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(p, q)-Nuclear Operators in Case of $0<p<1$

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The notion and properties of the nuclear operators on Banach spaces, first given by A. Grothendieck [2], have been generalized in various ways and many investigations have been developed on these operators and on ideals consisting of these operators ([5], [9], [1], [6], [7], [3]).

In [8] (resp. [7]) the p-nuclear (resp. (p, q)-nuclear) operators which, on Hilbert spaces and in case of $p=1$ (resp. $p=q=1$), are identical with the classical nuclear operators, were discussed. In [1], J. Cohen dealt with another kind of p-nuclear operators which, on Hilbert spaces are also identical with the classical nuclear operators. All these investigations were done under the assumption that $1 \leqq p \leqq \infty$. On the other hand, recently in [5], [6] and [3] the p-nuclear operators and the spaces L^p and l^p in case of $0<p<1$ have been studied. The aim of this paper is to define the (p, q)-nuclear and (p, q)-quasi-nuclear operators in case of $0<p<1, 0<q \leqq \infty$ and to develop their properties. Thus the present paper is partly the continuation of our previous work [7] concerning the (p, q)-nuclear operators in case of $1 \leqq p \leqq \infty$.

In [10] A. Pietsch defined many kinds of s-numbers of operators and discussed the interrelations among such numbers. On the other hand, in [3] C. H. Ha investigated the properties of some of these s-numbers and clarified the relations between the approximation numbers (resp. the Gelfand numbers) and the quasinorm of a p-nuclear (resp. p-quasi-nuclear) operator in case of $0<p<1$. It is the second aim to extend these Ha's results to the case of (p, q)-nuclear and (p, q)-quasi-nuclear operators.

In Section 1 we recall the definition and some properties of the Lorentz sequence space $l^{p,q}$ for our later use. Among other things, the Hardy, Littlewood and Pólya's lemma (later Lemma 3) cited there will be frequently used for the calculations of the quantities of $l^{p,q}$-quasi-norm. We shall there define the (p, q)-nuclear and (p, q)-quasi-nuclear operators in case of $0<p<1$, $0<q \leqq \infty$, which differ in form from those in case of $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ (cf. [7]). And we define also the operators of type $S_{p,q}^{app}$ and of type $S_{p,q}^{gel}$ by making use of the approximation and Gelfand numbers respectively. Section 2 is devoted to the investigation of the operators defined in Section 1 and the classes of such operators as operator ideals. In the final Section 3, following Pietsch [9] and Ha [3] we shall show that in case of $0<p<1$, $p \leqq q \leqq \infty$ the operator of type $S_{p,q}^{app}$ (resp.

$S^{gel}_{p,q}$) is (p, q)-nuclear (resp. (p, q)-quasi-nuclear).

1. Preliminaries

The concrete description of the Lorentz sequence space in literature, especially in case of $0<p<1$, seems to be very little. Therefore we first recall the notion and some properties of the Lorentz sequence spaces $l^{p,q}$ which will be frequently used in the following.

DEFINITION 1. For $0<p\leqq\infty, 0<q\leqq\infty$, the Lorentz sequence space $l^{p,q}$ or $l(p, q)$ is defined as the collection of the sequences $\lambda=\{\lambda_i\}_{1\leqq i\leqq\infty}\in c_0$ such that

$$\|\{\lambda_i\}\|_{l^{p,q}} := \begin{cases} (\sum_{i=1}^{\infty} i^{q/p-1}|\lambda_i|^{*q})^{1/q} & \text{if } q<\infty, \\ \sup_i i^{1/p}|\lambda_i|^* & \text{if } q=\infty, \end{cases}$$

is finite, where $|\lambda_j|^*$ denotes the j-th term in the non-increasing rearrangement of the sequence $\{|\lambda_i|\}$. In the following, unless otherwise stated, we are supposed to replace $(\sum_{i=1}^{\infty} i^{q/p-1}|\lambda_i|^{*q})^{1/q}$ by $\sup i^{1/p}|\lambda_i|^*$ when $q=\infty$.

LEMMA 1 ([4]). *For $0<p\leqq\infty, 0<q\leqq\infty$, $l^{p,q}$ is a quasi-normed space with respect to $\|\cdot\|_{l^{p,q}}$.*

LEMMA 2. (i) *If $0<p\leqq\infty$ and $0<q_1<q_2\leqq\infty$, then*

$$l^{p,q_1}\subset l^{p,q_2}$$

and

$$\|\lambda\|_{l^{p,q_2}}\leqq C\|\lambda\|_{l^{p,q_1}} \qquad \text{for each } \lambda\in l^{p,q_1}.$$

(ii) *If $0<p_1<p_2\leqq\infty$ and $0<q_1, q_2\leqq\infty$, then*

$$l^{p_1,q_1}\subset l^{p_2,q_2}$$

and

$$\|\lambda\|_{l^{p_2,q_2}}\leqq C\|\lambda\|_{l^{p_1,q_1}} \text{ for each } \lambda\in l^{p_1,q_1}.$$

Here C stands for a positive constant depending on the parameters p_1, p_2, q_1 and q_2 and independent on λ.

This is a direct consequence from the result concerning Lorentz sequence spaces $l^{p,q}$ obtained by the interpolation theory of Banach spaces (e.g. in [4]). However, since the interpolation theoretical discussions are usually applied to the Lorentz spaces $L^{p,q}$ of functions, we here give the proof.

PROOF. We first note that by Definition 1 the assertion (ii) in case of $q_1=q_2$ is clear and (i) was proved in [4]. These two results combined together yield the assertion (ii) in case of $0<p_1<p_2\leqq\infty$, $0<q_1\leqq q_2\leqq\infty$. Therefore it remains for proving (ii) to show (ii) in case of $0<p_1<p_2\leqq\infty$, $0<q_2<q_1\leqq\infty$. In this case, noting $q_1/q_2>1$ and putting $q_2/p_2=q_2/p_1-\varepsilon$ with an $\varepsilon>0$, we have, by Hölder's inequality,

$$\begin{aligned}
&\sum_i i^{q_2/p_2-1}|\lambda_i|^{*q_2} \\
&= \sum_i (i^{q_2/q_1-1-\varepsilon})(i^{q_2/p_1-q_2/q_1}|\lambda_i|^{*q_2}) \\
&< (\sum_i 1/i^{1+\varepsilon/(1-q_2/q_1)})^{1-q_2/q_1}(\sum_i i^{q_1/p_1-1}|\lambda_i|^{*q_1})^{q_2/q_1} \\
&< +\infty
\end{aligned}$$

which completes the proof.

