prop. 3.7] and its proof. The results for $K=NS^1$ or S^1 are [1, Prop. 3.8]. The others are seen easily. *q.e.d.*

§3. Actions with orbits of codimension 1

In this section, let G be a compact connected Lie group and M be a closed (compact and without boundary) connected smooth n manifold. Assume that there is given a non-trivial smooth action $G\times M\to M$ of G on M, and let (H) be the type of its principal isotropy subgroups, and d be its maximal orbit dimension, i.e., the dimension of the principal orbit G/H, (cf. [3, IV, 3]).

(3.1) [3, IV, Th. 3.3 and VI, Cor. 2.5] When every orbit is principal, M is a G/H-bundle over the closed manifold M/G with structure group NH/H, where NH is the normalizer of H in G.

(3.2) [3, IV, Lemma 4.1] The orbit space M/G is an $n-d$ manifold if $n-d\leqq 2$.

(3.3) If $d=n$, then M is equivariantly diffeomorphic to G/H.

For the case $d=n-1$, we consider the following situation:

(3.4) For $l=1, 2$, let $K_l\supsetneqq H$ be a closed subgroup of G and

$$\rho_l: K_l \longrightarrow O(k_l),\quad k_l = n-\dim G/K_l,$$

be a representation. Assume that the K_l-action on a unit disk D^{k_l} via ρ_l is transitive on the boundary ∂D^{k_l} and its isotropy subgroup $(K_l)_{p_l}$ is equal to H for some $p_l \in \partial D^{k_l}$, and also the G-manifold $G \times_{K_l} D^{k_l}$ has (H) as the type of principal isotropy subgroups.

Then, we can identify $G \times_{K_l} \partial D^{k_l} = G/H$ by the equivariant diffeomorphism

$$G \times_{K_l} \partial D^{k_l} \ni [g, p_l] \longleftrightarrow gH \in G/H,$$

and for any $\alpha \in NH$, we obtain the G-manifold

$$M_\alpha = G \times_{K_1} D^{k_1} \cup_\alpha G \times_{K_2} D^{k_2}, \tag{3.5}$$

where the attaching map $\alpha: G \times_{K_1} \partial D^{k_1} = G/H \rightarrow G/H = G \times_{K_2} \partial D^{k_2}$ is given by $\alpha(gH) = g\alpha^{-1}H$ $(g \in G)$.

PROPOSITION 3.6. [3, IV, Th. 8.2] *Assume that $d = n-1$ and there exist non-principal orbits.*

(i) *Then there exist K_l and ρ_l of* (3.4) *and*

$$M = M_\alpha \qquad \text{for some} \quad \alpha \in NH.$$

(ii) *Assume that there exist K_l and ρ_l of* (3.4) *satisfying the following*:

(3.7) *If $\rho'_l: K_l \rightarrow O(k_l)$ satisfies the assumption of* (3.4) *except the condition $H = (K_l)_{p_l}$, then there exists $\gamma_l \in NK_l$ such that $\rho'_l c_{\gamma_l}$ is equivalent to ρ_l, where $c_{\gamma_l}(k) = \gamma_l k \gamma_l^{-1}$ $(k \in K_l)$.*

Then, we can choose any such fixed K_l and ρ_l for those in (i).

PROOF. (i) It is sufficient to notice in the proof of [3, IV, Th. 8.2] that we can take $M_{f_{l-1}} = G \times_{K_l} D^{k_l}$ $(l = 1, 2)$, which follows immediately from the differentiable slice theorem (cf. [3, VI, Cor. 2.4]).

