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On Higher Coassociativity

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Introduction

Co-H-spaces are defined as generalizations of suspended spaces, and, to certain extent, they have dual properties of H-spaces which are considered as generalizations of loop spaces. For H-spaces the so-called Sugawara-Stasheff's sequence of fibrations plays an essential rôle, however, for co-H-spaces we have no such ones. On the other hand, as Ganea pointed out, the coretraction γ for the evaluation map ε seems to be important for co-H-spaces. The purpose of the present paper is to define A'_n-structures which are formal dual of Stasheff's A_n-form and some relevant notions, e. g., A'_n-maps and (weak-) homotopy-coalgebras, and then to consider how γ relates to these notions.

In § 1, we give the preliminary definitions and results concerning co-H-spaces and the coretraction γ. In §§ 2–3, we give the definitions of A'_n-spaces and A'_n-maps and some of their properties. In § 4, we define a generalized Hopf-homomorphism $H(f)$ of a map f of A'_2-spaces whose vanishing is equivalent to f being a q-A'_2-map.

Now, our main results are as follows.

THEOREM 5.7. *An A'_3-cogroup X is an s-A'_4-cogroup if and only if the corresponding coretraction γ is a q-A'_3-map.*

THEOREM 6.4. *If X is a simply-connected coalgebra of finite dimension, then X has a homotopy-type of a suspended space.*

THEOREM 6.20. *Let X be an s-A'_4-cogroup such that the corresponding γ is an A'_3-map, then X is a weak homotopy coalgebra of order* 3.

Our method is very elementary-homotopical, and the most difficulties arise from the fact that we must construct the (s-)homotopy of (s-)homotopies.

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§ 1. Preliminaries

In this section, we shall state preliminary facts which will be necessary in the subsequent sections. Throughout present paper, if otherwise not mentioned, all considerations will be carried out in the category $\boldsymbol{CW}_*$ of countable based CW-complexes and based continuous maps, therefore, homotopies are based homotopies.

NOTATIONS.

$W_n(X)=\underbrace{X\vee\cdots\vee X}_{n}$, the wedge product (i.e., the one point union) of n-copies of X,

$j_n\colon W_n(X)\to X^n$, the inclusion map,

$\nabla_n\colon W_n(X)\to X$, the folding map, i.e., $\nabla_n(*,\dots, x,\dots, *)=x$,

$\Delta\colon X\to X\times X$, the diagonal map, i.e., $\Delta(x)=(x, x)$,

$i'_k\colon X\to W_n(X)$, the inclusion map into the k-th factor,

$p'_k\colon W_n(X)\to X$, the projection onto the k-th factor,

$X*Y$, the join of spaces X and Y, whose typical point is $t_0x\oplus t_1y$, t_0, $t_1\geqq 0$, $t_0+t_1=1$,

$X\wedge Y$, the smash product of spaces X and Y,

Y^X, the space of base point free maps $f\colon X\to Y$ (equipped with the base point $*\colon X\to *_Y$),

$\{X; Y\}$, the space of based maps $f\colon (X, *)\to(Y, *)$,

$[X; Y]$, the set of all based homotopy classes of based maps $f\colon (X, *)\to(Y, *)$,

$\Omega_X(A, B)$, the space of paths in X whose starting points are in A and terminating points are in B,

S, the suspension functor,

Ω, the loop functor,

(categories will be denoted by bold-faced capital letters).

A *multiplicative set* M is the set with a multiplication $\mu\colon M\times M\to M$ having two-sided identity element e. We shall write $x\circ y$ for $\mu(x, y)$. A map $f\colon M\to M'$ of multiplicative sets is a homomorphism if it satisfies $f(x\circ y)=f(x)\circ f(y)$ for any x, $y\in M$ and $f(e)=e'$. Multiplicative sets and homomorphisms make up a category $\boldsymbol{M}$. A multiplicative set M is said to admit *inverses* if there exist two maps ν_R and ν_L of M into M such that $x\circ\nu_R(x)=e$ and $\nu_L(x)\circ x=e$ hold. A *loop* Λ is the multiplicative set satisfying the following conditions: for any a, $b\in\Lambda$, there exists a unique $x\in\Lambda$ such that $a\circ x=b$, and there exists a unique $y\in\Lambda$ such that $y\circ b=a$. Sometimes we shall write $a\backslash b$ and a/b for such x and y. Loops and homomorphisms make up a category $\boldsymbol{\Lambda}$.

A based space $(X, *)$ is a *co-H-space* if $[X;\ ]$ is a covariant functor of $\boldsymbol{TOP}_*$ into $\boldsymbol{M}$, or equivalently, there exists a based map $\mu'\colon X\to X\vee X$ such that $\nabla_2(1\vee *)\mu'\simeq 1\simeq\nabla_2(*\vee 1)\mu'$ hold, or $j_2\mu'\simeq\Delta$ holds, where $\simeq$ means that both sides are homotopic. μ' is the comultiplication and $*$ is the counit. A co-H-space is necessarily path-connected. We shall use the traditional notation $+$ in

$[X;\ ]$. Then, we have $\mu' \simeq i_1' + i_2'$.

A co-H-space X is said to admit *coinversions*, if there exist two maps v_R' and v_L': $X \to X$ such that $\nabla_2(1 \vee v_R')\mu' \simeq * \simeq \nabla_2(v_L' \vee 1)\mu'$ hold. A co-H-space X is an *h-coloop* if $[X;\ ]$ is a covariant functor of $\boldsymbol{TOP}_*$ into $\boldsymbol{\Lambda}$.

PROPOSITION 1.1. (cf. [7]). *Let (X, μ_X') be a given co-H-space.*

(1.1.1) *If X is simply connected, X admits coinversions.*

(1.1.2) *The following two conditions are equivalent:*

(i) *$[X; X \vee X]$ admits a loop-structure with respect to μ_X'.*

(ii) *X admits coinversions.*

DEFINITION 1.2. Given a triad $(f_1: X_1 \to B \leftarrow X_2: f_2)$, define its *fibred product* T_{f_1,f_2} by

$$T_{f_1,f_2} = \{(x_1, x_2, w) \in X_1 \times X_2 \times B^I \mid w(0) = x_1 \quad \text{and} \quad w(1) = x_2\}.$$

The projections π_i: $T_{f_1,f_2} \to X_i$, $i = 1, 2$, are defined by

$$\pi_i(x_1, x_2, w) = x_i.$$

LEMMA 1.3. *Let T_{f_1,f_2} be the fibred product of a given triad $(f_1: X_1 \to B \leftarrow X_2: f_2)$.*

(i) *The projections π_1 and π_2 are fibre maps.*

(ii) *For any homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{g_2} & X_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow[f_1]{} & B \end{array},$$

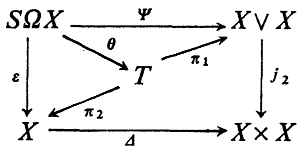
there exists a map k: $X \to T_{f_1,f_2}$ such that $\pi_1 k = g_1$ and $\pi_2 k = g_2$ hold.

Moreover, if X is an h-coloop and π_2 induces a monomorphism π_{2*}: $[X; T_{f_1,f_2}] \to [X; X_2]$ (this is the case when the homotopy-fibre of f_1 is contractible in X_1), then k is unique up to homotopy.

With abuse of language, we say that T_{f_1,f_2} is a *homotopy pull back of* $(f_1: X_1 \to B \leftarrow X_2: f_2)$.

Finally, we shall recall Ganea's theorems [3] for the subsequent considerations.

THEOREM 1.4. *Consider the h-pull back T_{Δ,j_2}, then there exists a homotopy-equivalence θ: $S\Omega X \to T_{\Delta,j_2} = T$ such that the following diagram is homotopy-commutative:*



where ε *is the evaluation map, i.e.,* $\varepsilon<a, l> = l(a)$, *and* Ψ *is the map defined by*

$$\Psi<a, l> = \begin{cases} (l(2a), *) & \textit{for} \quad 0 \leqq a \leqq 1/2, \\ (*, l(2a-1)) & \textit{for} \quad 1/2 \leqq a \leqq 1. \end{cases}$$

Moreover, Ψ *induces monomorphism of generalized homotopy groups, and therefore the totality of homotopy classes of comultiplications of* X *and the totality of homotopy classes of coretractions of* ε, *i.e., maps* $\gamma: X \to S\Omega X$ *satisfying* $\varepsilon\gamma \simeq 1$, *are in* 1 *to* 1 *correspondence. Finally, the homotopy fibre*)* *of* ε, *i.e., the fibre of* π_1, *is* $\Omega X * \Omega X$.

THEOREM 1.5. *Let* Φ_k: $W_{k-1}(S\Omega X) \to W_k(X)$ *be the map defined by*

$$\Phi_k(*,\dots, \underset{i-th}{<a, l>},\dots, *) = \begin{cases} (*,\dots, l(2a),\dots, *) & \textit{for} \quad 0 \leqq a \leqq 1/2, \\ (*,\dots, \underset{i-th}{*},\dots, l(2-2a)) & \textit{for} \quad 1/2 \leqq a \leqq 1. \end{cases}$$

Then, Φ_k *induces monomorphisms of generalized homotopy groups, and* W_{k-1} $(S\Omega X)$ *may be considered as the homotopy fibre of* ∇_k.

THEOREM 1.6. *Let* (X, μ'_X) *be an h-coloop, then* μ'_X *is homotopy coassociative if and only if the corresponding coretraction* γ *is a co-H-map.*

§2. A'_n-spaces

Let $(X, \mu', *)$ be a co-H-space. There are various ways of coassociating to define a map α: $X \to W_n(X)$ using μ' repeatedly. For $n=2$, there exists only one μ'; but for $n=3$, there are two ways, $(\mu' \vee 1)\mu'$ and $(1 \vee \mu')\mu'$; for $n=4$, there are 5 ways,.... Moreover, different ways of coassociating may define the same map, for example, $(\mu' \vee 1 \vee 1)(1 \vee \mu')\mu' = (1 \vee 1 \vee \mu')(\mu' \vee 1)\mu'$ $(=(\mu' \vee \mu')\mu')$: $X \to W_4(X)$.

For each α: $X \to W_n(X)$ we shall define a sequence σ_α of $(n-1)$ increasing integers by the following way.

For $n=2$, $\sigma_{\mu'} = \{1\}$. Assume that we have defined for n $(\geqq 2)$. Let

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*) In general, for a given map f: $X \to Y$, consider the following fibre space: $E_f = \{(x, W) \in X \times Y^I \mid w(0) = f(x)\}$, and the map p_f: $E_f \to Y$ defined by $p_f(x, W) = w(1)$; defining j: $X \to E_f$ by $j(x) = (x, *_{f(x)})$ and r: $E_f \to X$ by $r(x, w) = x$, we shall have $rj = 1$, $jr \simeq 1$, $p_f j = f$ and $fr \simeq p_f$. We call the fibre E_f of p_f the homotopy fibre of f. Notice that: $E_f = T_{\bar{p}, f}$, where $\bar{p}$: $\bar{P}Y \to Y$ is the well known path-space fibering (terminating at the base point).

$$\alpha = (1 \vee \cdots \vee \underset{k-th}{\mu'} \vee \cdots \vee 1)\alpha' \colon X \longrightarrow W_n(X) \longrightarrow W_{n+1}(X)$$

be a coassociating presentation of α, and $\sigma_{\alpha'} = \{i'_1, \dots, i'_{n-1}\}$. Put

$$(2.1) \qquad \begin{cases} i_\nu = 2i' & \text{for } \nu < k, \\ i_k = i'_{k-1} + i'_k & (i'_0 = 0 \text{ and } i'_n = 2^{n-1}), \\ i_\nu = 2i'_{\nu-1} & \text{for } \nu > k. \end{cases}$$

(Thus, σ_α corresponds to a process of taking successive midpoints in the interval $[0, 2^n]$.) As easily seen, α's and σ_α's are in 1 to 1 correspondence.