LEMMA 3 (*Hardy, Littlewood and Pólya, cf.* [7]). *Let* $\{c_i^*\}$ *and* $\{{}^*c_i\}$ *be the non-increasing and non-decreasing rearrangements of a finite sequence* $\{c_i\}_{1\leqq i\leqq n}$ *of positive numbers, respectively. Then for two sequences* $\{a_i\}_{1\leqq i\leqq n}$ *and* $\{b_i\}_{1\leqq i\leqq n}$ *of positive numbers we have*

$$\sum_i a_i^*\cdot{}^*b_i \leqq \sum_i a_ib_i \leqq \sum_i a_i^*\cdot b_i^*.$$

We here note that if the right hand side is convergent in these inequalities, these hold in case of infinite sequences too.

We now need to remember the following three types of s-numbers of operators on Banach spaces ([10]).

Let E be a normed or quasi-normed linear space, and $\|\cdot\|_E$ denotes its norm or quasi-norm in E. The set of all bounded linear operators from a Banach space E into a Banach space F is denoted by $L(E, F)$ equipped with the usual bounded operator norm $\|T\|=\sup_{\|x\|\leqq 1}\|Tx\|$. We denote by $L^{(n)}(E, F)$ the subspace of $L(E, F)$ of operators T of rank(T) (=dimension of the range of T)$<n$.

DEFINITION 2. For each operator $T\in L(E, F)$ and for $i=1, 2,\ldots,$ the approximation numbers $\alpha_i(T)$, the Gelfand numbers $\beta_i(T)$ and the Kolmogorov numbers $\gamma_i(T)$ are defined by

$$\alpha_i(T):=\inf\{\|T-A\|: A\in L^{(i)}(E, F)\},$$

$$\beta_i(T):=\inf\{\|T|M\|: M \text{ is a subspace of } E, \operatorname{codim} M<i\}$$

and

$$\gamma_i(T) := \inf\{\|Q_N^F T\| : N \text{ is a subspace of } F, \dim N < i\}$$

respectively, where $T|M$ denotes the restriction of an operator T to M and Q_N^F denotes the canonical map of F onto F/N.

For the general properties of these s-numbers we may refer to [5] and [10].

By making use of these s-numbers we define the following classes of operators generalizing those in [5] and [10].

DEFINITION 3. For $0<p, q\leqq\infty$, we define

$$S_{p,q}^{app}(E, F) := \{T\in L(E, F) : \{\alpha_i(T)\}\in l^{p,q}\}$$

and

$$S_{p,q}^{gel}(E, F) := \{T\in L(E, F) : \{\beta_i(T)\}\in l^{p,q}\}$$

and we put

$$a_{p,q}(T) = \|\{\alpha_i(T)\}\|_{l^{p,q}} \quad \text{for} \quad T\in S_{p,q}^{app}$$

and

$$b_{p,q}(T) = \|\{\beta_i(T)\}\|_{l^{p,q}} \quad \text{for} \quad T\in S_{p,q}^{gel}.$$

DEFINITION 4. For $0<p<1$, $0<q\leqq\infty$, the operator $T\in L(E, F)$ is said to be (p, q)-nuclear, if T can be written in the form

$$(1) \qquad Tx = \sum_{i=1}^{\infty} \lambda_i <x, x_i'> y_i \quad \text{for each} \quad x\in E,$$

with $x_i'\in E'$ (the dual of E) such that $\|x_i'\|_{E'}=1$, $y_i\in F$ such that $\|y_i\|_F = 1$ and $\{\lambda_i\}\in l^{p,q}$.

The operator $T\in L(E, F)$ is said to be (p, q)-quasi-nuclear, if there exist a sequence $\{\mu_i\}_{1\leqq i\leqq\infty}$ of positive numbers and a sequence $\{x_i'\}\subset E'$ such that $\{\mu_i\}\in l^{p,q}$, $\|x_i'\|=1$, $i=1, 2,\ldots$ and the inequality

$$(2) \qquad \|Tx\| \leqq \sum_{i=1}^{\infty} \mu_i |<x, x_i'>|$$

holds for each $x\in E$.

The collection of the (p, q)-nuclear (resp. (p, q)-quasi-nuclear) operators is denoted by $N_{p,q}(E, F)$ (resp. $N_{p,q}^Q(E, F)$) and we denote

$$\nu_{p,q}(T) = \inf \|\{\lambda_i\}\|_{l^{p,q}}$$

$$(\text{resp. } \nu_{p,q}^Q(T) = \inf \|\{\mu_i\}\|_{l^{p,q}}),$$

where the infimum is taken over all $\{\lambda_i\}$ (resp. $\{\mu_i\}$) satisfying (1) (resp. (2)).

REMARK. The (p, q)-nuclear and (p, q)-quasi-nuclear operators for $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$ were defined and investigated in [7]. However we can not leave the definition untouched in case of $0<p<1$, since then the Tx doesn't converge (cf. [7]). Therefore following [5] we define as above the (p, q)-nuclear and (p, q)-quasi-nuclear operators in case of $0<p<1$ so that the series (1) of Tx converges. The definitions coincide with those in [5] and [3] when $p=q$.

In this paper we deal with only the case $0<p<1$ as far as the (p, q)-nuclear and (p, q)-quasi-nuclear operators are treated.