(ii) In the same way, we can write $M \approx G \times_{(K_1, \rho'_1)} D^{k_1} \cup G \times_{(K_2, \rho'_2)} D^{k_2}$, where K_l acts on D^{k_l} via ρ'_l. Then by (3.7) it is easy to see that $G \times_{(K_l, \rho'_l)} D^{k_l}$ is equivariantly diffeomorphic to $G \times_{K_l} D^{k_l}$, and we have the desired result. *q.e.d.*

Now, we consider the following condition for the situation (3.4):

(3.8) For any $\gamma_l \in NH \cap NK_l$, the right translation ψ_{γ_l} of $G \times_{K_l} D^{k_l}$ is smooth, where ψ_{γ_l} is given by

$$\psi_{\gamma_l}[g, tp_l] = [g\gamma_l^{-1}, tp_l] \qquad (g \in G,\ 0 \leqq t \leqq 1).$$

PROPOSITION 3.9. *Under the assumption* (3.8), *M_α and M_β of* (3.5) *for $\alpha, \beta \in NH$ are equivariantly diffeomorphic if and only if there exist $\gamma_l \in NH \cap NK_l$ $(l = 1, 2)$ such that γ_1 and $\beta^{-1}\gamma_2\alpha$ are contained in the same component of NH/H.*

PROOF. (Sufficiency) We can choose a smooth path $\theta: [0, 1] \to NH/H$ from γ_1 to $\beta^{-1}\gamma_2\alpha$, which is locally constant at 0 and 1. Then, by considering

$$(*) \qquad M_\tau = G \times_{K_1} D^{k_1} \cup (G/H \times [0, 1]) \cup G \times_{\tau^{-1}K_2\tau} D^{k_2} \qquad (\tau = \alpha, \beta),$$

the desired diffeomorphism $\psi: M_\alpha \approx M_\beta$ is given by

$$(**) \qquad \psi[g, tp_1] = [g\gamma_1^{-1}, tp_1], \quad \psi[g, tp_2] = [g\alpha^{-1}\gamma_2^{-1}\beta, tp_2]$$

and $\psi(gH, s) = (g\theta(s)^{-1}H, s)$ for $g \in G$, t, $s \in [0, 1]$.

(Necessity) Assume that M_α and M_β are equivariantly diffeomorphic. In the same way as the proof of [3, V, Th. 5.1], we can choose an equivariant homeomorphism $\psi: M_\alpha \approx M_\beta$ such that M_α and M_β have the forms of (*), and ψ maps $G \times_{K_1} D^{k_1}$ and $G \times_{\alpha^{-1}K_2\alpha} D$ to $G \times_{K_1} D^{k_1}$ and $G \times_{\beta^{-1}K_2\beta} D^{k_2}$, respectively, satisfying (**) for some $\gamma_l \in NH \cap NK_l$ $(l=1, 2)$. Then γ_1 and $\beta^{-1}\gamma_2\alpha$ are connected by the path

$$\theta: [0, 1] \subset G/H \times [0, 1] \xrightarrow{\psi} G/H \times [0, 1] \longrightarrow G/H. \qquad q.e.d.$$

For the condition (3.8), we have the following

LEMMA 3.10. *Assume that the representation $\rho_l: K_l \to O(k_l)$ satisfies $\rho_l(K_l) \supset SO(k_l)$. Then* (3.8) *holds, if ρ_l is equivalent to $\rho_l c_{\gamma_l}$ for any $\gamma_l \in NH \cap NK_l$.*

PROOF. We use the notations omitting the index l. ψ_γ in (3.8) is the bundle map of the disk bundle $D^k \to G \times_K D^k \to G/K$ onto itself, inducing $\tilde{\psi}: G/K \to G/K$, $\tilde{\psi}(gK) = g\gamma^{-1}K$. Therefore, it is sufficient to show that $f = \psi_\gamma | D^k$ is linear. From the definition of ψ_γ, we see that f is given by

$$f(t\rho(g)p) = t\rho(\gamma g \gamma^{-1})p \qquad \text{for} \quad g \in K.$$

There exists $A' \in GL(k)$ such that $\rho c_\gamma = c_{A'}\rho$ by the assumption, where $c_A(X) = AXA^{-1}$ $(X \in O(k))$. Then it is easy to see that $A' = tA$ for some $A \in O(k)$ and $t \neq 0$, since $\rho(K) \supset SO(k)$. Thus we have $\rho c_\gamma = c_A \rho$ and so