Now, let $\partial_k(n+1-s, s)$ be the parenthesizing $x_1 \cdots (x_k \cdots x_{k+s-1}) \cdots x_n$ of n-letters word $x_1 x_2 \cdots x_n$. For each $\sigma_\alpha = \{i_1, \dots, i_{n-1}\}$, we shall define a set of $(n-2)$-parenthesizings $\partial_{k_i}(n+1-s_i, s_i)$ by the followings:

(2.2.a) If $i_k - i_{k-1} = i_{k+1} - i_k = 2^\nu$ ($\nu = 0, 1, \dots,$ or 2^{n-2}) and $2^\nu | i_k$ but $2^{\nu+1} \nmid i_k$, then we say that $\{i_{k-1}, i_k, i_{k+1}\}$ defines $\partial_k(n-1, 2)$.

(2.2.b) If $\{i_{k-1}, \dots, i_{k+s-1}\}$ defines $\partial_k(n-s+1, s)$, $i_{k+s-1} - i_{k-1} = 2^\mu$, $2^\mu | i_{k+s-1}$ but $2^{\mu+1} \nmid i_{k+s-1}$, or if $\{i_k, \dots, i_{k+s}\}$ defines $\partial_{k+1}(n-s+1, s)$, $i_{k+s} - i_k = i_k - i_{k-1} = 2^\mu$, $2^\mu | i_k$ but $2^{\mu+1} \nmid i_k$, then we say that $\{i_{k-1}, i_k, \dots, i_{k+s}\}$ defines $\partial_k(n-s, s+1)$.

(2.2.c) If $\{i_{k-1}, \dots, i_{k+s-1}\}$ defines $\partial_k(n-s+1, s)$ and $\{i_{k+s-1}, \dots, i_{k+s+t-1}\}$ defines $\partial_{k+s}(n-t+1, t)$ and $i_{k+s+t-1} - i_{k+s-1} = i_{k+s-1} - i_{k-1} = 2^\mu$, $2^\mu | i_{k+s-1}$ but $2^{\mu+1} \nmid i_{k+s-1}$, then we say that $\{i_{k-1}, \dots, i_{k+s-1}, \dots, i_{k+s+t-1}\}$ defines $\partial_k(n-s-t, s+t)$.

Thus, to each α, we have defined the unique set of $(n-2)$-parenthesizings of the n-letters word, and then applying these $(n-2)$-parenthesizings we have a "complete" parenthesizing. On the other hand, these complete parenthesizings and vertices of Stasheff's complex K_n are in 1 to 1 correspondence.
Therefore, the totality of α's and the vertices set of K_n are in 1 to 1 correspondence.

Here, we recall the definition of K_n [10].

$$K_n = \{(t_1, \dots, t_{n-2}) \in I^{n-2} | \forall j,\ 2^j t_1 \cdots t_j \geqq 1\},\ n \geqq 2,$$

$$\partial K_n = L_n = \{(t_1, \dots, t_{n-2}) \in K_n | \exists j,\ 2^j t_1 \cdots t_j = 1 \text{ or } t_j = 1\}.$$

There exist face maps $\partial_k(r, s) \colon K_r \times K_s \to K_n$, $r+s = n+1$, $1 \leqq k \leqq r$, and degeneracy maps $s_j \colon K_n \to K_{n-1}$, $1 \leqq j \leqq n$, $n \geqq 3$, and these maps are subject to the following commutation laws:

$$(2.3.a) \qquad \partial_j(r, s+t-1)(1 \times \partial_k(s, t)) = \partial_{j+k-1}(r+s-1, t)(\partial_j(r, s) \times 1);$$

$$(2.3.b) \qquad \partial_{j+s-1}(r+s-1, t)(\partial_k(r, s) \times 1)$$

$$= \partial_k(r+t-1, s)(\partial_j(r, t)\times 1)(1\times T), \qquad j > k,$$

where T: $K_s\times K_t \to K_t\times K_s$ is the switching map;

(2.3.c) $$s_j s_k = s_k s_{j+1} \qquad \text{for} \quad k \leqq j;$$

(2.3.d) $$s_j\partial_k(r, s)$$

$$= \partial_{k-1}(r-1, s)(s_j\times 1) \qquad \text{for} \quad j<k \quad \text{and} \quad r>2,$$

$$= \partial_k(r, s-1)(1\times s_{j-k+1}) \qquad \text{for} \quad s>2,\ k\leqq j<k+s,$$

$$= \partial_k(r-1, s)(s_{j-s+1}\times 1) \qquad \text{for} \quad k+s\leqq j;$$

(2.3.e) $$s_j\partial_k(n-1, 2) = \pi_1 \quad \text{for} \quad 1\leqq j=k<n, \quad 1<j=k+1\leqq n,$$

$$s_1\partial_2(2, n-1) = s_n\partial_1(2, n-1) = \pi_2,$$

where π_1 and π_2 are projections onto the first and the second factors.

Since K_n is a convex cell complex which is homeomorphic to I^{n-2}, starting with s_1, s_2, s_3: $K_3\to K_2=\{*\}$ and using (2.3.d~e), we may define s_j by induction on n.

Now, we define the vertices transformations

$$\bar{\partial}_k(r, s)\colon K_r\times K_s \longrightarrow K_n\,,$$

for $1\leqq k\leqq r$ and $r+s=n+1$ by the following way:

(2.4) $$\bar{\partial}_k(r, s)(\xi, \eta)\colon X \xrightarrow[\xi]{} W_r(X) \xrightarrow[\eta(k)]{} W_n(X)\,,$$

for any $\xi\in K_r$ and $\eta\in K_s$, where $\eta(k)=1\vee\cdots\vee \underset{k-th}{\eta} \vee\cdots\vee 1$. If $\xi=\{\xi_1,\dots,\xi_{r-1}\}$ and $\eta=\{\eta_1,\dots,\eta_{s-1}\}$, then we have

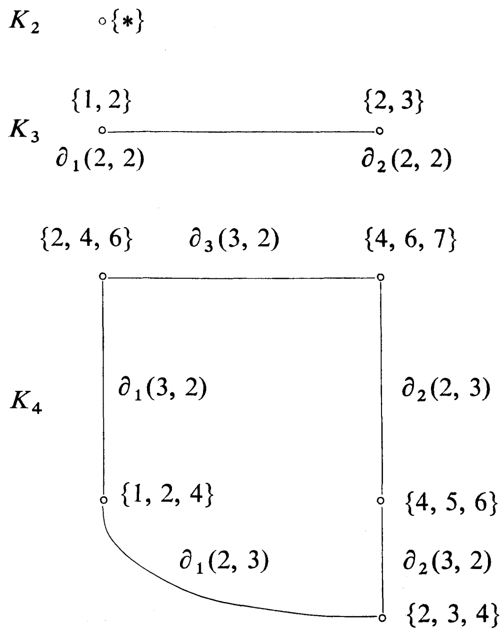
(2.5) $$\eta(k) = \{2^{s-1}\xi_1,\dots, 2^{s-1}\xi_{k-1}, a\eta_1+b,\dots, a\eta_{s-1}+b,$$
$$2^{s-1}\xi_k,\dots, 2^{s-1}\xi_{r-1}\}\,,$$

where $a=\xi_k-\xi_{k-1}$ and $b=2^{s-1}\xi_{k-1}$.

LEMMA 2.6. *$\bar{\partial}_k(r, s)$'s satisfy the commutation laws* (2.3.a) *and* (2.3.b).

Therefore, we may regard K_n as the cell complex defined by coassociatings. Fig. I shows K_2, K_3 and K_4.

Fig. 1



DEFINITION 2.7. A based space $(X, *)$ is said to admit an *A'_n-structure*, if there exist maps M'_i: $X \times K_i \to W_i(X)$, $2 \leqq i \leqq n$, satisfying the following conditions:

(2.7.1) μ': $X \to X \vee X$, defined by $\mu'(x) = M'_2(x, \{1\})$ for all $x \in X$, is a comultiplication, and $*$ is a counit;

(2.7.2) for any $(\rho, \sigma) \in K_r \times K_s$, $r+s=i+1$, it holds

$$M'_i(\ ;\partial_k(r, s)(\rho, \sigma)) = M'_s(\ ;\sigma)(k) \circ M'_r(\ ;\rho),$$

where $M'_s(\ ;\sigma)(k) = 1 \vee \cdots \vee \underset{k-th}{M'_s(\ ;\sigma)} \vee \cdots \vee 1$;

(2.7.3) for $i \geqq 3$, it holds $M'_{i-1}(\ ;s_j(\pi)) \simeq p_j M'_i(\ ;\tau)$, where $p_j = \nabla_2(j-1) \circ *(j) = \nabla_2(j) \circ *(j)$.

If X admits an A'_n-structure, we call X an *A'_n-space*. If X admits an A'_n-structure for every n, we say that X admits an *A'_∞-structure*.

DEFINITION 2.7′. A based space $(X, *)$ is said to admit a *w-A'_n-structure*, if in the above Definition 2.7, the condition (2.7.2) is replaced by:

(2.7.2′) there exist maps $\bar{\partial}_k(r, s) \simeq 1 \times \partial_k(r, s)$: $X \times K_r \times K_s \to X \times K_i$, $r+s=i+1$, and it holds

$$M'_i(\bar{\partial}_k(r, s)(x;(\rho, \sigma)) = M'_s(\ ;\sigma)(k) \circ M'_r(x;\rho)$$

for any $(x; (\rho, \sigma)) \in X \times K_r \times K_s$.

REMARK 2.8. If X is homotopically non-trivial, then X cannot be strictly coassociative, i.e., $(\mu' \vee 1)\mu' = (1 \vee \mu')\mu'$ does not hold. On the contrary, assume that μ' is strictly coassociative. Put $X_- = \mu'^{-1}(X \times \{*\})$, $X_+ = \mu'^{-1}(\{*\} \times X)$ and $X_0 = X_- \cap X_+$. Since X is homotopically non-trivial, we have $X_- - X_0 \neq \emptyset$ and $X_+ - X_0 \neq \emptyset$. Let x be an element of $X_+ - X_0$. Then, $(1 \vee \mu')\mu'(x)$ is of the form $(*, *, x')$. Thus, we have

$$(\mu' \vee 1)\mu'(X_+) \subset \{*\} \times \{*\} \times X,$$

and

$$p_1^\wedge p_3^\wedge(\mu' \vee 1)\mu' \simeq *.$$

On the other hand, since $*$ is the counit, we have

$$p_1^\wedge p_3^\wedge(\mu' \vee 1)\mu' \simeq 1,$$

which contradicts to non-triviality.

2.9. We recall the definition of A_n-form [10] before we give Theorem 2.10.

A based space (X, e) is said to admit an *A_n-form* if there exist maps M_i: $X^i \times K_i \to X$ for $2 \leqq i \leqq n$ satisfying the following conditions:

(2.9.1) $M_2(e, x; \{1\}) = M_2(x, e; \{1\}) = x$ for all $x \in X$;

(2.9.2) for any $(\rho, \sigma) \in K_r \times K_s$, $r+s=i$, we have

$$\begin{aligned} &M_i(x_1,\dots, x_i; \partial_k(r, s)(\rho, \sigma)) \\ &\quad = M_r(x_1,\dots, x_{k-1}, M_s(x_k,\dots, x_{k+s-1}; \sigma), x_{k+s},\dots, x_i; \rho); \end{aligned}$$

(2.9.3) for $\tau \in K_i$, $i>2$, we have

$$\begin{aligned} &M_i(x_1,\dots, x_{j-1}, e, x_{j+1},\dots, x_i; \tau) \\ &\quad = M_{i-1}(x_1,\dots, x_{j-1}, x_{j+1},\dots, x_i; s_j(\tau)). \end{aligned}$$

THEOREM 2.10. *Let X be a finite CW-complex, then the following two conditions are equivalent:*

(2.10.1) *X admits an A'_n-structure.*

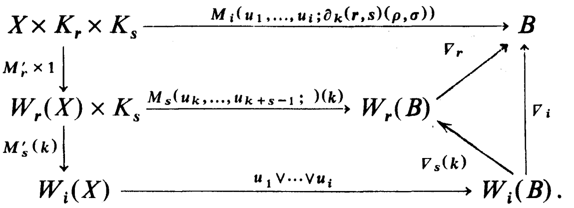
(2.10.2) *For any based CW-complex B, the mapping space $\{X; B\}$ admits a natural (i.e., functorial) A_n-form.*

PROOF. (2.10.1) implies (2.10.2). Suppose $(X, *)$ has an A'_n-structure $\{M'_i\}$, $2 \leqq i \leqq n$. For any B, define M_i: $\{X; B\} \to \{X; B\}$ by

(2.11) $$[M_i(u_1,\dots, u_i; \tau)](x) = \nabla_i(u_1 \vee \cdots \vee u_i)M'_i(x; \tau),$$

for any $(u_1,\dots, u_i; \tau) \in \{X; B\}^i \times K_i$. Let e: $X \to *$, then we have $M_2(u, e; \{1\})$

$\simeq u \simeq M_2(e, u; \{1\})$, but since we work in $\boldsymbol{CW}$, we may assume that $M_2(u, e; \{1\}) = u = M_2(e, u; \{1\})$. Let $(\rho, \sigma) \in K_r \times K_s$, $r+s=i+1$, then (2.7.2) implies (2.9.2) as in the diagram below:



Similarly, (2.7.3) implies (2.9.3).