2. Ideals of operators $N_{p,q}$, $N^Q_{p,q}$, $S^{app}_{p,q}$ and $S^{gel}_{p,q}$

In this section we mainly investigate the properties of $N_{p,q}$, $N^Q_{p,q}$, $S^{app}_{p,q}$ and $S^{gel}_{p,q}$ as the ideals of operators (cf. [10]). We begin with

PROPOSITION 1. *Assume $0<p<1$ and $0<q\leqq\infty$. Then, for each $T\in N_{p,q}(E, F)$, the series $\sum_{i=1}^{\infty}\lambda_i<x, x_i'>y_i$ in (1) is convergent. Especially when $0<p<1$ and $p\leqq q$, the inequality*

$$\|T\| \leqq C_{p,q}\nu_{p,q}(T)$$

holds for each $T\in N_{p,q}(E, F)$, with a positive constant

$$C_{p,q} = \begin{cases} 1 & \text{if } 0<p<1,\ p\leqq q\leqq 1, \\ (\sum_{i=1}^{\infty} i^{(1/q-1/p)q'})^{1/q'} & \text{if } 0<p<1<q,\ 1/q+1/q'=1. \end{cases}$$

PROOF. By the definition of $T\in N_{p,q}(E, F)$ we have

$$Tx = \sum_{i=1}^{\infty}\lambda_i<x, x_i'>y_i \qquad \text{for each} \quad x\in E,$$

with $\|x_i'\|=\|y_i\|=1$ and $\sum_{i=1}^{\infty} i^{q/p-1}|\lambda_i|^{*q}<\infty$.

We first deal with the case of $q<p$. We here remark the fact that if $\{a_i\}$ is a monotonically decreasing sequence of positive numbers and $\sum_{i=1}^{\infty}a_i<\infty$, then $\lim_{i\to\infty} i a_i=0$. By this remark we have $\lim_{i\to\infty} i^{q/p}|\lambda_i|^{*q}=0$, whence $|\lambda_i|^*=o(i^{-1/p})$ with $0<p<1$, and therefore we have

$$\|Tx\| \leqq \sum_{i=1}^{\infty}|\lambda_i|\,\|x\| = \sum_{i=1}^{\infty}|\lambda_i|^*\|x\| < \infty.$$

Next we assume $0<p<1$ and $p\leqq q$. Then we have

$$\|Tx\| \leqq \|x\| \sum_{i=1}^{\infty} |\lambda_i|^*$$

$$\leqq \begin{cases} (\sum_{i=1}^{\infty} i^{q/p-1}|\lambda_i|^{*q})^{1/q} \|x\| & \text{if } 0<p<1,\ p \leqq q \leqq 1, \\ (\sum_{i=1}^{\infty} i^{(1/q-1/p)q'})^{1/q'} (\sum_{i=1}^{\infty} i^{q/p-1}|\lambda_i|^{*q})^{1/q}\|x\| & \\ & \text{if } 0<p<1<q,\ 1/q+1/q'=1. \end{cases}$$

Accounting $(1/q-1/p)q'<-1$, this completes the proof.

By Definitions 3 and 4 and by Lemma 2, the next proposition is easily proved. So the proof is omitted.

PROPOSITION 2. (i) *If* $0<p<1$ *and* $0<q_1<q_2\leqq\infty$, *then*

$$N_{p,q_1}(E, F) \subset N_{p,q_2}(E, F) \quad (\textit{resp. } N^Q_{p,q_1}(E, F) \subset N^Q_{p,q_2}(E, F))$$

and

$$\nu_{p,q_2}(T) \leqq C\nu_{p,q_1}(T) \quad (\textit{resp. } \nu^Q_{p,q_2}(T) \leqq C\nu^Q_{p,q_1}(T))$$

for each $T\in N_{p,q_1}(E, F)$ (*resp.* $T\in N^Q_{p,q_1}(E, F)$).
(ii) *If* $0<p_1<p_2<1$ *and* $0<q_1,\ q_2\leqq\infty$, *then*

$$N_{p_1,q_1}(E, F) \subset N_{p_2,q_2}(E, F) \quad (\textit{resp. } N^Q_{p_1,q_1}(E, F) \subset N^Q_{p_2,q_2}(E, F))$$

and

$$\nu_{p_2,q_2}(T) \leqq C\nu_{p_1,q_1}(T) \quad (\textit{resp. } \nu^Q_{p_2,q_2}(T) \leqq C\nu^Q_{p_1,q_1}(T))$$

for each $T\in N_{p_1,q_1}(E, F)$ (*resp.* $T\in N^Q_{p_1,q_1}(E, F)$).
(iii) *If* $0<p\leqq\infty$ *and* $0<q_1<q_2\leqq\infty$, *then*

$$S^{app}_{p,q_1}(E, F) \subset S^{app}_{p,q_2}(E, F) \quad (\textit{resp. } S^{gel}_{p,q_1}(E, F) \subset S^{gel}_{p,q_2}(E, F))$$

and

$$a_{p,q_2}(T)\leqq Ca_{p,q_1}(T) \quad (\textit{resp. } b_{p,q_2}(T)\leqq Cb_{p,q_1}(T))$$

for each $T\in S^{app}_{p,q_1}(E, F)$(*resp.* $T\in S^{gel}_{p,q_1}(E, F)$).
(iv) *If* $0<p_1<p_2<\infty$ *and* $0<q_1,\ q_2\leqq\infty$, *then*

$$S^{app}_{p_1,q_1}(E, F) \subset S^{app}_{p_2,q_2}(E, F) \quad (\textit{resp. } S^{gel}_{p_1,q_1}(E, F) \subset S^{gel}_{p_2,q_2}(E, F))$$

and

$$a_{p_2,q_2}(T) \leqq Ca_{p_1,q_1}(T) \quad (\textit{resp. } b_{p_2,q_2}(T) \leqq Cb_{p_1,q_1}(T))$$

for each $T \in S^{app}_{p_1,q_1}(E, F)$ *(resp.* $T \in S^{gel}_{p_1,q_1}(E, F)$*). Here* C *stands for the same constant in Lemma* 2.