$$(*) \qquad f(Xp) = AXA^{-1}p \qquad \text{for any} \quad X \in \rho(K).$$

Consider the isotropy subgroup $\rho(K)_p = \{X \in \rho(K); Xp = p\}$. Then we see easily that $A\rho(K)_p A^{-1} \subset \rho(K)_p$ by (*) and so A belongs to the normalizer $N(\rho(K)_p)$ in $O(k)$. On the other hand, we see easily that $N(\rho(K)_p) = \{B \in O(k); Bp = \pm p\}$ since $\rho(K) = O(k)$ or $SO(k)$. Therefore we see $Ap = \pm p$, and hence

$$f(Xp) = AXA^{-1}p = \pm AXp \qquad \text{for any} \quad X \in \rho(K).$$

This shows that f is linear as desired. *q.e.d.*

The following lemma for the special case that $G=S^3$ is used in § 5.

LEMMA 3.11. *When* $n=4$, $H=1$, $G=K_l=S^3$ *and* ρ_l *is* $\eta_1: S^3 \to O(4)$ *in Lemma* 2.5, *the condition* (3.8) *holds.*

PROOF. For this case, $G \times_{K_l} D^{k_l}=D^4$ and $\psi_{\gamma_l}: D^4 \to D^4$ is given by $\psi_{\gamma_l}(p) = p\gamma_l^{-1}$, and hence we have the lemma. *q.e.d.*

§ 4. Proofs of Theorems 1.2 and 1.3

In this section, we apply the results of the previous sections for the case that $G=S^3$ and $\dim M \leqq 3$.

PROOF OF THEOREM 1.2. Since $\dim H \leqq 1$ by (1.1), we have $d=\dim S^3/H \geqq 2$. Also $\dim M/S^3=2-d \geqq 0$ by (3.2). Thus $d=2$ and $\dim H=1$, and the result is clear from (1.1) and (3.3). *q.e.d.*

PROOF OF THEOREM 1.3. In the same way as the above proof, the maximal orbit dimension d satisfies $2 \leqq d \leqq 3$.

When $d=3=\dim M$, M is equivariantly diffeomorphic to S^3/H by (3.3), where H is finite.

When $d=2=\dim M-1$, we have $H=S^1$ or NS^1 since $\dim H=1$. For the case (a), M is an S^3/H-bundle over S^1 with structure group NH/H by (3.1). If H is NS^1, then $NH/H=1$ and so M is equivariantly diffeomorphic to $(S^3/NS^1) \times S^1=P_2(R) \times S^1$. If H is S^1, then $S^2=S^3/S^1$ and M is an S^2-bundle over S^1 with structure group $NS^1/S^1=Z_2$. Thus M is equivariantly diffeomorphic to $S^2 \times S^1$ or $(S^2 \times S^1)/((q, z) \equiv (-q, -z))$.

For the case (b), we apply Proposition 3.6. Lemma 2.5 shows that there do not exist K_l and ρ_l satisfying (3.4) for $H=NS^1$. If $H=S^1$, then Lemma 2.5 shows that K_l and ρ_l of (3.4) are given by

$$K_l = S^3,\ \rho_l = \eta_2 \quad \text{or} \quad K_l = NS^1,\ \rho_l = \nu,$$

and the condition (3.7) holds. Therefore, it is sufficient to classify M_α by Proposition 3.6. Since the condition (3.8) holds by Proposition 3.10, we see easily that $M_1 \approx M_j$, and hence $M_\alpha \approx M_\beta$ for any $\alpha, \beta \in NS^1$ by Proposition 3.9. Thus, M is determined uniquely by $H=S^1$ and $(K_1, K_2)=(NS^1, NS^1)$, (NS^1, S^3) or (S^3, S^3). *q.e.d.*

§ 5. The case that dim $M=4$ and H is finite

In this section, we assume that M is a closed S^3-manifold of dimension 4, and its principal isotropy subgroup H is finite.