(2.10.2) implies (2.10.1). Put $\mu' = i'_1 + i'_2 \in \{X; X \vee X\}$, then we have $j\mu' = i_1 + i_2 \simeq \Delta$, where i_1 and i_2 are the inclusion maps of X into the first and the second factors. Thus, μ' is a comultiplication, i.e., (2.7.1) is satisfied.

Define $M'_v\colon X \times K_v \to W_v(X)$ by

$$M'_v(x; \tau) = [M_v(i'_1, \dots, i'_v; \tau)](x).$$

Then, for any $(\rho, \sigma) \in K_r \times K_s$, $r+s=v+1$, we have

$$\begin{aligned} & M'_v(x; \partial_k(r, s)(\rho, \sigma)) \\ &= [M_r(i'_1, \dots, i'_{k-1}, M_s(i'_k, \dots, i'_{k+s-1}; \sigma), i'_{k+s}, \dots, i'_v; \rho)](x) \\ &= M'_s(\ ; \sigma)(k) \circ M'_r(x; \rho). \end{aligned}$$

Thus, we have (2.7.2). Similarly, we have (2.7.3).

Proposition 2.12. *SX admits an A'_∞-structure.*

Proof. For any vertex $\alpha = \{i_1, \dots, i_{n-1}\} \in K_n$ and $<t, x> \in SX$, put

$$\tilde{M}'_n(<t, x>; \alpha) = (*, \dots, <(2^{n-1}t - i_{k-1})/(i_k - i_{k-1}), x>, \dots, *),$$

for $i_{k-1} \leqq t2^{n-1} \leqq i_k$, $k = 1, 2, \dots, n$.

Since K_n may be regarded as a convex polyhedron which is a cone over L_n, it can be triangulated adding suitable vertices; then $M'_n\colon SX \to W_n(SX)$ can be defined as a linear extension of $\tilde{M}'_n$.

§3. A'_n-maps and Mapping Cones

At first we shall fix a notation. Let $F\colon X \times I \to Y$ be a homotopy satisfying $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Then, we shall denote F by $H(f_0, f_1)$.

DEFINITION 3.1. A map $f: X \to Y$ of A'_n-spaces is a *homomorphism* if $W_i(f)M'_{X,i} = M'_{Y,i}(f \times 1)$ holds for any $i \leqq n$.

For example, $Sf: SX \to SY$ is a homomorphism (with respect to μ'_0) for any map $f: X \to Y$.

DEFINITION 3.2. A map $f: X \to Y$ of A'_n-spaces is an A'_n-*maps*, provided that there exist homotopies

$$H'_i = H(W_i(f)M'_{X,i},\ M'_{Y,i}(f \times 1)): X \times K_i \times I \longrightarrow W_i(Y), \quad 2 \leqq i \leqq n,$$

which are subject to the following conditions:

(3.2.1) for any $\partial_k(r, s)$, $r+s=i+1$, there exists a homeomorphism $\tilde{\partial}_k(r, s)$ of $K_r \times K_s \times I$ into $K_i \times I$ which preserves level and satisfies

$$H'_i(\tilde{\partial}_k(r, s)((x; (\rho, \sigma)), t) =$$

$$\begin{cases} H'_s((\ ;\sigma), ((2^{i-1}-1)t)/(2^{s-1}-1))(k)\circ M'_{X,r}(x; \rho) \\ \qquad\qquad\qquad \text{for} \quad 0 \leqq t \leqq (2^{s-1}-1)/(2^{i-1}-1), \\ M'_{Y,s}(\ ;\sigma)(k)\circ H'_r((x; \rho), ((2^{i-1}-1)t+1-2^{s-1})/(2^{i-1}-2^{s-1})) \\ \qquad\qquad\qquad \text{for} \quad (2^{s-1}-1)/(2^{i-1}-1) \leqq t \leqq 1, \end{cases}$$

for any $(\rho, \sigma) \in K_r \times K_s$, $i \geqq 3$;

(3.2.2) there exist homotopies

$$F'_R = H(fE'_{X,R} \dotplus p_{\hat{1}}H'_2,\ E'_{Y,R}f),$$

$$F'_L = H(fE'_{X,L} \dotplus p_{\hat{2}}H'_2,\ E'_{Y,L}f),$$

where $E'_{X,R} = H(1_X, p_{\hat{1}}\mu'_X)$ and $E'_{X,L} = H(1_X, p_{\hat{2}}\mu'_X)$ and dotted plus $\dotplus$ implies addition with respect to homotopy parameter;

(3.2.3) there exist homotopies

$$H(p_j H'_i \dotplus D'_{Y,i,j}(f \times 1),\ W_{i-1}(f)D'_{X,i,j} \dotplus H'_{i-1}(1 \times s_j)):$$

$$X \times K_i \times I \times I \longrightarrow W_{i-1}(Y),$$

where $D'_{X,i,j} = H(p_j M'_{X,i},\ M'_{X,i-1}(1 \times s_j))$ and so on.

REMARK 3.3. 1) Homeomorphisms $\tilde{\partial}_k(r, s)$'s are very complicated. For $i=4$, $\tilde{\partial}_1(3, 2)$ is given in the following Fig. 2.

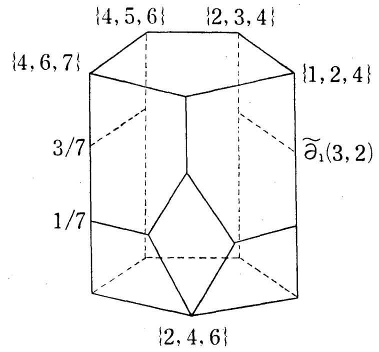


Fig. 2

2) Define $D'_X=H(\Delta, j\mu'_X)$ by $D'_X(x, t)=(E'_{X,L}(x, t), E'_{X,R}(x, t))$, then (3.2.2) is equivalent to

(3.2.2′) there exists a homotopy $F=H((f\times f)D'_X\dotplus j_Y\circ H'_2, D'_Yf)$, which is also equivalent to

(3.2.2″) there exists a homotopy $G: X\times I\times I\to Y\times Y$ satisfying the following conditions

$$G(x, t, 0) = (f\times f)D'_X(x, t),\ G(x, t, 1) = D'_Y(f(x), t),$$

$$G(x, 1, s) = j_YH'_2(x, s) \quad\text{and}\quad G(x, 0, s) = \Delta_Yf(x).$$

DEFINITION 3.2′. A map $f: X\to Y$ of A'_n-spaces is an *quasi-A'_n-map* (abb. q-A'_n-map) if homotopies H_i satisfy only the condition (3.2.1).

LEMMA 3.4. *If X is an h-coloop, then ν'_R and ν'_L are homotopy equivalences.*

PROOF. Since $1+\nu'_R\simeq 0$, we have $\nu'_L+\nu'_L\nu'_R\simeq 0$; then by the cancellation law we have $\nu'_L\nu'_R\simeq 1$. Similarly, we have $\nu'_R\nu'_L\simeq 1$.

LEMMA 3.5. *Let $f: X\to Y$ be an A'_2-map of h-coloops, then $\nu'_Yf\simeq f\nu'_X$.*

PROOF. We shall obtain

$$f+\nu'_{Y,R}f \simeq \nabla(1\vee \nu'_{Y,R})\mu'_Yf \simeq 0,$$

$$f+f\nu'_{X,R} = f\nabla(1\vee \nu'_{X,R})\mu'_X \simeq 0.$$

Then, by the cancellation law, we have $\nu'_{Y,R}f\simeq f\mu'_{X,R}$.

NOTATIONS 3.6. (i) $N'_R(f)=H(\nu'_{Y,R}f, f\nu'_{X,R})$.

(ii) $\bar{N}'_R(f)=H(f+f\nu'_{X,R}, f+\nu'_{Y,R}f)$

$$=fN'_{X,R}\dot{-}N'_{Y,R}f\dot{-}(1\vee \nu'_{Y,R})H'_2,$$

where $N'_{X,R}=H(\nabla(1 \vee \nu'_{X,R})\mu'_X, *)$ and so on.

PROPOSITION 3.7. *Let $f\colon X\to Y$ be an A'_n-map of A'_n-spaces, then the mapping cone C_f has a canonical w-A'_n-structure, i.e., the inclusion map $i\colon Y\to C_f$ is a homomorphism.*

PROOF. Let $\{M'_{X,i}\}_{2\leqq i\leqq n}$ and $\{M'_{Y,i}\}_{2\leqq i\leqq n}$ be A'_n-structures of X and Y, respectively. Define $M'_i\colon C_f\times K_i\to W_i(C_f)$ by

$$M'_i(y;\tau) = M'_{Y,i}(y;\tau) \quad \text{for} \quad (y;\tau)\in Y\times K_i,$$

$$M'_i((t,x);\tau) = \begin{cases} (2^{i-1}t,\ M'_{i,X}(x;\tau)) & \text{for} \quad (t,x)\in CX,\ 0\leqq t\leqq 1/2^{i-1}, \\ H'_i((x;\tau),\ (2^{i-1}t-1)/(2^{i-1}-1)) & \text{for} \quad (t,x)\in CX, \\ & \qquad 1/2^{i-1}\leqq t\leqq 1. \end{cases}$$

Next, define $D'\colon C_f\times I\to C_f\times C_f$ by

$$D'(y,s) = D'_Y(y,s),$$

$$D'((t,x),s) = \begin{cases} (2t/(2-s),\ D'_X(x,s)) & \text{for} \quad 0\leqq t\leqq (2-s)/2, \\ G(x,s,(2t+s-2)/2) & \text{for} \quad (2-s)/2\leqq t\leqq 1, \end{cases}$$

where D's and G in the right hand sides are homotopies defined in 3.2 and 3.3. Then, we shall have $D'=H(\Delta, j\mu')$ for $\mu'=M'_2$; thus μ' is a comultiplication, i.e., (2.7.1) is satisfied.

To examine the condition (2.7.2′), we shall define the maps $\bar{\partial}_k(r,s)\colon C_f\times K_r\times K_s \longrightarrow C_f\times K_i$, $r+s=i+1$, by

$$\bar{\partial}_k(r,s)(y;(\rho,\sigma)) = (y;\partial_k(r,s)(\rho,\sigma)) \quad \text{for} \quad (y;(\rho,\sigma))\in Y\times K_r\times K_s,$$

$$\bar{\partial}_k(r,s)((t,x);(\rho,\sigma)) = \begin{cases} ((t,x);\partial_k(r,s)(\rho,\sigma)) & \text{if} \quad i\leqq 3 \quad \text{or} \quad t\leqq 1/2^{i-1}, \\ (x;\bar{\partial}_k(r,s)(\rho,\sigma),t) & \text{if} \quad i\geqq 4 \quad \text{and} \quad t\geqq 1/2^{i-1}. \end{cases}$$

As easily seen, (2.7.2′) holds for any $(y;(\rho,\sigma))$ and $((t,x);(\rho,\sigma))$, $0\leqq t\leqq 1/2^{i-1}$. If $t\geqq 1/2^{i-1}$, put $t'=(2^{i-1}t-1)/(2^{i-1}-1)$, then we have

$$(2^{i-1}-1)t'/(2^{s-1}-1) = (2^{i-1}t-1)/(2^{s-1}-1) \quad \text{for} \quad 1/2^{i-1}\leqq t\leqq 1/2^{r-1},$$

$$((2^{i-1}-1)t'+1-2^{s-1})/(2^{i-1}-2^{s-1}) = (2^{r-1}t-1)/(2^{r-1}-1) \quad \text{for} \quad 1/2^{r-1}\leqq t\leqq 1.$$

Thus (2.7.2′) is a direct consequence of (3.2.1). The remaining conditions may be obtained easily.