Next, we shall show that the classes $N_{p,q}$, $N^Q_{p,q}$, $S^{app}_{p,q}$ and $S^{gel}_{p,q}$ make the linear spaces. Since it is obvious that the scalar multiples of the operator belonging to each class belong to the same class, it remains to show the following

PROPOSITION 3. (i) *Let* $0<p<1$, $0<q<\infty$. *If* $T_1, T_2 \in N_{p,q}(E, F)$ *(resp.* $N^Q_{p,q}(E, F)$*), then* $T_1+T_2 \in N_{p,q}(E, F)$ *(resp.* $N^Q_{p,q}(E, F)$*) and*

$$\nu_{p,q}(T_1+T_2) \leqq C_{p,q}\{\nu_{p,q}(T_1)+\nu_{p,q}(T_2)\}$$

$$(resp.\ \nu^Q_{p,q}(T_1+T_2) \leqq C_{p,q}\{\nu^Q_{p,q}(T_1)+\nu^Q_{p,q}(T_2)\}),$$

where $C_{p,q}=\max(2^{1/p-1}, 2^{1/q-1}, 2^{1/p-1/q})$.
(ii) *Let* $0<p\leqq\infty$, $0<q\leqq\infty$. *If* $T_1, T_2 \in S^{app}_{p,q}(E, F)$ *(resp.* $S^{gel}_{p,q}(E, F)$*), then* $T_1+T_2 \in S^{app}_{p,q}(E, F)$ *(resp.* $S^{gel}_{p,q}(E, F)$*) and*

$$a_{p,q}(T_1+T_2) \leqq C'_{p,q}\{a_{p,q}(T_1)+a_{p,q}(T_2)\}$$

$$(resp.\ \ b_{p,q}(T_1+T_2) \leqq C'_{p,q}\{b_{p,q}(T_1)+b_{p,q}(T_2)\}),$$

where $C'_{p,q}=\max(2,\ 2^{2/q-1}, 2^{1/p-1/q+1}, 2^{1/p+1/q-1})$.

PROOF. (i) We shall show this by making use of the same way as in the proof of Theorem 1 in [7]. By the definition, for any $\varepsilon>0$ and $k=1, 2$, T_k can be written in the form

$$T_k x = \sum_{i=1}^{\infty} \lambda_{k,i} <x, x'_{k,i}> y_{k,i}$$

with

$$\|x'_{k,i}\| = \|y_{k,i}\| = 1$$

and

$$\sum_{i=1}^{\infty} i^{q/p-1} |\lambda_{k,i}|^{*q} < \{\nu_{p,q}(T_k)+\varepsilon\}^q,$$

where $|\lambda_{k,i}|^*$ denotes the i-th term in the non-increasing rearrangement of $\{|\lambda_{k,i}|\}_{1\leqq i\leqq\infty}$. Now, let N be any positive integer and let $|\lambda_{k,i}|$ be the $n(k, i)$-th term in the non-increasing rearrangement of $\{|\lambda_{1,i}|, |\lambda_{2,i}|\}_{1\leqq i\leqq N}$. Then we have

$$\sum_{k=1}^{2}\sum_{i=1}^{N} n(k, i)^{q/p-1}|\lambda_{k,i}|^q$$

$$\leqq \begin{cases} \displaystyle\sum_{k=1}^{2}\sum_{i=1}^{N} (2i-k+1)^{q/p-1}|\lambda_{k,i}|^{*q} & \text{if } q \geqq p \text{ (by Lemma 3)}, \\ \displaystyle\sum_{k=1}^{2}\sum_{i=1}^{N} i^{q/p-1}|\lambda_{k,i}|^{*q} & \text{if } q < p, \end{cases}$$

$$\leqq \max(2^{q/p-1},\ 1)\sum_{k=1}^{2}\sum_{i=1}^{N} i^{q/p-1}|\lambda_{k,i}|^{*q}$$

$$\leqq \max(2^{q/p-1},\ 1)\{(v_{p,q}(T_1)+\varepsilon)^q+(v_{p,q}(T_2)+\varepsilon)^q\}.$$

We here note the inequality

$$(\xi+\eta)^a \leqq \{\max(2^{a-1},\ 1)\}(\xi^a+\eta^a)$$

for any $\xi,\ \eta\geqq 0$ and $a>0$.

By making use of this inequality and considering that $\varepsilon>0$ and the integer $N>0$ are arbitrarily taken, we get

$$\begin{aligned} & v_{p,q}(T_1+T_2) \\ & \quad \leqq \max(2^{1/p-1/q},\ 1)\cdot\max(2^{1/q-1},\ 1)(v_{p,q}(T_1)+v_{p,q}(T_2)) \\ & \quad = C_{p,q}(v_{p,q}(T_1)+v_{p,q}(T_2)) \end{aligned}$$

with $C_{p,q}=\max(2^{1/p-1},\ 2^{1/q-1},\ 2^{1/p-1/q})$. On account of $(T_1+T_2)x=\sum_{k=1}^{2}\sum_{i=1}^{\infty}\lambda_{k,i}<x,\ x'_{k,i}>y_{k,i}$, this completes the proof in case of $T_1,\ T_2\in N_{p,q}$.