For the case that every orbit is principal, (3.1) shows that M is an S^3/H-bundle over $M/S^3=S^1$ with structure group NH/H, and we have its characteristic class χ in $\hat{\pi}_0(NH/H)$ of (2.4) by the classification theorem [4, Th. 18.5]. These show the following

PROPOSITION 5.1. *When every orbit is principal, M is determined by H and $\chi \in \hat{\pi}_0(NH/H)$.*

LEMMA 5.2. *Theorem* 1.5 (i) *holds.*

PROOF. When H is a finite subgroup of (i) in Theorem 1.5, we have $\sharp\hat{\pi}_0(NH/H)=1$ by Lemma 2.3, and hence the desired result by the above proposition. *q.e.d.*

To study the case (ii) of Theorem 1.5, we consider the relation between the characteristic class χ and $H_1(M)$.

LEMMA 5.3. *Let A_l ($l=1, 2$) be a connected space such that $A_1 \cap A_2 = A_0$. Then the first integral homology group $H_1(A_1 \cup_\varphi A_2)$, of an attaching space $A_1 \cup_\varphi A_2$ by a homeomorphism $\varphi: A_0 \to A_0$, is given by*

$$H_1(A_1 \cup_\varphi A_2) = \mathrm{Coker}(i_{1*}, -(i_2\varphi)_*) + \tilde{H}_0(A_0), \tag{5.4}$$

where $i_l: A_0 \to A_l$ is the inclusion and $(i_{1*}, -(i_2\varphi)_*): H_1(A_0) \to H_1(A_1) + H_1(A_2)$.

PROOF. (5.4) follows immediately from the Mayer-Vietoris exact sequence of $(A_1 \cup_\varphi A_2; A_1, A_2)$. *q.e.d.*

The following lemma is clear.

LEMMA 5.5. *Let H be a finite subgroup of S^3, $D(H)$ be the commutator subgroup of H, and $\alpha \in NH$. Then, we have the commutative diagram*

$$\begin{array}{ccc} H_1(S^3/H) & \xrightarrow{\alpha_*} & H_1(S^3/H) \\ \| & & \| \\ H/D(H) & \xrightarrow{c_{\alpha\sharp}} & H/D(H), \end{array}$$

where $\alpha: S^3/H \to S^3/H$ is the right transformation given by $\alpha(gH) = g\alpha^{-1}H$ and $c_\alpha: H \to H$ is the automorphism given by $c_\alpha(h) = \alpha h \alpha^{-1}$.

In the above lemma, we see easily the following

LEMMA 5.6. *If H is a subgroup in* (ii) *of Theorem* 1.5, *then it holds the following table*:

H	$H/D(H)$	α	$c_{\alpha\#}$
$D^*(4m)$ (m: even $\geqq 3$)	$Z_2<a_{2m}>+Z_2<j>$	a_{4m}	$a_{2m}\to a_{2m}$, $j\to a_{2m}+j$
$D^*(4m)$ (m: odd $\geqq 3$)	$Z_4<j>$	a_{4m}	$j\to -j$
$D^*(8)$	$Z_2<i>+Z_2<j>$	a_8	$i\to i$, $j\to i+j$
		ee'	$i\to j$, $j\to i+j$
Z_n ($n\geqq 3$)	$Z_n<a_n>$	j	$a_n\to -a_n$
T^*	$Z_3<ee'>$	a_8	$ee'\to -ee'$

where $Z_n<a>$ *is a cyclic group* Z_n *generated by* a, *and* $ee'=(1+i+j+k)/2$.

Now, we are ready to prove (ii) of Theorem 1.5.