REMARK 3.8. For $n \leqq 3$, C_f admits an A'_n-structure. Moreover the projection $p: C_f \to SX$ is an A'_n-map.

PROPOSITION 3.9. *Let $f: X \to Y$ be an A'_3-map of A'_3-cogroups, then C_f is also an A'_3-cogroup.*

PROOF. It is sufficient to show that C_f admits a coinversion ν'. Let ν'_X and ν'_Y be coinversions of X and Y, respectively. Define $\nu': C_f \to C_f$ by

$$\nu'(y) = \nu'_Y(y),$$

$$\nu'(t, x) = \begin{cases} (2t, \nu'_X(x)) & \text{for } (t, x) \in CX \text{ and } 0 \leqq t \leqq 1/2, \\ N'(f)(x, 2t-1) & \text{for } (t, x) \in CX \text{ and } 1/2 \leqq t \leqq 1. \end{cases}$$

Then, $\nabla(1 \vee \nu')\mu': C_f \to C_f$ is homotopic to the map $\tilde{\nu}'$ of the following form:

$$\tilde{\nu}'|\{(t, x)|0 \leqq t \leqq 1/4\} = (4t, \nabla(1 \vee \nu'_X)\mu'_X(x)),$$

$$\tilde{\nu}'|\{(t, x)|1/4 \leqq t \leqq 1/2\} = \bar{N}'(f)(x, 4t-1),$$

$$\tilde{\nu}'|\{(t, x)|1/2 \leqq t \leqq 1\} = \nabla(1 \vee \mu'_Y)H'_2(x, 2t-1),$$

$$\tilde{\nu}'|Y = \nabla(1 \vee \nu'_Y)\mu'_Y.$$

Since $\bar{N}'(f) + \nabla(1 \vee \nu'_Y)H'_2 \simeq fN'_X \dot{-} N'_Y f \simeq *$, we obtain $\tilde{\nu}' \simeq *$.

§4. Some Invariants

Given a map $f: X \to Y$ of A'_n-spaces, it will be the first problem to determine whether or not f is an A'_2-map, i.e., f satisfies

$$(f \vee f)\mu'_X \simeq \mu'_Y f \tag{4.1}$$

and then

$$(f \times f)D'_X + j_Y H'_2 \simeq D'_Y f. \tag{4.2}$$

If both X and Y are suspended spaces, say $X = SA$ and $Y = SB$, then so called Hopf-homomorphisms $H_k: [SA; SB] \to [SA; S(\overset{k}{\wedge} B)]$ are useful to show (4.1), especially if both X and Y are spheres, only H_2 and H_3 are necessary (cf. [4]).

LEMMA 4.3. *Let X be an h-coloop, then for any space Y, we have the following exact sequence of loops:*

$$0 \longrightarrow [CX, X; Y \times Y, Y \vee Y] \xrightarrow[r_*]{} [X; Y \vee Y] \xrightarrow[j_*]{} [X; Y \times Y] \longrightarrow 0.$$

PROOF. Let μ'_X be the comultiplication of X, then $\mu'(t, x)=(t, \mu'_X(x))$ *) gives an A'_2-structure of CX. Then, proof may be carried out by the routine way as in homotopy groups.

Now, let f: $X\to Y$ be a given map of h-coloops, and X be a finite CW-complex, then (4.1) is equivalent to

$$\alpha(f) = (f \vee f)\mu'_X - \mu'_Y f \simeq *.$$

Obviously, $j_*[\alpha(f)]=0$; therefore we have the unique element $[g]\in[CX, X; Y\times Y, Y\vee Y]$ such that $r_*[g]=[\alpha(f)]$ holds. Moreover, we have isomorphisms (cf. [3])

$$[CX, X; Y\times Y, Y\vee Y] \approx [X; \Omega_{Y\times Y}(*, Y\vee Y)]$$
$$\approx [X; \Omega Y * \Omega Y].$$

DEFINITION 4.4. Let $H(f)$ be the image of $[g]$ under the composition of the above isomorphisms.

If $Y=SB$, then $H(f)\in[X; \Omega SB*\Omega SB]=[X; S(\Omega SB\wedge\Omega SB)]\approx[X; S(B_\infty \wedge B_\infty)]$, where B_∞ denotes the reduced product of B. Thus, we may consider $H(f)$ as a modification of generalized Hopf homomorphisms.

By definition, we have

PROPOSITION 4.5. *f satisfies* (4.1) *if and only if* $H(f)=0$.

REMARK 4.6. Being f an A'_2-map, f has to satisfy the condition (4.2). Generally, for a q-A'_2-map f of an h-coloop X to an A'_2-space Y, we may define functions Υ_L and Υ_R of Ker. H into $[SX; Y]$, and their vanishing is equivalent to the condition (4.2). Moreover, we may show that any q-A'_2-map f defined on a suspended space is an A'_2-map, (cf. [9]).

REMARK 4.7. Define $H_*(f)\in[X; S\Omega Y]$ by $H_*(f)=[S\Omega f\circ\gamma_X-\gamma_Y\circ f]$, then we shall have $\Psi_*H_*(f)=i'_*H(f)$, where Ψ: $S\Omega Y\to Y\vee Y$ and i': $\Omega Y*\Omega Y\to Y\vee Y$ are maps defined in § 1 and Ψ^* and i'_* are monomorphisms.

DEFINITION 4.8. A co-H-space (X, μ'_X) is said to be *homotopy-cocommutative* if it holds $T\mu'_X\simeq\mu'_X$: $X\to X\vee X$.

PROPOSITION 4.9. *Let (A, μ'_A) and (B, μ'_B) be co-H-spaces, and X be the smash product of them, then we have*

i) *$\mu'_1=\mu'_A\wedge 1_B$ and $\mu'_2=1_A\wedge\mu'_B$ are comultiplications of X,*

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*) More generally, if X is a co-H-space with comultiplication μ'_X, then for any space Y, $X\wedge Y$ is a co-H-space with comultiplication $\mu'_X\wedge 1_Y$.

ii) μ'_1 *is homotopic to* μ'_2; *therefore they define a unique comultiplication of* X, *and finally,*

iii) μ'_X *is homotopy-cocommutative.*

PROOF. i) $E'_{A,R} \wedge 1_B$ gives a homotopy from 1_X to $\nabla(1 \vee *)\mu'_1$ and $N'_{A,R} \wedge 1_B$ gives a homotopy from $\nabla(1 \vee \nu'_{1,R})\mu'_1$ to $*$, where $\nu'_{1,R} = \nu'_{A,R} \wedge 1_B$. Notice that $*: X \to *_X$ is the common counit of μ'_1 and μ'_2.

ii) As easily seen, it holds

$$(\mu'_2 \vee \mu'_2)\mu'_1 = (1 \vee T \vee 1)(\mu'_1 \vee \mu'_1)\mu'_2.$$

Then applying $(\nabla \vee \nabla)(1 \vee * \vee * \vee 1)$ by the left-hand side, we have $\mu'_1 \simeq \mu'_2$.

iii) Since $\mu'_X = i'_1 + i'_2$, we shall obtain

$$\mu'_X \simeq (* \underset{1}{+} i'_1) \underset{2}{+} (i'_2 \underset{1}{+} *) \simeq i'_2 \underset{2}{+} i'_1 = T\mu'_X.$$

THEOREM 4.10. *If* X *is a homotopy cocommutative h-cogroup, i.e.,* A'_3-*space with coinversion, then*

$$H: [X;\, Y] \longrightarrow [X;\, \Omega Y * \Omega Y]$$

is a homomorphism.

PROOF. It is sufficient to show that

$$[X;\, Y] \ni [f] \longrightarrow [\alpha(f)] \in [X;\, Y \vee Y]$$

is a group-homomorphism.

At first, we shall mention that

$$\mu'_Y \nabla = \nabla_{Y \vee Y}(\mu'_Y \vee \mu'_Y).$$

Then, we have

$$\begin{aligned}\mu'_Y(f_1 + f_2) &= \nabla_{Y \vee Y}(\mu'_Y f_1 \vee \mu'_Y f_2)\mu'_X \\ &= \mu'_Y f_1 + \mu'_Y f_2.\end{aligned}$$

Using homotopy-coassociativity and -cocommutativity, we shall have

$$\begin{aligned}&((f_1 + f_2) \vee (f_1 + f_2))\mu'_X \\ &\qquad \simeq \nabla_{Y \vee Y}((f_1 \vee f_1) \vee (f_2 \vee f_2))(\mu'_X \vee \mu'_X)\mu'_X.\end{aligned}$$

On the other hand, since it holds

$$(f_i \vee f_i)\mu'_X \simeq \alpha(f_i) + \mu'_Y f_i, \qquad i = 1, 2,$$

we have

$$\begin{aligned}
&((f_1+f_2)\vee(f_1+f_2))\mu'_X \\
&\simeq \nabla_{Y\vee Y}((\alpha(f_1)+\mu'_Y f_1)\vee(\alpha(f_2)+\mu'_Y f_2))\mu'_X \\
&\simeq \nabla_{Y\vee Y}(\nabla_{Y\vee Y}\vee\nabla_{Y\vee Y})(\alpha(f_1)\vee\alpha(f_2)\vee\mu'_Y f_1\vee\mu'_Y f_2)(\mu'_X\vee\mu'_X)\mu'_X \\
&= (\alpha(f_1)+\alpha(f_2))+\mu'_Y(f_1+f_2).
\end{aligned}$$

Therefore, we shall obtain

$$\alpha(f_1+f_2)\simeq\alpha(f_1)+\alpha(f_2).$$

EXAMPLES 4.10. Our invariant $H(f)$ is not necessarily easy to determine its vanishing, however, in some cases we can do it.

(4.10.1) If $\alpha\in\pi_6(S^3)$ is an element of order 3, $H(f)$ belongs to $\pi_6(\Omega S^3 * \Omega S^3)\approx\pi_6(S^5)\approx \mathbf{Z}_2$, then we have $H(\alpha)=0$ by Theorem 4.9.

(4.10.2) If $\beta\in\pi_{15}(S^5)$ is an element of order 9, then $H(\beta)$ belongs to $\pi_{15}(\Omega S^5 * \Omega S^5)\approx\pi_{15}(S^9\cup e^{13})$. Since there exists an exact sequence

$$\pi_{15}(S^9)\longrightarrow\pi_{15}(S^9\cup e^{13})\longrightarrow\pi_{15}(S^{13})$$

and $\pi_{15}(S^9)\approx \mathbf{Z}_2\approx\pi_{15}(S^{13})$, we shall obtain $H(\beta)=0$.

(4.10.3) Let ξ be the non-zero element of $[S^3\cup_\alpha e^7; S^5]\approx\pi_7(S^5)\approx \mathbf{Z}_2$, then $H(\xi)$ belongs to $[S^3\cup_\alpha e^7; \Omega S^5 * \Omega S^5][S^3\cup_\alpha e^7; S^9]=0$; therefore we have $H(\xi)=0$. The same argument holds for $\xi'\in[S^3\cup_\alpha e^7; S^6]$.

(4.10.4) Let ξ be the non-zero element of $[S^3\cup_\alpha e^7; S^5\cup_\beta e^{16}]\approx\pi_7(S^5)\approx Z_2$, then $H(\xi)$ belongs to $[S^3\cup_\alpha e^7; \Omega(S^5\cup_\beta e^{16}) * \Omega(S^5\cup_\beta e^{16})]\approx\pi_7(S^9)=0$; therefore we have $H(\xi)=0$.

§5. A'_4-spaces and q-A'_3-maps

Theorem 1.4 says that the homotopy classes of comultiplications of X are in 1 to 1 correspondence with the homotopy classes of coretractions. Therefore we may give guess that the coretraction $\gamma: X\to S\Omega X$ may characterize A'_n-structure of X.