In the same way, by making use of the definition of the $(p,\ q)$-quasi-nuclear operator we can show that if $T_1,\ T_2\in N^Q_{p,q}(E,\ F)$, then $T_1+T_2\in N^Q_{p,q}(E,\ F)$ and

$$v^Q_{p,q}(T_1+T_2)\leqq C_{p,q}(v^Q_{p,q}(T_1)+v^Q_{p,q}(T_2)).$$

(ii) We next assume that $0<p\leqq\infty,\ 0<q\leqq\infty$ and $T_1,\ T_2\in S^{app}_{p,q}(E,\ F)$. We notice that the sequences $\{\alpha_i(T)\}$ and $\{\beta_i(T)\}$ are non-increasing and they are the additive s-numbers (Theorem 9.5 in [10]), that is,

$$\alpha_{i+j-1}(T_1+T_2)\leqq\alpha_i(T_1)+\alpha_j(T_2)\ \text{ and }\ \beta_{i+j-1}(T_1+T_2)\leqq\beta_i(T_1)+\beta_j(T_2),$$
$$i,\ j=1,\ 2,\dots.$$

Therefore we have

$$\begin{aligned} & \sum_{i=1}^{\infty} i^{q/p-1}\alpha_i(T_1+T_2)^q \\ & \quad \leqq \max(2,\ 2^{q/p})\sum_{i=1}^{\infty} i^{q/p-1}\alpha_{2i-1}(T_1+T_2)^q \\ & \quad \leqq \max(2,\ 2^{q/p})\sum_{i=1}^{\infty} i^{q/p-1}(\alpha_i(T_1)+\alpha_i(T_2))^q \\ & \quad \leqq \max(2,\ 2^{q/p})\max(2^{q-1},\ 1)\sum_{i=1}^{\infty} i^{q/p-1}\{\alpha_i(T_1)^q+\alpha_i(T_2)^q\}. \end{aligned}$$

Hence we obtain

$$
\begin{aligned}
& a_{p,q}(T_1+T_2) \\
& \quad \leqq \max(2^{1/q}, 2^{1/p}) \max(2^{1-1/q}, 2^{1/q-1})\{a_{p,q}(T_1)+a_{p,q}(T_2)\} \\
& \quad = C'_{p,q}\{a_{p,q}(T_1)+a_{p,q}(T_2)\}
\end{aligned}
$$

with $C'_{p,q}=\max(2, 2^{2/q-1}, 2^{1/p-1/q+1}, 2^{1/p+1/q-1})$.

In the same way, we can show that if $T_1, T_2 \in S^{gel}_{p,q}(E, F)$, then $T_1+T_2 \in S^{gel}_{p,q}(E, F)$ and

$$b_{p,q}(T_1+T_2) \leqq C'_{p,q}\{b_{p,q}(T_1)+b_{p,q}(T_2)\}.$$

This finishes the proof.

For completing the proof that the classes $N_{p,q}$, $N^{Q}_{p,q}$, $S^{app}_{p,q}$ and $S^{gel}_{p,q}$ make the ideals of operators in L it is sufficient to show the following

PROPOSITION 4. *Let E, F, G and H be Banach spaces and let $R \in L(E, F)$, $S \in N_{p,q}(F, G)$* (resp. $N^{Q}_{p,q}(F, G)$, $S^{app}_{p,q}(F, G)$, $S^{gel}_{p,q}(F, G)$) *and $T \in L(G, H)$. Then we have*

$$TSR \in N_{p,q}(E, H) \ (\textit{resp. } N^{Q}_{p,q}(E, H),\ S^{app}_{p,q}(E, H),\ S^{gel}_{p,q}(E, H))$$

and

$$
\begin{aligned}
& \nu_{p,q}(TSR) \leqq \|T\| \cdot \nu_{p,q}(S) \cdot \|R\| \\
& (\textit{resp. } \nu^{Q}_{p,q}(TSR) \leqq \|T\| \cdot \nu^{Q}_{p,q}(S) \cdot \|R\|, \\
& a_{p,q}(TSR) \leqq \|T\| \cdot a_{p,q}(S) \cdot \|R\|, \\
& b_{p,q}(TSR) \leqq \|T\| \cdot b_{p,q}(S) \cdot \|R\|).
\end{aligned}
$$

PROOF. We shall prove that if $R \in L(E, F)$, $S \in N_{p,q}(F, G)$ and $T \in L(G, H)$, then $TSR \in N_{p,q}(E, H)$. By the definition for any $\varepsilon>0$ $TSRx$ can be written in the form

$$TSRx = \sum_{i=1}^{\infty} \lambda_i \langle Rx, y'_i \rangle Tz_i \qquad \text{for each} \quad x \in E,$$

with $\|y'_i\|_{F'} = 1, \quad \|z_i\|_G = 1 \quad \text{and} \quad \|\{\lambda_i\}\|_{l^{p,q}} < \nu_{p,q}(S)+\varepsilon.$

Therefore, denoting by R' the adjoint of R and putting

$$\mu_i = \lambda_i \|R'y'_i\| \, \|Tz_i\|,$$

$$x'_i = R'y'_i / \|R'y'_i\|$$

and $u_i = Tz_i/\|Tz_i\|$, $i=1, 2, \ldots$, we obtain

$$TSRx = \sum_{i=1}^{\infty} \mu_i < x, x_i' > u_i \qquad \text{for each} \quad x \in E,$$

with

$$\|x_i'\|_{E'} = 1, \quad \|u_i\|_H = 1$$

and

$$\|\{\mu_i\}\|_{l^{p,q}} \leqq \|T\| \cdot \|\{\lambda_i\}\|_{l^{p,q}} \cdot \|R\|$$
$$< \|T\|(\nu_{p,q}(S)+\varepsilon)\|R\| .$$

This shows $TSR \in N_{p,q}(E, H)$ and

$$\nu_{p,q}(TSR) \leqq \|T\| \cdot \nu_{p,q}(S) \cdot \|R\| .$$

Since the rest of the proof can be proceeded in the same way, only pay the attention to the inequality due to [10]

$$\alpha_i(TSR) \leqq \|T\| \alpha_i(S) \|R\|$$

and

$$\beta_i(TSR) \leqq \|T\| \beta_i(S) \|R\|, \qquad i = 1, 2, \ldots .$$

By Definition 4, the inclusion relation

$$N_{p,q}(E, F) \subset N^Q_{p,q}(E, F)$$

and

$$\nu^Q_{p,q}(T) \leqq \nu_{p,q}(T) \qquad \text{for each} \quad T \in N_{p,q}(E, F)$$

are obvious. We note that this fact was proved for the case of $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ in [7, Proposition 14]. On the other hand, by making use of the inequality $\beta_i(T) \leqq \alpha_i(T)$ for each $T \in L(E, F)$, $i=1, 2, \ldots$, shown in Theorem 3.2 in [10], we get $S^{app}_{p,q}(E, F) \subset S^{gel}_{p,q}(E, F)$. Summalizing these results we obtain the following

PROPOSITION 5. *Let* $0 < p \leqq \infty, 0 < q \leqq \infty$. *Then we have* $N_{p,q}(E, F) \subset N^Q_{p,q}(E, F)$,

$$\nu^Q_{p,q}(T) \leqq \nu_{p,q}(T) \qquad \textit{for each} \quad T \in N_{p,q}(E, F)$$

and

$$S^{app}_{p,q}(E, F) \subset S^{gel}_{p,q}(E, F),$$

$$b_{p,q}(T) \leqq a_{p,q}(T) \qquad \textit{for each} \quad T \in S^{app}_{p,q}(E, F).$$

We now show some examples of the (p, q)-nuclear operator and the operator belonging to the class $S^{gel}_{p,q}$.