LEMMA 5.7. *Theorem* 1.5 (ii) *holds.*

PROOF. Let M be given by H and $\chi=[\alpha]$, $\alpha\in NH$, in Proposition 5.1. Denote $E_l=\{\exp(t\pi i);\ l-1\leqq t\leqq l\}\subset S^1$ ($l=1, 2$), and set $A_l=\pi^{-1}(E_l)$, where $\pi\colon M\to M/S^3=S^1$ is the projection of the bundle. Then

$$A_0 = A_1 \cap A_2 = S^3/H \cup S^3/H \qquad \text{(disjoint union)},$$

and the definition of χ shows that $M=A_1\cup_\varphi A_2$ and the diagram

$$\begin{array}{ccccc} A_l & \xleftarrow{\ i_l\ } & A_0 & \xrightarrow{\ \varphi\ } & A_0 \\ {\scriptstyle i}\uparrow & & \| & & \| \\ S^3/H & \xleftarrow{\ f\ } & S^3/H\cup S^3/H & \xrightarrow{\ 1\cup\alpha\ } & S^3/H \end{array}$$

is commutative, where the inclusion i is a homotopy equivalence and f is the folding map. These facts and Lemma 5.5 show that $(i_{1*}, -(i_2\varphi)_*)$ in (5.4) is equal to the homomorphism

$$H'+H' \longrightarrow H'+H', \quad (a, b) \longrightarrow (a+b, -a-c_{\alpha\#}(b)),$$

$(H'=H/D(H))$. Thus we have easily

$$H_1(M) = H_1(A_1 \cup_\varphi A_2) \cong H'/\mathrm{Im}(1-c_{\alpha\#})+Z$$

by Lemma 5.3, and hence we obtain (ii) of Theorem 1.5 by the above lemma. *q.e.d.*

Now, we study the case (iv) in Theorem 1.5, by applying the results in § 3.

LEMMA 5.8. *If H is finite, then we can take H and K_l ($l=1, 2$) in the tables of* (iv) *of Theorem* 1.5 *as the ones in Proposition* 3.6 (ii).

PROOF. By Lemma 2.5, it is sufficient to show that (3.7) holds for $H=D^*(4m)$, $K_l=D^*(8m)$, $\rho_l=\chi_2$ and $\rho_l'=\chi_3$. This is clear, since $\chi_2=\chi_3 c_{a_{8m}}$ by definitions of χ_2 and χ_3. *q.e.d.*

LEMMA 5.9. *Theorem* 1.5 (iv) *holds when $H\neq D^*(8)$.*

PROOF. Let (H, K_1, K_2) be given in the table of (iv) of Theorem 1.5. By the above lemma and Proposition 3.6, it is sufficient to classify M_α for $\alpha\in NH$. We see that (3.8) holds by Lemmas 2.5, 3.10 and 3.11, and so we can classify M_α by Proposition 3.9.

If $H=1$ or Z_2, then $NH/H=S^3/H$ is path-connected, and so $M_1\approx M_\alpha$ for any $\alpha\in NH$ by Proposition 3.9.

Let $H=Z_n$ ($n\geqq 3$), $D^*(4m)$ ($m\geqq 3$) or T^*. Then we see that $NH=NS^1$, $D^*(8m)$ or O^* and $NK_l\supset D^*(8)$, $D^*(16m)$ or O^*, respectively, by Lemma 2.3. Consider the element $\alpha_0\in NH\cap NK_l$, given by $\alpha_0=j$, a_{4m} or e, respectively. Then we see that $M_1\approx M_{\alpha_0}$ and hence $M_\alpha\approx M_\beta$ for any α, $\beta\in NH$ by Proposition 3.9. Thus M is determined uniquely by (H, K_1, K_2). *q.e.d.*

Next we consider the case that $H=D^*(8)$.

LEMMA 5.10. *Theorem* 1.5 (iv) *is also valid when $H=D^*(8)$.*

PROOF. In the same way as the above proof, it is sufficient to classify M_α ($\alpha\in ND^*(8)$) by Proposition 3.9, where $K_l=D^*(16)$ or NS^1.