At first we shall make a remark: let X be an A'_3-cogroup with the A'_3-structure $\{\mu'_X, M'_{X,3}\}$, then by Theorem 1.6, the corresponding coretraction γ is an q-A'_2-map, which defines a new A'_3-structure $\{\mu'_X, M'_{X,3}\}$, but we have no guarantee that $M'_{X,3}$ and $M''_{X,3}$ are homotopic relative $X\times\dot{K}_3$.

DEFINITION 5.1. An A'_4-cogroup X is said to be an *s-A'_4-cogroup*, provided that $W_4(\varepsilon)(1\wedge\nu'_0\vee 1\vee\nu'_0)(1\vee 1\vee\mu'_0)M'_{0,3}\circ(\gamma\times 1)$ is homotopic to $(1\vee\nu'\vee 1\vee\nu')(1\vee 1\vee\mu')M'_{X,3}$ relative to $X\times\dot{K}_3$, i.e., the homotopy satisfies the condition induced from (3.2.1).

As easily seen, any suspended space is an s-A'_4-cogroup with respect to its natural A'_4-structure.

Proposition 5.2. *Let X be an A'_3-cogroup such that the corresponding coretraction γ is an q-A'_3-map, then X is an s-A'_4-cogroup.*

Proof. Let $\{M'_{0,i}\}$ be the natural A'_4-structure of $S\Omega X$. Define H'_4: $X \times K_4 \times \{1\} \cup X \times L_4 \times I \to W_4(X)$ by the followings:

$$\tilde{H}'_4|X \times K_4 \times \{1\} = W_4(\varepsilon)\cdot M'_{0,4}(\gamma \times 1);$$

$$\tilde{H}'_4|X \times \bar{\partial}_k(K_3 \times K_2 \times I) =$$

$$\begin{cases} W_4(\varepsilon)\cdot H'_2(\ ;7t)(k)\cdot M'_{X,3} & \text{for } 0 \leqq t \leqq 1/7, \\ W_4(\varepsilon)\cdot \mu'_0(k)\cdot H'_3(\ ;(7t-1)/6) & \text{for } 1/7 \leqq t \leqq 1, \end{cases} \quad k = 1, 2, 3;$$

$$H'_4|X \times \bar{\partial}_k(K_2 \times K_3 \times I) =$$

$$\begin{cases} W_4(\varepsilon)H'_3(\ ;7t/3)(k)M'_{X,2} & \text{for } 0 \leqq t \leqq 3/7, \\ W_4(\varepsilon)\mu'_0(k)H'_2(\ ;(7t-3)/4) & \text{for } 3/7 \leqq t \leqq 1, \end{cases} \quad k = 1, 2.$$

The remaining part of $L_4 \times I$ is the tetragon $T = P_0P'_1P_3P''_1$ in the Fig. 3.

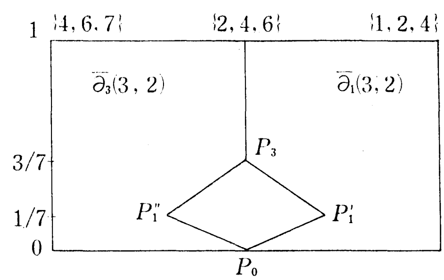


Fig. 3

On the edge of T, $\tilde{H}'_4$ is of the following forms:

$$\tilde{H}'_4|P_0P'_1 = W_4(\varepsilon)(H'_2(\ ;7t) \vee \gamma \vee \gamma)(1 \vee \mu'_X)\mu'_X \qquad \text{for } 0 \leqq t \leqq 1/7;$$

$$\tilde{H}'_4|P'_1P_3 = W_4(\varepsilon)(\mu'_0 \vee 1 \vee 1)(\gamma \vee H'_2(\ ;(7t-1)/2)\mu'_X$$

$$\text{for } 1/7 \leqq t \leqq 3/7;$$

$$\tilde{H}'_4|P_0P''_1 = W_4(\varepsilon)(\gamma \vee \gamma \vee H'_2(\ ;7t)(\mu'_X \vee 1)\mu'_X \qquad \text{for } 0 \leqq t \leqq 1/7;$$

$$H'_4|P''_1P_3 = W_4(\varepsilon)(1 \vee 1 \vee \mu'_0)(H'_2(\ ;(7t-1)/2) \vee \gamma)\mu'_X$$

for $\quad 1/7 \leqq t \leqq 3/7$.

Now, put

$$\tilde{H}'_4|P_0P_3 = W_4(\varepsilon)(H'_2(\ ;7t/3)H'_2(\ ;7t/3))\mu'_X \qquad \text{for} \quad 0 \leqq t \leqq 3/7.$$

Then, we shall have $\tilde{H}'_4|P_0P_3 \simeq \tilde{H}'_4|P_0P'_1P_3$ and $\tilde{H}'_4|P_0P_3 \simeq \tilde{H}'_4|P_0P''_1P_3$. In fact, put

$$H'_{2,L}(\ ;t,s) = \begin{cases} H'_2(\ ;7t/(3-2s)) & \text{for} \quad 0 \leqq t \leqq (3-2s)/7, \\ \mu'_0 \cdot \gamma & \text{for} \quad (3-2s)/7 \leqq t \leqq 1; \end{cases}$$

$$H'_{2,R}(\ ;t,s) = \begin{cases} (\gamma \vee \gamma)\mu'_X & \text{for} \quad 0 \leqq t \leqq s/7, \\ H'_2(\ ;(7t-s)/(3-s)) & \text{for} \quad s/7 \leqq t \leqq 1; \end{cases}$$

$$F(\ ;t,s) = W_4(\varepsilon)(H'_{2,L}(\ ;t,s) \vee H'_{2,R}(\ ;t,s))\mu'_4.$$

Then, F is a homotopy from $H'_4|P_0P_3$ to $H'_1|P_0P'_4P_3$. Similarly, we may define a homotopy F' from $H'_4|P_0P_3$ to $H|''_1P_0P'_4P_3$. These homotopies define $H'_4|T$.

Let M'_4 be the extension of H'_4 over $X \times K_4 \times I$, and put $M'_{X,4} = M'_4|X \times K_4 \times \{0\}$, then $M'_{X,4}\colon X \times K_4 \to W_4(X)$ together with $\{\mu'_X, M'_{X,3}\}$ gives an A'_4-structure on X.

The following homotopy-commutative diagram shows that X is an s-A'_4-cogroup:

$$\begin{array}{ccccccc}
S\Omega X \times K_3 & \xrightarrow{M'_{0,3}} & W_3(S\Omega X) & \xrightarrow{1 \vee 1 \vee \mu'_0} & W_4(S\Omega X) & \xrightarrow{1 \vee \nu'_0 \vee 1 \vee \nu'_0} & W_4(S\Omega X) \\
\uparrow{\scriptstyle \gamma \times 1} & H'_3 & \uparrow{\scriptstyle W_3(\gamma)} & H'_2(3) & \uparrow{\scriptstyle W_4(\gamma)} & N'(\gamma)(2,4) & \uparrow{\scriptstyle W_4(\gamma)} \\
X \times K_3 & \xrightarrow[M'_{X,3}]{} & W_3(X) & \xrightarrow[1 \vee 1 \vee \mu']{} & W_4(X) & \xrightarrow[1 \vee \nu \vee 1 \vee \nu]{} & W_4(X)
\end{array}$$

To prove the converse of Proposition 5.2, we shall need certain computative lemmas.

LEMMA 5.3. *Let X be an A'_2-cogroup, and define $\rho\colon W_6(X) \to W_4(X)$ by the composition $\rho = (1 \vee \nabla \vee 1 \vee 1)(1 \vee T \vee 1 \vee 1)(1 \vee \nabla \vee 1 \vee 1 \vee 1)$, then we have*

$$\Phi_4(1 \vee \nu'_0 \vee \nu'_0) = (1 \vee 1 \vee T)\rho(\Phi_2 \vee \Phi_2 \vee \Phi_2).$$

PROOF. Put $\tilde{\Phi}_2 = \Phi_2\nu'_0\colon S\Omega X \to X \vee X$, then we shall have

$$\Phi_3 = (1 \vee T)(1 \vee \nabla \vee 1)(\Phi_2 \vee \tilde{\Phi}_2),$$

$$\Phi_4 = (1 \vee 1 \vee T)(1 \vee 1 \vee \nabla \vee 1)(\Phi_3 \vee \tilde{\Phi}_2).$$

Therefore, we have

$$\begin{aligned}\Phi_4 &= (1\vee 1\vee T)(1\vee 1\vee \nabla\vee 1)(\Phi_3\vee\tilde{\Phi}_2)\\ &= (1\vee 1\vee T)(1\vee 1\vee \nabla\vee 1)((1\vee T)(1\vee\nabla\vee 1)(\Phi_2\vee\tilde{\Phi}_2))\vee\tilde{\Phi}_2)\\ &= (1\vee 1\vee T)\rho(\Phi_2\vee\Phi_2\vee\Phi_2)(1\vee\nu'_0\vee\nu'_0).\end{aligned}$$

Since $(\nu'_0)^2=1$, we have the desired result.

LEMMA 5.4. *Let X be an A'_4-cogroup, then we have*

(5.4.1) $\rho(\bar{\mu}'_X\vee\bar{\mu}'_X\vee\bar{\mu}'_X)M'_{X,3}\simeq(1\vee\nu'_X\vee 1\vee\nu'_X)(1\vee 1\vee\mu'_X)M'_{X,3}$,

where $\bar{\mu}'_X=(1\vee\nu'_X)\mu'_X$.

PROOF. $\rho(\bar{\mu}'_X\vee\bar{\mu}'_X\vee\bar{\mu}'_X)M'_{X,3}$

$\simeq\rho(1\vee\nu'_X\vee 1\vee\nu'_X\vee 1\vee\nu'_X)(1\vee\mu'_X\vee 1\vee 1\vee 1)(\mu'_X\vee 1\vee 1\vee 1)(\mu'_X\vee 1\vee 1)(-M'_{X,3})$

(by A'_3 and A'_4, $(-M'_{X,3})(x,t)=M'_{X,3}(x,1-t)$)

$\simeq(1\vee 1\vee\nabla\vee 1)(1\vee T\vee 1\vee 1)(1\vee *\vee\nu'_X\vee 1\vee\nu'_X)(\mu'_X\vee 1\vee 1\vee 1)(\mu'_X\vee 1\vee 1)$

$\circ(-M'_{X,3})$

$\simeq(1\vee 1\vee\nabla\vee 1)(1\vee\nu'_X\vee *\vee 1\vee\nu'_X)(1\vee\mu'_X\vee 1\vee 1)(1\vee\mu'_X\vee 1)(-M'_{X,3})$

$\simeq(1\vee\nu'_X\vee 1\vee\nu'_X)(1\vee\mu'_X\vee 1)(-M'_{X,3})$

$\simeq(1\vee\nu'_X\vee 1\vee\nu_X)M'_{X,3}$.

LEMMA 5.5. *Let X be an A'_4-cogroup. Define $\Pi_X\colon X\times K_3\to W_4(X)$ by $\Pi_X=(1\vee\nu'_X\vee 1\vee\nu'_X)(1\vee 1\vee\mu'_X)M'_{X,3}$, then we shall have*

(5.5.1) $\Phi_4(1\vee\nu'_0\vee\nu'_0)W_3(\gamma)M'_{X,3}\simeq(1\vee 1\vee T)\Pi_X$ *rel.* $X\times\dot{K}_3$,

(5.5.2) $\Phi_4(1\vee\nu'_0\vee\nu'_0)M'_{0,3}\simeq(1\vee 1\vee T)W_4(\varepsilon)\Pi_0$ *rel.* $X\times\dot{K}_3$.