EXAMPLE 1. Let $\{\delta_i\} \in l^{p,q}$, $0<p<1$ and $p \leqq q \leqq \infty$. We define the multiplication operator D by $\{\delta_i\}$ from l^∞ into $l^{p,q}$ by

$$D(\{\xi_j\}) = \{\delta_j \xi_j\} \qquad \text{for each} \quad \{\xi_j\} \in l^\infty.$$

Then, defining $x'_i \in (l^\infty)'$, $i=1, 2,\dots$, by $<\{\xi_j\}, x'_i> = \xi_i$ and putting $e_i = \{\overbrace{0, 0,\dots, 0, 1}^{i}, 0,\dots\}$ we have

$$D(\{\xi_j\}) = \sum_{i=1}^{\infty} \delta_i <\{\xi_j\}, x'_i> e_i.$$

Hence

$$D \in N_{p,q}(l^\infty, l^{p,q})$$

and

$$\nu_{p,q}(D) \leqq \|\{\delta_i\}\|_{l^{p,q}}.$$

On the other hand, we see

$$D(e) = \{\delta_i\} \qquad \text{for} \quad e = \{1, 1,\dots\}$$

and

$$\|\{\delta_i\}\|_{l^{p,q}} = \|D(e)\|_{l^{p,q}} < \|D\| < C_{p,q} \nu_{p,q}(D),$$

where the last inequality is obtained by Proposition 1. Thus

$$C'_{p,q} \|\{\delta_i\}\|_{l^{p,q}} \leqq \nu_{p,q}(D) \leqq \|\{\delta_i\}\|_{l^{p,q}}$$

with

$$C'_{p,q} = 1/C_{p,q}.$$

EXAMPLE 2. Let $0<p, q \leqq \infty$ and let $\{\delta_i\}$ be a monotonically decreasing sequence of positive numbers such that $\{\delta_i\} \in l^{p,q}$. We define the operator $D_0 \in L(l^{p,q}, l^{p,q})$ by

$$D_0(\{\xi_j\}) = \{\delta_j \xi_j\} \qquad \text{for each} \quad \{\xi_j\} \in l^{p,q}.$$

Then we shall show $D_0 \in S^{gel}_{p,q}(l^{p,q}, l^{p,q})$. To show this, it is sufficient to prove $\beta_i(D_0) \leqq \delta_i$, $i=1, 2,\dots$.

Let $M_i=\{\{\xi_j\}\in l^{p,q}\colon \xi_j=0$ for $1\leqq j<i\}$ and $N_i=l^{p,q}_{(i)}$, i-dimensional $l^{p,q}$ space. Furthermore, we define the operator $T\in L(N_i, l^{p,q})$ by

$$T(\{\xi_j\}_{1\leqq j\leqq i}) = \{\xi_1,\dots, \xi_i, 0, 0,\dots\}.$$

Then M_i is a subspace of $l^{p,q}$ of codimension $<i$, and

$$\|D_0|M_i\| = \delta_i,\ i = 1, 2,\dots, \text{ thus } \beta_i(D_0) \leqq \delta_i.$$

Furthermore, in case of $0<q\leqq p\leqq\infty$, we have $\beta_i(D_0)=\delta_i$, $i=1, 2,\dots$. In fact $\|T\|=1$ and

$$\begin{aligned}&\|D_0T(\{\xi_j\}_{1\leqq j\leqq i})\|\\ &= (\sum_{j=1}^{i} j^{q/p-1}|\delta_j\xi_j|^{*q})^{1/q}\\ &\geqq \delta_i(\sum_{j=1}^{i} j^{q/p-1}|\xi_j|^{*q})^{1/q},\end{aligned}$$

where the last inequality is obtained by accounting $\delta_1\geqq\delta_2\geqq\cdots\geqq\delta_i$ and by making use of Lemma 3 considering $p\geqq q$. Thus we have

$$\begin{aligned}\beta_i(D_0)\|T\| &\geqq \beta_i(D_0T)\\ &= \inf\|D_0T(\{\xi_j\}_{1\leqq j\leqq i})\|\\ &\geqq \delta_i,\end{aligned}$$

where the infimum is taken over all $\{\xi_j\}_{1\leqq j\leqq i}\in N_i$ such that $\|\{\xi_j\}_{1\leqq j\leqq i}\|=1$. Hence $\beta_i(D_0)=\delta_i$, $i=1, 2,\dots$.

Utilizing Example 1, in a similar way as in [7], we can state the factorization theorem of (p, q)-nuclear operators as follows.

PROPOSITION 6. *Let $0<p<1$ and $p\leqq q\leqq\infty$. Then $T\in L(E, F)$ is (p, q)-nuclear if and only if T can be factorized in the form $T=QDP$, where $P\in L(E, l^{\infty})$, $Q\in L(l^{p,q}, F)$ and D is the multiplication operator by a $\{\delta_i\}\in l^{p,q}$ mentioned in Example* 1.