By the proof of Lemma 2.3, we have

$$ND^*(8)/D^*(8) = D(6) = \{1, x, xy, xy^2, y, y^2\},$$

where $x=eD^*(8)$, $y=ee'D^*(8)$. Since $NH\cap NK_l=D^*(16)$, Proposition 3.9 and the easy calculation show that

$$M_1 \approx M_x \not\approx M_y \approx M_{xy} \approx M_{xy^2} \approx M_{y^2}.$$

Now, we calculate $H_1(M_\alpha)$ for $\alpha=1$ or ee' by Lemma 5.3. By (3.5),

$$M_\alpha = A_1 \cup_\alpha A_2, \quad A_0 = A_1 \cap A_2 = S^3/H \qquad (H = D^*(8)),$$

where $A_l=S^3\times_{K_l}D^{k_l}$, and we have the commutative diagram

$$\begin{array}{ccccc} A_l & \xleftarrow{\ i_l\ } & A_0 & \xrightarrow{\ \alpha\ } & A_0 \\ {\scriptstyle p}\downarrow & & \| & & \| \\ S^3/K_l & \xleftarrow[\ p_l\ ]{} & S^3/H & \xrightarrow{\ \alpha\ } & S^3/H \end{array}$$

where p and p_l are the projections and p is a homotopy equivalence. By identifying

$$H_1(A_0) = H/D(H) = Z_2\langle i\rangle + Z_2\langle j\rangle\,,$$

$$H_1(A_l) = K_l/D(K_l) = \begin{cases} Z_2\langle e\rangle + Z_2\langle j\rangle & \text{if } K_l = D^*(16), \\ Z_2\langle j\rangle & \text{if } K_l = NS^1, \end{cases}$$

(cf. Lemma 5.6), the above facts and Lemma 5.5 show that $((i_1)_*, -(i_2\alpha)_*)$ in (5.4) is equal to the homomorphism given by

$$i \longrightarrow (0, 0), \quad j \longrightarrow (j, -j) \quad \text{if } \alpha = 1,$$

$$i \longrightarrow (0, -j), \quad j \longrightarrow (j, -j) \quad \text{if } \alpha = ee'.$$

Then (5.4) and the easy calculation show that $H_1(M_\alpha)$ is given by the second table in (iv) of Theorem 1.5. *q.e.d.*

§6. The case that dim $M=4$ and dim $H=1$

In this section, we assume that M is a closed S^3-manifold of dimension 4 and $\dim H=1$.

LEMMA 6.1. *If H is NS^1, then every orbit is principal, and the first half of Theorem* 1.5 (iii) *holds.*

PROOF. The first half of lemma follows immediately from Lemma 2.5. Then, M is a trivial S^3/NS^1-bundle over $N=M/S^3$ by (3.1), and N is a connected closed surface by (3.2). *q.e.d.*

In the rest of this section, we assume that $H=S^1$.

When the fixed point set $F(S^3, M)$ is non-empty, any point $x \in F(S^3, M)$ has an invariant neighborhood which is equivariantly diffeomorphic to D^4 with the S^3-action given by η_2 (cf. Lemma 2.5). Thus we have

$$F(S^3, M) = F_1 \cup \cdots \cup F_k, \quad F_l = S^1 \qquad (1 \leqq l \leqq k).$$

Further, $F(S^3, M)$ has a closed invariant tubular neighborhood U, which is a D^3-bundle over $F(S^3, M)$, and so

(6.2) $U = U_1 \cup \cdots \cup U_k, \quad U_l = D^3 \times S^1 \quad \text{or} \quad D^3 \times S^1/((q, z) \equiv (-q, -z)),$

where S^3 acts on D^3 via η_2. On the other hand, the S^3-manifold $M'=M-\operatorname{Int} U$ is

(6.3) $M' = M - \operatorname{Int} U \approx (S^3/S^1) \times_{Z_2} F(S^1, M')$

by [5, Lemma 4.2], where $F(S^1, M')$ is the one of Proposition 1.4.