PROOF. (5.5.1) Define the homotopy $\bar{H}'_2\colon X\times I\to S\Omega X\vee S\Omega X$ from $(\gamma\vee\gamma)\cdot\bar{\mu}'_X$ to $\bar{\mu}'_0\cdot\gamma$ by

$$\bar{H}'_2(x;t)=\begin{cases}(1\vee N'(\gamma)(\ ;2t))\mu'_X & \text{for } 0\leqq t\leqq 1/2,\\ (1\vee\nu'_0)H'_2(x;2t-1) & \text{for } 1/2\leqq t\leqq 1.\end{cases}$$

Then, we shall have

$$\begin{aligned}&\Phi_4(1\vee\nu'_0\vee 1\vee\nu'_0)W_3(\gamma)M'_{X,3}\\ &\quad=(1\vee 1\vee T)\rho(\Phi_2\vee\Phi_2\vee\Phi_2)W_3(\gamma)M'_{X,3} && \text{(by (5.3))}\\ &\quad\simeq(1\vee 1\vee T)\rho W_6(\varepsilon)W_6(\gamma)(\bar{\mu}'_X\vee\bar{\mu}'_X\vee\bar{\mu}'_X)M'_{X,3} && \text{(by } \bar{H}'_2\vee\bar{H}'_2\vee\bar{H}'_2)\end{aligned}$$

$$\simeq (1\vee 1\vee T)\rho(\bar{\mu}'_X\vee\bar{\mu}'_X\vee\bar{\mu}'_X)M'_{X,3}$$

$$\simeq (1\vee 1\vee T)(1\vee\nu'_X\vee 1\vee\nu'_X)(1\vee 1\vee\mu'_X)M'_{X,3}\quad \text{(by (5.4))}.$$

(5.5.2) may be shown similarly using Lemmas (5.3) and (5.4).

PROPOSITION 5.6. *Let X be an s-A'_4-cogroup, then γ is a q-A'_3-map.*

PROOF. Consider the following diagram:

$$\begin{array}{ccccccc} S\Omega X & \xrightarrow{M'_{0,3}} & W_3(S\Omega X) & \xrightarrow{1\vee\nu'_0\vee\nu'_0} & W_3(S\Omega X) & \xrightarrow{\Phi_4} & W_4(X). \\ {\scriptstyle\gamma\times 1}\uparrow & & \uparrow{\scriptstyle W_3(\gamma)} & & & & \\ X\times K_3 & \xrightarrow[M'_{X,3}]{} & W_3(X) & & & & \end{array}$$

By Lemma 5.5, we have

$$\Phi_4(1\vee\nu'_0\vee\nu'_0)W_3(\gamma)M'_{X,3}\simeq(1\vee 1\vee T)\Pi_X,$$

$$\Phi_4(1\vee\nu'_0\vee\nu'_0)M'_{0,3}(\gamma\times 1)\simeq(1\vee 1\vee T)W_3(\varepsilon)\Pi_0(\gamma\times 1).$$

On the other hand, since X is an s-A'_4-cogroup, $(1\vee 1\vee T)\Pi_X$ is homotopic to $(1\vee 1\vee T)W_3(\varepsilon)\Pi_0(\gamma\times 1)$; thus we have

$$\Phi_4(1\vee\nu'_0\vee\nu'_0)W_3(\gamma)M'_{X,3}\simeq\Phi_4(1\vee\nu'_0\vee\nu'_0)M'_{0,3}(\gamma\times 1)$$

relative $X\times\dot{K}_3$. Since Φ_4 is a homotopy-monomorphism, and ν'_0 is a homotopy-equivalence, we shall obtain the desired result.

Combining Propositions 5.2 and 5.6, we have

THEOREM 5.7. *Let X be an A'_3-cogroup, then γ is an q-A'_3-map if and only if X is an s-A'_4-cogroup.*

§6. Homotopy-Coalgebras and Suspensions

In this section, we consider from a little different point of view. We begin with the special case.

DEFINITION 6.1. An A'_3-cogroup X is a *coalgebra* if there exists a coretraction γ satisfying the following condition (Γ_∞)

$$\gamma_0\gamma = S\Omega\gamma\cdot\gamma.$$

REMARK 6.2. If X is a suspended space, then X is a coalgebra with respect to its canonical coretraction.

REMARK 6.3. Obviously, γ is a q-A'_2-homomorphism, i.e., $(\gamma\vee\gamma)\mu'_X=\mu'_0\gamma$ for $\mu'_X=\Psi\gamma$.

THEOREM 6.4. *If X is a simply-connected coalgebra of finite dimension, then X has a homotopy-type of a suspended space.*

To prove this theorem, we need some preparations.

Given a triad $(f\colon X\to B\leftarrow Y\colon g)$, define its topological pull-back $P_{f,g}$ by $P_{f,g}=\{(x,y)\in X\times Y; f(x)=g(y)\}$. Define $\Theta'\colon SP_{f,g}\to P_{Sf,Sg}$ by $\Theta'<a,(x,y)>=(<a,x>, <a,y>)$, then Θ' is a homeomorphism. Next, define $\Theta\colon ST_{f,g}\to T_{Sf,Sg}$ by $\Theta<a,(x,y,w)>=(<a,x>, <a,y>, \ll a,w\gg)$, where $\ll a,w\gg$ is the path of SB defined by $\ll a,w\gg(t)=<a,w(t)>$, and define $i_{f,g}\colon P_{f,g}\to T_{f,g}$ by $i_{f,g}(x,y)=(x,y,w_b)$, where w_b is the path of B defined by $w_b(t)=b=f(x)=g(y)$. Then, we have the following (strictly) commutative diagram:

$$(6.5)\qquad \begin{array}{ccc} ST_{f,g} & \xrightarrow{\Theta} & T_{Sf,Sg} \\ {\scriptstyle Si_{f,g}}\uparrow & & \uparrow{\scriptstyle i_{Sf,Sg}} \\ SP_{f,g} & \xrightarrow{\Theta'} & P_{Sf,Sg}. \end{array}$$

PROPOSITION 6.6. *Let X be a coalgebra, then starting with $D_1=\Omega X$ and $\gamma_1=\gamma$, we have a sequence of maps $\gamma_k\colon X\to SD_k$ such that the following diagram is homotopy-commutative:*

$$(6.7)_k\qquad \begin{array}{ccccccc} & & SD_{k+1} & \xrightarrow{\qquad\gamma_k\qquad} & X & & \\ & /\!/ & & \searrow^{\theta_k} & \downarrow{\scriptstyle\kappa_k} & \searrow^{\gamma} & \\ T_{\iota_k,\Omega\gamma}=D_{k+1} & \xrightarrow{p_{k,2}} & \Omega X & \quad T_{S\iota_k,S\Omega\gamma_k}= & W_k & \xrightarrow{\pi_{k,2}} & S\Omega X \\ {\scriptstyle p_{k,1}}\downarrow & & \downarrow{\scriptstyle\Omega\gamma_k} & & {\scriptstyle\pi_{k,1}}\downarrow & & \downarrow{\scriptstyle S\Omega\gamma_k} \\ D_k & \xrightarrow{\iota_k} & \Omega SD_k & & SD_k & \xrightarrow{S\iota_k} & S\Omega SD_k \end{array}$$

where $\iota_k\colon D_k\to\Omega SD_k$ is the natural inclusion defined by $\iota_k(\delta^{(k)})(t)=<t,\delta^{(k)}>$ for any $\delta^{(k)}\in D_k$ and $t\in I$.

PROOF. If $\gamma(x)\neq *$, put $\gamma(x)=<a_x, l_x>$, then we have $S\iota_1\cdot\gamma(x)=<a_x, (s\to<s,l_x>)>$ and $S\Omega\gamma\cdot\gamma(x)=<a_x, (s\to<a_{x,s}, l_{x,s}>)>$, where $<a_{x,s}, l_{x,s}>=\gamma(l_x(s))$ and $(s\to<a_s, l_s>)$ denotes the loop of $S\Omega X$ which sends s to $<a_s, l_s>$. Then the condition (Γ_∞) implies that

$$(6.8)\qquad a_{x,s}=s \quad\text{and}\quad l_{x,s}=l_x \qquad \text{for all}\quad s\in I.$$

Therefore, we may define a homotopy $\Gamma\colon X\times I\to S\Omega D_1$ by

$$\Gamma(x, u) = \begin{cases} <a_x, (s \longrightarrow <s, l_x>)> & \text{if } \gamma(x) \neq *, \\ * & \text{if } \gamma(x) = *, \end{cases}$$

and $\kappa_1 : X \to W_1$ by $\kappa_1(x) = (\gamma(x), \gamma(x), w(x))$, where $w(x)$ is the path in $S\Omega D_1$ defined by $w(x)(u) = \Gamma(x, u)$, and finally $\gamma_2 : X \to SD_2$ by

$$\gamma_2(x) = \begin{cases} <a_x, (l_x, l_x, \omega_x^{(1)})> & \text{if } \gamma(x) \neq *, \\ * & \text{if } \gamma(x) = *, \end{cases}$$

where $\omega_x^{(1)}$ is the path in ΩSD_1 such that $\ll a_x, \omega_x^{(1)} \gg = w(x)$ holds. Then the diagram $(6.7)_1$ is homotopy-commutative. Put $\gamma_2(x) = <a_x, \delta^{(2)}(x)>$ for $x \notin \Sigma = \{x \in X; \gamma(x) = *\}$. Then, we obtain $S\iota_2 \cdot \gamma_2(x) = <a_x, (r \to <r, \delta^{(2)}(x)>)>$ and $S\Omega\gamma_2 \cdot \gamma(x) = <a_x, \gamma_2 \circ l_x> = <a_x, (r \to <r, \delta^{(2)}(x)>)>$ by (6.8). Define $\lambda^{(2)}: X - \Sigma \to \Omega SD_2$ and $\delta^{(3)}: X - \Sigma \to D_3$ by $\lambda^{(2)}(x)(r) = <r, \delta^{(2)}(x)>$ and $\delta^{(3)}(x) = (\delta^{(2)}(x), l_x, \omega^{(2)}(x))$, where $\omega^{(2)}(x)$ is the path of ΩSD_2 defined by $\omega^{(2)}(x)(u) = \lambda^{(2)}(x)$ for all $u \in I$.

Then, we may define $\kappa_2 : X \to W_2$ and $\gamma_3 : X \to SD_3$ by $\kappa_2(x) = (\gamma_2(x), \gamma(x), \ll a_x, \omega^{(2)}(x) \gg)$, and

$$\gamma_3(x) = \begin{cases} <a_x, \delta^{(3)}(x)> & \text{for } x \notin \Sigma, \\ * & \text{for } x \in \Sigma, \end{cases}$$

and it holds $\Theta_2 \gamma_3 = \kappa_2$.

Now, assume that we have defined maps $\gamma_i : X \to SD_i$, $i = 1, 2, \ldots, k$ $(k \geqq 3)$, such that it holds

$$\gamma_i(x) = \begin{cases} <a_x, \delta^{(i)}(x)> & \text{for } x \notin \Sigma, \\ * & \text{for } x \in \Sigma, \end{cases}$$

where $\delta^{(i)}(x) = (\delta^{(i-1)}(x), l_x, \omega^{(i-1)}(x))$ and $\omega^{(i-1)}(x)$ is the path of ΩSD_{i-1} defined by $[\omega^{(i-1)}(x)(u)](t) = <t, \delta^{(i-1)}(x)>$, moreover it holds $\delta^{(i-1)}(l_x(t)) = \delta^{(i-1)}(x)$ for all $t \in I$.

Then, we obtain

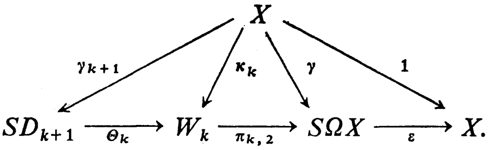
$$S\iota_k \cdot \gamma_k(x) = <a_x, (t \to <t, \delta^{(k)}(x)>)> = S\Omega\gamma_k \cdot \gamma(x).$$

Therefore, we may define $\delta^{(k+1)}: X - \Sigma \to D_{k+1}$ by $\delta^{(k+1)}(x) = (\delta^{(k)}(x), l_x, \omega^{(k)}(x))$, where $\omega^{(k)}(x)$ is the path of ΩSD_k defined by $[\omega^{(k)}(x)(u)](t) = <t, \delta^{(k)}(x)>$, and $\gamma_{k+1} : X \to SD_{k+1}$ by

$$\gamma_{k+1}(x) = \begin{cases} <a_x, \delta^{(k+1)}(x)> & \text{for } x \notin \Sigma, \\ * & \text{for } x \in \Sigma, \end{cases}$$

and $\kappa_k: X \to W_k$ by $\kappa_k(x)=(\gamma_k(x), \gamma(x), \ll a_x, \omega^{(k)}(x)\gg)$. Obviously, it holds $\Theta_k \cdot \gamma_{k+1}=\kappa_k$ and γ_{k+1}, $\delta^{(k+1)}(x)$ satisfy the required conditions.