PROOF. The sufficiency is evident by Proposition 4. The necessity is proved by virtue of the definition of $T\in N_{p,q}(E, F)$:

$$Tx = \sum_{i=1}^{\infty} \delta_i\langle x, x'_i\rangle y_i$$

with $\|x'_i\|=\|y_i\|=1$ and $\{\delta_i\}\in l^{p,q}$. By making use of these sequences we define the operators P, D and Q by

$$Px = \{<x, x'_i>\} \in l^\infty \qquad \text{for each} \quad x \in E,$$

$$D(\{\xi_i\}) = \{\delta_i \xi_i\} \in l^{p,q} \qquad \text{for each} \quad \{\xi_i\} \in l^\infty$$

and

$$Q(\{\eta_i\}) = \sum_{i=1}^{\infty} \eta_i y_i \in F \qquad \text{for each} \quad \{\eta_i\} \in l^{p,q}.$$

Then we can write in the form

$$Tx = QDPx$$

with $\|P\| \leqq 1$ and $\|Q\| \leqq C_{p,q}$, where $C_{p,q}$ is the constant in Proposition 1. This completes the proof.

3. Relations between $N_{p,q}$ and $S^{app}_{p,q}$ and between $N^Q_{p,q}$ and $S^{gel}_{p,q}$

In this section we shall investigate the relationships between $N_{p,q}(E, F)$ and $S^{app}_{p,q}(E, F)$ and between $N^Q_{p,q}(E, F)$ and $S^{gel}_{p,q}(E, F)$ to obtain the inequalities concerning $\nu_{p,q}$ and $a_{p,q}$ and concerning $\nu^Q_{p,q}$ and $b_{p,q}$. These are the (p, q)-version extending Theorem 3 in [9] and Theorem 3.6 in [3], and the proofs might be done as in those in [9] and in [3]. These results might be successful in case of $0<p<1$ and $p \leqq q$, but those in case of $q<p<1$ are yet unknown to us.

To begin with we recall the well-known Auerbach's lemma:

Let M be an n-dimensional normed linear space. Then there exists a basis $\{x_1, x_2,\dots, x_n\}$ for M and a subset $\{u'_1, u'_2,\dots, u'_n\}$ of M' such that

$$x = \sum_{i=1}^{n} <x, u'_i> x_i \qquad \text{for each} \quad x \in M,$$

with $\|u'_i\| = \|x_i\| = 1$ and $<x_i, u'_j> = \delta_{i,j}$, $i, j = 1, 2,\dots, n$.

THEOREM 1. *Let E and F be Banach spaces and let $0<p<1$, $p \leqq q \leqq \infty$. Then we have*

$$S^{app}_{p,q}(E, F) \subset N_{p,q}(E, F)$$

and

$$\nu_{p,q}(T) \leqq 2^{4/p-1/q+2} a_{p,q}(T) \qquad \textit{for each} \quad T \in S^{app}_{p,q}(E, F).$$

PROOF. By Definition 2, for $i=1, 2,\dots$ there exists an $A_i \in L^{(2^i-1)}(E, F)$ such that

$$\|T - A_i\| \leqq 2\alpha_{2^i-1}(T).$$

We now put

$$B_i = A_{i+1} - A_i,$$

$$\dim \mathscr{R}(B_i) = d_i \qquad (\mathscr{R}(B_i) \text{ denotes the range of } B_i),$$

$$i_0 = 0, \quad i_r = \sum_{i=1}^{r} d_i,$$

and

$$I_r = \{\text{the integers in } [i_{r-1}+1, i_r]\}, \qquad r = 1, 2, \dots.$$

Then, since the sequence $\{\alpha_j(T)\}$ is decreasing, we have

$$\|B_i\| \leqq 4\alpha_{2^i-1}(T).$$

And, since $d_i < 2^{i+1} - 1 + 2^i - 1 < 2^{i+2}$, we have

$$i_r < 2^3(2^r - 1) < 2^{3+r}, \qquad r = 1, 2, \dots.$$

By Auerbach's lemma, there exist $\{u'_i\}_{i\in I_r} \subset F'$ and $\{y_i\}_{i\in I_r} \subset \mathscr{R}(B_r)$ such that $\|u'_i\| = 1$, $\|y_i\| = 1$ and

$$B_r x = \sum_{i\in I_r} \langle B_r x, u'_i \rangle y_i, \qquad r = 1, 2, \dots,$$

for each $x \in E$. Putting

$$x'_i = B'_r u'_i / \|B'_r u'_i\|,$$

$$\lambda_i = \|B'_r u'_i\| \leqq \|B_r\|, \qquad \text{for} \quad i \in I_r, \quad r = 1, 2, \dots,$$

we have

$$B_r x = \sum_{i\in I_r} \lambda_i \langle x, x'_i \rangle y_i, \qquad r = 1, 2, \dots.$$

By making use of these $\{x'_i\}_{i\in I_r}$, $\{y_i\}_{i\in I_r}$, $r = 1, 2, \dots$, we can write

$$Tx = \lim_{r\to\infty} A_{r+1} x = \sum_{r=1}^{\infty} B_r x$$

$$= \sum_{r=1}^{\infty} \sum_{i\in I_r} \lambda_i \langle x, x'_i \rangle y_i \qquad \text{for each} \quad x \in E,$$

with $\|x'_i\|_{E'} = 1$, $\|y_i\|_F = 1$, $i = 1, 2, \dots$. Therefore, on account of $p \leqq q$, by Lemma 3 we get

$$\{\nu_{p,q}(T)\}^q \leqq \sum_{i=1}^{\infty} i^{q/p-1} \lambda_i^{*q}$$

$$
\begin{aligned}
&\leqq \sum_{i=1}^{\infty} i^{q/p-1}\lambda_i^q \\
&= \sum_{r=1}^{\infty} \sum_{i\in I_r} i^{q/p-1}\lambda_i^q \\
&\leqq \sum_{r=1}^{\infty} \sum_{i\in I_r} i_r^{q/p-1}\|B_r\|^q \\
&\leqq \sum_{r=1}^{\infty} 2^{r+2}\cdot 2^{(3+r)(q/p-1)}(4\alpha_{2^r-1}(T))^q \\
&= 2^{3+4(q/p-1)+2q}\sum_{r=1}^{\infty} 2^{r-1}(2^{r-1})^{q/p-1}(\alpha_{2^r-1}(T))^q \\
&\leqq 2^{4q/p+2q-1}\sum_{r=1}^{\infty}\sum_{n=2^{r-1}}^{2^r-1} n^{q/p-1}\alpha_n(T)^q .
\end{aligned}
$$

Hence

$$\nu_{p,q}(T) \leqq 2^{4/p-1/q+2}a_{p,q}(T),$$

which finishes the proof.