Lemma 6.4. *Proposition* 1.4 (i) *holds.*

Proof. $M'/S^3 = F(S^1, M')/Z_2$ is connected since M' is so. Because the isotropy subgroup at $x \in \partial M' = \partial U$ is conjugate to S^1, the Z_2-action is free on $\partial F(S^1, M')$. We see that $F(S^1, M')$ is a surface by (6.3). *q.e.d.*

We prepare the following lemma to show (ii) of Proposition 1.4.

Lemma 6.5. *Let G be a compact Lie group, H be its closed subgroup, and assume that $gHg^{-1} \subset NH$ $(g \in G)$ implies $g \in NH$. Let X be a smooth manifold with a smooth NH/H-action. For any G-equivariant diffeomorphism $f\colon Y \to Y$, $Y = (G/H) \times_{NH/H} X$, there is an NH/H-equivariant diffeomorphism $f'\colon X \to X$ such that*

$$f[gH, x] = [gH, f'(x)] \qquad (gH \in G/H,\ x \in X).$$

Proof. For any $x \in X$, we set

$$f[H, x] = [f_1(x)H, f_2(x)] \qquad \text{in} \quad Y.$$

Since f is H-equivariant, we have

$$f[gH, x] = [gf_1(x)H, f_2(x)] \qquad \text{for any} \quad g \in G.$$

If $g \in H$, the above two equalities show that $f_1(x)NH = gf_1(x)NH$. Hence $f_1^{-1}(x)Hf_1(x) \subset NH$, and so $f_1(x) \in NH$ by the assumption. Thus we have

$$f[gH, x] = [gH, f_1(x)H \cdot f_2(x)] \qquad \text{for any} \quad g \in G,$$

which shows that f is a bundle map of the bundle $X \to Y \to G/H$ onto itself. Therefore $f' = f|X$ is the NH/H-equivariant diffeomorphism of X onto itself, and we see easily the desired equality. *q.e.d.*

Since the assumption of the above lemma holds for $G = S^3$ and $H = S^1$, we have the following

Corollary 6.6. *Any S^3-equivariant diffeomorphism of $S^2 \times S^1/((q, z) \equiv (-q, -z)) = S^2 \times_{Z_2} S^1$ or $S^2 \times S^1 = S^2 \times_{Z_2} (Z_2 \times S^1)$ onto itself can be extended to an S^3-equivariant diffeomorphism of $D^3 \times S^1/((q, z) \equiv (-q, -z))$ or $D^3 \times S^1$ onto itself.*

Proof of Proposition 1.4. (i) is proved in Lemma 6.4. (ii) follows immediately from the above corollary, (6.2) and (6.3). *q.e.d.*

Now, we have Theorem 1.5 completely.

PROOF OF THEOREM 1.5. (i), (ii) and (iv) are proved in Lemmas 5.2, 5.7, 5.9 and 5.10. (iii) and (v) follow immediately from Proposition 1.4. *q.e.d.*

In the last of this section, we give some examples of manifolds in Theorem 1.5.

EXAMPLE 6.7. The following are manifolds in (ii) of Theorem 1.5, which are not product bundles:

$$(S^3/H)\times_{Z_2}S^1 \qquad \text{for } H = Z_n\ (n \geqq 3),\ D^*(4m)\ (m \geqq 2) \text{ and } T^*,$$

where Z_2 acts on S^1 by the antipodal map and on S^3/H by $\alpha\colon gH\to g\alpha^{-1}H$ for $\alpha=j$, a_{4m} and e, respectively;

$$(S^3/H)\times_{Z_3}S^1 \qquad \text{for } \ H = D^*(8),$$

where Z_3 acts on S^1 by the rotation and on S^3/H by $gH\to g(ee')^{-1}H$.