Now, let X be an $(n-1)$-connected coalgebra and consider the following homotopy-commutative diagram:



Since conn. γ*) $=2n-2$ and conn. $\Omega X=n-2$, using Lemmas 3.1 and 3.2 in [3], we obtain

(6.9.1) $\quad \text{conn.}\,\gamma_k = (k+1)(n-2)+2,$

(6.9.2) $\quad \text{conn.}\,\Theta_k = (k+3)(n-2)+3,$

(6.9.3) $\quad \text{conn.}\,(\varepsilon\circ\pi_{k,2}) = (k+2)(n-2)+3,$

(7.9.4) $\quad \text{conn.}\,D_k = n-2.$

PROOF OF THEOREM 6.4. For a sufficiently large k, we have $\dim X \leqq (k+2)(n-2)+3$. Fix such a k, and put $N=(k+2)(n-2)+3$. Since conn. $(\varepsilon\circ\pi_{k,2}\circ\Theta_k)=(k+2)(n-2)+3$, by J. H. C. Whitehead's theorem, $(\varepsilon\circ\pi_{k,2}\circ\Theta_k)_*: H_N(SD_{k+1}) \to H_N(X)$ is an epimorphism. On the other hand, since $\dim X \leqq N$, $H_N(X)$ is free, using Berstein-Hilton's homology decomposition (Theorem 6.1 in [2]) we obtain a CW-complex Y and a map f': $Y \to D_{k+1}$ satisfying the following conditions:

(6.10.1) $f'_*: H_q(Y) \longrightarrow H_q(D_{k+1})$ is an isomorphism for $q < N-1$.

(6.10.2) $(\varepsilon\circ\pi_{k,2}\circ\Theta_k\circ Sf')_*: H_N(SY) \longrightarrow H_N(X)$ is an isomorphism.

(6.10.3) $H_q(Y)=0 \quad$ for $\quad q>N$.

Since $H_q(X)=0$ for $q>N$, $f=\varepsilon\circ\pi_{k,2}\circ\Theta_k\circ Sf'$: $SY \to X$ is a homotopy equivalence.

LEMMA 6.11. *For the homotopy equivalence f in Theorem 6.2, $\tilde{f}=f\circ\gamma$ is homotopic to a suspended map.*

PROOF. By definitions, we have the followings:

---

*) For a based map $f: X \to Y$, we denote conn. $f=n$ if $\pi_i(f)=0$ for $i \leqq n$, which is equivalent to say that $f_*: \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i<n$, and $f_*: \pi_n(X) \to \pi_n(Y)$ is an epimorphism.

$$\pi_{i,1}\circ\Theta_i = Sp_{i,1} \qquad \text{for} \quad 1 \leqq i \leqq k;$$

$$\pi_{i,2}\circ\Theta_i\circ\gamma_{i+1} \simeq \gamma \qquad \text{for} \quad 1 \leqq i \leqq k-1;$$

$$\gamma_i\circ\varepsilon = \varepsilon_{0,i}\circ S\Omega\gamma_i \qquad \text{for} \quad 1 \leqq i \leqq k;$$

$$\varepsilon_{0,i}\circ S\iota = 1_{SD_i};$$

where $\varepsilon_{0,i}$: $S\Omega SD_i \to SD_i$ is the map defined by

$$\varepsilon_{0,i}<a, (r \longrightarrow <b_r, \delta_r^{(i)}>)> = <b_a, \delta_a^{(i)}> .$$

Then, we obtain

$$\gamma\circ f \simeq \gamma\circ\varepsilon\circ\pi_{1,2}\circ\Theta_1\circ\gamma_2\circ\varepsilon\circ\pi_{2,2}\circ\Theta_2\circ\gamma_3\circ\cdots\circ\gamma_k\circ\varepsilon\circ\pi_{k,2}\circ\Theta_k\circ Sf'$$

$$\simeq \varepsilon_0\circ S\iota_1\circ Sp_{1,1}\circ\varepsilon_{0,2}\circ S\iota_2\circ Sp_{2,1}\circ\cdots\circ\varepsilon_{0,k}\circ S\iota_k\circ Sp_{k,i}\circ Sf'$$

$$= S(p_{1,1}\circ\cdots\circ p_{k,1}\circ f') .$$

COROLLARY 6.12. *The homotopy equivalence f in Theorem* 6.2 *is a q-A'_2-map.*

PROOF. Let γ'_0 be the canonical coretraction of SY, then we have $\gamma_0\circ\gamma\circ f \simeq S\Omega\gamma\circ S\Omega f\circ\gamma'_0$, and then applying ε_0 by the left we obtain $\gamma\circ f \simeq S\Omega(\varepsilon\circ\gamma)\circ S\Omega f\circ\gamma'_0$. Therefore f is a q-A'_2-map.

Being X a coalgebra is a sufficient condition for X to be a homotopy-suspended space, however, this characterization is not homotopically invariant, and then we attempt to put it in the homotopy-version.

Define maps ε_i: $SD_i \to X$, $i \geqq 2$, by $\varepsilon_i<a, (\delta^{(i-1)}, l, \omega^{(i-1)})> = l(a)$.

DEFINITION 6.13. i) A space X is a *homotopy-coalgebra of order* 1 (abbr. *HCAL*–1), if it admits a coretraction γ, i.e., X is an A'_2-space. A map f: $X \to Y$ of HCAL–1's is an *HCAL–1-map* if there exists a homotopy $\Gamma_1(f) = H(\gamma_Y\circ f, S\Omega f\circ\gamma_X)$.

ii) An HCAL–1 X is a *homotopy-coalgebra of order* 2 (abbr. *HCAL*–2) if it admits a coretraction γ_2 for ε_2, i.e., it holds $\varepsilon_2\circ\gamma_2 \simeq 1$. An HCAL–1-map f: $X \to Y$ of HCAL–2's is an *HCAL–2-map* if there exists a homotopy $\Gamma_2(f) = H(\gamma_{2,Y}\circ f, SD_2(f)\circ\gamma_{2,X})$.

REMARK 6.14. i) Let X be an HCAL–1 with a coretraction γ, f: $X \to Y$ be a homotopy-equivalence with a homotopy-inverse g, then $\gamma' = S\Omega f\circ\gamma\circ g$ is a coretraction of Y and f and g are HCAL–1-maps with respect to these coretractions.

ii) Let X be an HCAL–2, and f: $X \to Y$ be a homotopy-equivalence with a homotopy-inverse g. Since f and g are HCAL–1-maps, we may define $D_2(f)$:

$D_2(X)\to D_2(Y)$ and $D_2(g)\colon D_2(Y)\to D_2(X)$ such that we have $D_2(g)\circ D_2(f)\simeq 1$, $D_2(f)\circ D_2(g)\simeq 1$, $\pi_{1,1}\circ D_2(f)=\Omega f\circ\pi_{1,1}$ and $\pi_{1,2}\circ D_2(f)=\Omega f\circ\pi_{1,2}$ and so on. Similarly, we may define $SD_2(f)\colon SD_2(X)\to SD_2(Y)$, $W_1(f)\colon W_1(X)\to W_1(Y)$, $SD_2(g)$ and $W_1(g)$ satisfying the similar conditions as above, and moreover, we have the following homotopy-commutative diagram:

$$\begin{array}{ccccccc} SD_2(X) & \xrightarrow{\Theta_{1,X}} & W_1(X) & \xrightarrow{\pi_{2,1}} & S\Omega X & \xrightarrow{\varepsilon_X} & X \\ {\scriptstyle SD_2(f)}\downarrow\uparrow{\scriptstyle SD_2(g)} & & {\scriptstyle W_1(f)}\downarrow\uparrow{\scriptstyle W_1(g)} & & {\scriptstyle S\Omega f}\downarrow\uparrow{\scriptstyle S\Omega g} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle g} \\ SD_2(Y) & \xrightarrow[\Theta_{1,Y}]{} & W_1(Y) & \xrightarrow[\pi_{2,1}]{} & S\Omega Y & \xrightarrow[\varepsilon_Y]{} & Y \end{array}.$$

Notice that $D_2(g)\circ D_2(f)\simeq 1$ is shown by the fact that the exact presentations of homotopies $\Gamma_1(f)$ and $\Gamma_1(g)$ are given by the aid of $F=H(g\circ f, 1)$. The essential part is shown in the following Figure 4, where the thick arrows represent altogether the third component ω^* of $D_2(g)\circ D_2(f)(l', l'', \omega)$.

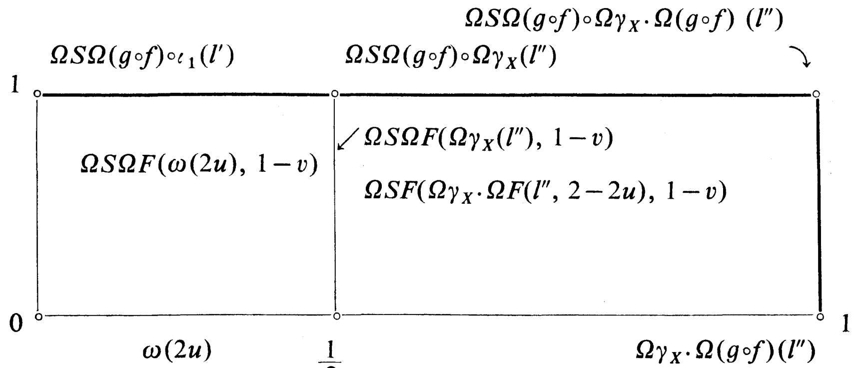


Fig. 4

Define $\gamma_{2,Y}\colon Y\to SD_2(Y)$ by $\gamma_{2,Y}=SD_2(f)\circ\gamma_{2,X}\circ g$, then $\gamma_{2,Y}$ is a coretraction for $\varepsilon_{2,Y}=\varepsilon_Y\circ\pi_{2,1}\circ\Theta_{1,Y}$ and f and g are HCAL–2-maps with respect to $\gamma_{2,X}$ and $\gamma_{2,Y}$.

Fix a map $f\colon X\to SY$ and set $f(x)=<a_x, y_{f,x}>$. Let $\{X; SY\}(f)$ be the totality of maps $g\colon X\to SY$ such that we have $g(x)=<a_x, y_{g,x}>$. Then two maps g_0 and g_1 of $\{X; SY\}(f)$ are said to be *s-homotopic* if there exists an s-homotopy $G={}^sH(g_0, g_1)\colon X\times I\to SY$, i.e., G has the presentation $G(x, u)=<a_x, y_{x,u}>$; in notation, $g_0\underset{s}{\simeq} g_1$.

PROPOSITION 6.15. *An HCAL–1 is an HCAL–2 if and only if there exists a coretraction γ for which we can find an s-homotopy $\tilde{\Gamma}={}^sH(S\iota_1\circ\gamma, S\Omega\gamma\circ\gamma)$. Therefore, an HCAL–2 X is an A'_3-cogroup. Further if an HCAL–2 X is $(n-1)$-connected and of dimension $\leqq 4n-5$, then X has the HCAL–1 homotopy-type of a suspended space.*

PROOF. Sufficiency is easily seen, and we show necessity. Put $\Xi=H(\varepsilon\circ\gamma, 1)$, $\Xi_2=H(\varepsilon_2\circ\gamma_2, 1)$, $\kappa_1'=\Theta_1\circ\gamma_2$, $\gamma'=\pi_{1,1}\circ\kappa_1'$, $\gamma''=\pi_{1,2}\circ\kappa_1'$ and $\gamma_2(x)=<a_x, (l_x', l_x', \omega_x)>$. Define an s-homotopy $\tilde{\Gamma}'={}^sH(S\iota_1\circ\gamma', S\Omega\gamma\circ\gamma'')$ by $\tilde{\Gamma}'(x, u)=\ll\tilde{a}_x, \omega_x\gg(u)$. Then, we have $\Gamma_0'=H(\gamma', \gamma)=\varepsilon_0\circ\tilde{\Gamma}'\dotplus\gamma\circ\Xi_2$, $\Gamma_0'''={}^sH(\gamma', \gamma'')=S\Omega\varepsilon\circ\tilde{\Gamma}'\dotplus S\Omega\Xi\circ(\gamma''\times 1)$ and $\Gamma_0''=H(\gamma'', \gamma)=\dot{-}\Gamma_0'''\dotplus\Gamma_0'$. Therefore, we may define $\tilde{\Gamma}''={}^sH(S\iota_1\circ\gamma'', S\Omega\gamma'', \gamma'')$ by $\tilde{\Gamma}''=\dot{-}S\iota_1\circ\Gamma_0'''\dotplus\tilde{\Gamma}'\dotplus S\Omega\Gamma_0''\circ(\gamma''\times 1)$. Thus we have obtained the first assertion. The remainders are easily obtained (cf. the proof of Theorem 6.2).

DEFINITION 6.16. An HCAL–2 X is an *HCAL*–3 if there exists a coretraction $\gamma_3: X\to SD_3$ for ε_3. An HCAL–2-map $f: X\to Y$ of HCAL–3's is an *HCAL–3-map* if there exists a homotopy $\Gamma_3(f)=H(\gamma_{3,Y}\circ f, SD_3(f)\circ\gamma_{3,X})$.

By the same argument as in Proposition 6.15, we obtain

PROPOSITION 6.17. *If X is an HCAL–3, then we have a homotopy $H(S\iota_2\circ\gamma_2, S\Omega\gamma_2\circ\gamma)$, and then γ_2 is a q-A_2'-map. Moreover, we can define a map κ_2: $X\to W_2$ such that it holds $\Theta_2\circ\gamma_3\simeq\kappa_2$. Obviously, an $(n-1)$-connected HCAL–3 of dimension$\leqq 5n-7$ has the homotopy type of a suspended space.*

By the similar argument as in Remarks 6.14 (ii), we see that being an HCAL–3 is a homotopy-invariant.

We conclude this section by considering the relation between HCAL–3's and s-A_4'-spaces. We begin with

PROPOSITION 6.18. *Let X be an HCAL–2 satisfying the following condition $[ss-\tilde{\Gamma}_2(\gamma)]$:*

There exists a homotopy $\tilde{\Gamma}_2(\gamma)$: $X\times I\times I\to S\Omega S\Omega S\Omega X$ such that we have

$$\tilde{\Gamma}_2(\gamma)(x, u, 0) = S\Omega\gamma_0\circ\tilde{\Gamma}(x, u),$$

$$\tilde{\Gamma}_2(\gamma)(x, u, 1) = S\Omega S\Omega\gamma\circ\tilde{\Gamma}(x, u),$$

$$\tilde{\Gamma}_2(\gamma)(x, 0, v) = \gamma_{00}\circ\tilde{\Gamma}(x, v),$$

$$\tilde{\Gamma}_2(\gamma)(x, 1, v) = S\Omega\tilde{\Gamma}(\gamma(x), v),$$

$$\tilde{\Gamma}_2(\gamma)(x, u, v) = <a_x, (r\to<\tilde{b}_{x,v,r}, (s\to< \ , \ >)>)> \quad \textit{for } (u, v)\in(0, 1),$$

where $\tilde{\Gamma}(x, u)={}^sH(S\iota_1\circ\gamma, S\Omega\gamma\circ\gamma)(x, u)=<a_x, (s\to<\tilde{b}_{x,u,s}, \tilde{l}_{x,u,s}>)>$ for the presentation $\gamma(x)=<a_x, l_x>$.

Then, X is an HCAL–3.

PROOF. Notice that we have

$$S\iota_2\circ\gamma_2(x) = <a_x, (r \longrightarrow <r, (l_x, l_x, \omega_x)>)>$$
$$= <a_x, (r \longrightarrow <r, \delta^{(2)}(x)>)> ,$$

$$S\Omega\gamma_2\circ\gamma(x) = \langle a_x, (r \longrightarrow \langle a_{x,r}, (l_{x,r}, l_{x,r}, \omega_{x,r})\rangle)\rangle ,$$

where $\langle\langle a_x, \omega_x\rangle\rangle (u)=\tilde{\Gamma}(x, u)$ and $\langle a_{x,r}, l_{x,r}\rangle = \gamma(l_x(r))$. Then,, we may define a homotopy $\tilde{\Gamma}_2\colon X\times I\to S\Omega SD_2$ by

$$\tilde{\Gamma}_2(x, v) = \langle a_x, (r \longrightarrow \tilde{b}\langle_{x,v,r}, (\tilde{l}_{x,v,r}, \tilde{l}_{x,v,r}, \omega_{x,v,r})\rangle)\rangle ,$$

where $\omega_{x,v,r}$ is the path of $\Omega S\Omega X$ such that it holds

$$\tilde{\Gamma}_2(\gamma)(x, u, v) = \langle a_x, (r \longrightarrow \langle\langle \tilde{b}_{x,v,r}, \omega_{x,v,r}\rangle\rangle (u)\rangle .$$

Then, it holds $\tilde{\Gamma}_2={}^sH(S\iota_2\circ\gamma_2, S\Omega\gamma_2\circ\gamma)$, and we obtain a lift $\gamma_3\colon X\to SD_3$ by

$$\gamma_3(x) = \langle a_x, (\delta^{(2)}(x), l_x, \omega^{(2)}(x))\rangle ,$$

where $\omega^{(2)}(x)$ is the path of ΩSD_2 such that we have $\langle\langle a_x, \omega^{(2)}(x)\rangle\rangle (v)=\tilde{\Gamma}_2(x, v)$. Obviously, we obtain $\varepsilon_3\circ\gamma_3=\varepsilon\circ\gamma\simeq 1$ and $\Theta_2\circ\gamma_3=\kappa_2$.

DEFINITION 6.19. We call an A'_3-cogroup a *weak-homotopy-coalgebra of order* 2 (abbr. *WHCAL*–2) in the sense that there exists a homotopy $\tilde{\Gamma}(\gamma)=H(S\iota_1\circ \gamma, S\Omega\gamma\circ\gamma)$.

A WHCAL–2 X is a *WHCAL*–3 if there exists a homotopy $\tilde{\Gamma}_2(\gamma)\colon X\times I\times I \to S\Omega S\Omega S\Omega X$ satisfying the first four conditions of $[ss-\tilde{\Gamma}_2(\gamma)]$ with respect to $\tilde{\Gamma}(\gamma)$.

THEOREM 6.20. *Let X be an s-A'_4-cogroup such that the corresponding γ is an A'_3-map, then X is a WHCAL–3.*

To prove this theorem, we make some preparations.

Given an A'_3-cogroup A, a finite CW-complex Z and any space Y, let $\{A\times Z; Y\}_l$ be the space of all maps $f\colon (A\times Z, *_X\times Z)\to(Y, *)$ and $[A\times Z; Y]_l$ be the corresponding homotopy set. Then, we have

LEMMA 6.21. (i) *$\{A\times Z; Y\}_l$ is an A_3-group under the multiplication induced by μ'_A.*

(ii) *$\Phi_{k*}\colon [A\times Z; W_{k-1}(S\Omega X)]_l\to[A\times Z; W_k(X)]_l$ and $\Psi_*\colon [A\times Z; S\Omega X]_l \vee [A\times Z; X\to X]_l$ are monomorphisms.*

Using Lemma 6.21 (ii), we obtain

LEMMA 6.22. *For a q-A'_2-map $f\colon X\to Y$ of A'_3-cogroups, the following two conditions are equivalent:*

[WHCAL–2] *There exists a homotopy $\tilde{\Gamma}_2(f)\colon X\times I\times I\to S\Omega S\Omega Y$ satisfying the following conditions:*

$$\bar{\Gamma}_2(f)(x, u, 0) = S\Omega S\Omega f\circ\bar{\Gamma}_X(x, u),$$

$$\bar{\Gamma}_2(f)(x, u, 1) = \bar{\Gamma}_Y(f(x), u),$$

$$\bar{\Gamma}_2(f)(x, 0, v) = \begin{cases} S\Omega\bar{\Gamma}(f)(\gamma_X(x), 2v) & \text{for } 0 \leqq v \leqq 1/2, \\ S\Omega\gamma_Y\circ\bar{\Gamma}(f)(x, 2v-1) & \text{for } 1/2 \leqq v \leqq 1, \end{cases}$$

$$\bar{\Gamma}_2(f)(x, 1, v) = \begin{cases} S\Omega S\Omega f\circ\gamma_{0,X}\gamma_X = \gamma_{0,Y}\circ S\Omega f\circ\gamma_X & \text{for } 0 \leqq v \leqq 1/2, \\ \gamma_{0,Y}\circ\bar{\Gamma}(f)(x-2v-1) & \text{for } 1/2 \leqq v \leqq 1, \end{cases}$$

where $\bar{\Gamma}_X = H(S\Omega\gamma_X\circ\gamma_X, \gamma_{0,X}\circ\gamma_X)$ *and* $\bar{\Gamma}(f) = H(S\Omega f\circ\gamma_X, \gamma_Y\circ f)$.

[WHCAL–2′] *There existsa homotopy* $\Gamma'_2(f)\colon X\times I\times I\to S\Omega Y\vee S\Omega Y$ *satisfying the following conditions*:

$$\Gamma'_2(f)(x, u, 0) = (S\Omega f\vee S\Omega f)\circ\Gamma'_X(x, u),$$

$$\Gamma'_2(f)(x, u, 1) = \Gamma'_Y(f(x), u),$$

$$\Gamma'_2(f)(x, 0, v) =$$

$$\begin{cases} (\bar{\Gamma}(f)(\ , 2v)\vee\bar{\Gamma}(\ , 2v))\circ\mu'_X(x) & \text{for } 0 \leqq v \leqq 1/2, \\ (\gamma_Y\vee\gamma_Y)\circ H'_2(f)(x-2v-1) & \text{for } 1/2 \leqq v \leqq 1, \end{cases}$$

$$\Gamma'_2(f)(x, 1, v) =$$

$$\begin{cases} (S\Omega f\vee S\Omega f)\circ\mu'_{0,X}\gamma_X(x) = \mu'_{0,Y}S\Omega f\circ\gamma_X(x) & \text{for } 0 \leqq v \leqq 1/2, \\ \mu'_{0,Y}\circ\bar{\Gamma}(f)(x, 2v-1) & \text{for } 1/2 \leqq v \leqq 1, \end{cases}$$

where $\Gamma'_X = H((\gamma_X\vee\gamma_X)\circ\mu'_X, \mu'_{0,X}\circ\gamma_X)$ *and* $H'_2(f) = H((f\vee f)\circ\mu'_X, \mu'_Y f)$.

As easily seen, an A'_3-cogroup is a WHCAL–3 if and only if the coretraction satisfies the condition [WHCAL–2].

PROPOSITION 6.23. *Let* $f\colon X\to SY$ *be an* A'_3*-map of* A'_3*-cogroups. If* $f\vee f$ *is a homotopy-monomorphism, then* f *satisfies the condition* [WHCAL–2].

PROOF. Recall the Ganea's proof of [3: Theorem 2.2], where the homotopy $H((\gamma\vee\gamma)\circ\mu', \gamma'_0\circ\gamma)$ is constructed via homotopies $N = H(\nabla(v'\vee 1)\mu', *)$, $\Gamma = H(\Phi_2\circ\bar{\gamma}, \bar{\mu}')$, $E = H(\nabla(1\vee *)\circ\mu', 1)$ and $Z = H((\mu'\vee\mu')\circ\mu', (1\vee\mu'\vee 1)\circ(1\vee\mu')\circ\mu')$. Since f is an A'_3-map, f is compatible with Z, N and E. Moreover, since $f\vee f$ is a homotopy-monomorphism, f is compatible with Γ. Therefore, we may construct the desired homotopy $\Gamma_2(f)$.

PROOF OF THEOREM 6.20. As easily seen, $\gamma\vee\gamma$ is a homotopy-monomorphism, then we can obtain the result by Proposition 6.23 and Lemma 6.22.

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