EXAMPLE 6.8. The following are manifolds in (iv) of Theorem 1.5, where the equation in the parentheses indicates the S^3-action:

$$S^4,\ P_2(C),\ P_4(R) \qquad (q\cdot[p, x] = [qp, x])\,;$$

$$P_2(C) \qquad (q\cdot[p\otimes p'] = [qp\otimes qp'])\,;$$

$$S^2\times S^2/((p, r)\equiv(-r, p)) \qquad (q\cdot[p, r] = [qpq^{-1}, qrq^{-1}])\,;$$

$$S^3\times_{S^1}S^2,\ S^3\times_{S^1}P_2(R),$$

where S^1 acts on S^2 or $P_2(R)$ by $b\cdot[a, x]=[b^n a, x]$;

$$S^3\times_{NS^1}S^2, \quad S^3\times_{NS^1}U^2 \qquad (U^2\text{: the Klein bottle}),$$

where NS^1 acts on S^2 or U^2 by $b\cdot(a, x)=(b^{2m}a, x)$, $j\cdot(a, x)=(\bar{a}, -x)$;

$$S^3\times_{NS^1}S^2, \quad S^3\times_{NS^1}P_2(R), \quad S^3\times_{O^*}S^1$$

where NS^1 acts on S^2 or $P_2(R)$ by $b\cdot[a, x]=[b^{2m}a, x]$, $j\cdot[a, x]=[\bar{a}, x]$, and O^* acts on S^1 by $O^*\to O^*/D^*(8)\to D(6)\subset O(2)$.

§7. Z_2-actions on surfaces

In this section, we classify Z_2-surfaces which appear in (iii) and (v) of Theorem 1.5.

We consider the following Z_2-surfaces:

The cylinder $C=[-1, 1]\times S^1$ with the Z_2-action $(t, x)\to(-t, x)$.

The unit disk $D=D^2$ with the Z_2-action $x\to -x$.

The Möbius band $B=[-1, 1]\times S^1/((t, x)\equiv(-t, -x))$ with the Z_2-action $[t, x]\to[-t, x]$, whose boundary is $\partial B=1\times S^1=S^1$ with the Z_2-action $x\to -x$.

By using these surfaces, we can construct the following Z_2-surfaces:

(7.1) Let N' be a connected compact surface such that $\partial N' = S^1 \times \{1, \ldots, k\}$, $k \geqq 0$. Then we have

$$Z_2 \times N' \cup C \times \{1, \ldots, k'\}, \qquad 0 \leqq k' \leqq k.$$

(7.2) Let N' be a connected compact surface admitting a free Z_2-action such that $\partial N' = Z_2 \times S^1 \times \{1, \ldots, k\} \cup S^1 \times \{1, \ldots, m\}$, $k \geqq 0$, $m \geqq 0$, (where Z_2 acts on $Z_2 \times S^1$ by $(\pm 1, x) \to (\mp 1, x)$, and on S^1 by $x \to -x$). Then for $0 \leqq k' \leqq k$, $0 \leqq m_1 \leqq m_1 + m_2 \leqq m$, we have

$$N' \cup C \times \{1, \ldots, k'\} \cup D \times \{1, \ldots, m_1\} \cup B \times \{m_1 + 1, \ldots, m_1 + m_2\}.$$

Then, we have the following

THEOREM 7.3. *Let N be a compact surface admitting a non-trivial Z_2-action such that Z_2 acts freely on its boundary ∂N and N/Z_2 is connected. Then N is equivariantly diffeomorphic to a Z_2-surface of* (7.1) *or* (7.2). *Any Z_2-surface N' in* (7.2) *is characterized by the classification theorem* [2, Th. 1.3].

PROOF. If N is not connected, then we see easily that $N \approx Z_2 \times N'$, which is the one of (7.1) for $k' = 0$.

Assume that N is connected, and consider the fixed point set $F(Z_2, N)$ whose component is a point or a circle. Each component has an invariant tubular neighborhood D if it is a point, and C or B if it is a circle. Therefore, we have easily the theorem by considering $N' = N - \mathrm{Int}\, U$, where U is a closed invariant tubular neighborhood of $F(Z_2, N)$. *q.e.d.*

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Department of Mathematics,
Faculty of Science,
Hiroshima University