Corresponding to this theorem the analogous relations will be shown for $N^Q_{p,q}$ and $S^{gel}_{p,q}$ or for $\nu^Q_{p,q}(\cdot)$ and $b_{p,q}(\cdot)$. Before proving these we here note the following facts due to Ha [3].

Let E and F be Banach spaces. We first recall the Kolmogorov's i-th diameter $d_i(D)$ of a bounded subset D in E defined by

$d_i(D)=\inf\{d>0\colon D\subset dV+G,\ G$ is a subspace of E, $\dim G<i\}$, $i=1, 2,\dots,$

where V is the closed unit ball in E.

Fact I. Then $\beta_i(T)=\gamma_i(T')=d_i(T'(U^0))$, for each $T\in L(E, F)$, $i=1, 2,\dots,$ where $T'\in L(F', E')$ is the adjoint of T and U denotes the closed unit ball in F (Theorem 2.7 in [3]).

Fact. II. Let M be a bounded subset which is contained in an n-dimensional subspace of E. Then there exist n elements $x_1, x_2,\dots, x_n$ in E with $\|x_i\|=d_1(M)$ such that

$$M\subset\{\sum_{i=1}^{n}\mu_i x_i\colon |\mu_i|\leqq 1,\ i=1, 2,\dots, n\}$$

(Corollary of Auberbach's lemma).

Fact III. Let M be a precompact subset of E. Then there exists a sequence $\{D_i\}_{1\leqq i\leqq\infty}$ of subsets of E such that:

(a) $d_1(D_i)\leqq 4d_{2^i-1}(M)$.

(b) D_i is contained in a subspace of dimension $\leqq 2^{i+2}$.

(c) The set $D_1+D_2+\cdots$ is dense in M

(Lemma 3.5 in [3]).
By making use of these facts we can show the following

THEOREM 2. *Let E, F be Banach spaces and let $0<p<1$, $p\leqq q\leqq\infty$. Then we have*

$$S_{p,q}^{gel}(E, F) \subset N_{p,q}^{Q}(E, F)$$

and

$$\nu_{p,q}^{Q}(T) \leqq 2^{4/p-1/q+2} b_{p,q}(T)$$

for each $T\in S_{p,q}^{gel}(E, F)$.

PROOF. Let $T\in S_{p,q}^{gel}(E, F)$ and let $M=T'(U^{\circ})$, where U stands for the closed unit ball in F. Then M is a precompact subset of E' and by Fact I

$$\beta_i(T) = d_i(M), \qquad i = 1, 2,\dots.$$

Putting

$$i_0 = 0, \quad i_r = 2^3(2^r-1) < 2^{3+r}$$

and

$$I_r = \{\text{the integers in } [i_{r-1}+1,\ i_r]\}, \qquad r = 1, 2,\dots.$$

Then, by Facts III and II, there exist a sequence $\{D_i\}_{1\leqq i\leqq\infty}$ of subsets of E' satisfying (a), (b) and (c) in Fact III and finite elements $\{u_i'\}$ in E' with $\|u_i'\|\leqq 4d_{2^r-1}(M)$ such that

$$D_r \subset \{\sum_{i\in I_r} \mu_i u_i' : |\mu_i| \leqq 1,\ i\in I_r\}, \qquad r = 1, 2,\dots.$$

Therefore each element $x'\in D_1+D_2+\cdots\subset M$ can be written in the form

$$x' = \sum_{r=1}^{\infty}\sum_{i\in I_r} \mu_i u_i' = \sum_{i=1}^{\infty} \mu_i u_i' = \sum_{i=1}^{\infty} \lambda_i x_i'$$

with $\lambda_i=\mu_i\|u_i'\|$ and $x_i'=u_i'/\|u_i'\|$, $i=1, 2,\dots$. Hence for each $x\in E$

$$\begin{aligned}\|Tx\| &= \sup_{\|y'\|\leqq 1} |<Tx, y'>| \\ &\leqq \sup_{x'\in M} |<x, x'>| \\ &= \sup_{x'\in D_1+D_2+\cdots} |<x, x'>| \qquad \text{(by (c) of Fact III)}\end{aligned}$$

$$\leqq \sum_{i=1}^{\infty} |\lambda_i| \, |<x, x'_i>|$$

with $\|x'_i\|=1$ and $|\lambda_i| \leqq \|u'_i\|$, $i=1, 2, \ldots$.

In view of these preparations, in account of the condition $q \geqq p$, we can calculate as follows:

$$\begin{aligned}
&\{\nu^{Q}_{p,q}(T)\}^q \\
&\leqq \sum_{i=1}^{\infty} i^{q/p-1} |\lambda_i|^{*q} \\
&\leqq \sum_{i=1}^{\infty} i^{q/p-1} |\lambda_i|^q \qquad \text{(by Lemma 3)} \\
&= \sum_{r=1}^{\infty} \sum_{i \in I_r} i^{q/p-1} |\lambda_i|^q \\
&\leqq \sum_{r=1}^{\infty} \sum_{i \in I_r} i_r^{q/p-1} (4d_{2^r-1}(M))^q \\
&\leqq \sum_{r=1}^{\infty} 2^{r+2} (2^{3+r})^{q/p-1} (4d_{2^r-1}(M))^q \\
&= 2^{3+4(q/p-1)+2q} \sum_{r=1}^{\infty} 2^{r-1} (2^{r-1})^{q/p-1} (d_{2^r-1}(M))^q \\
&\leqq 2^{4q/p+2q-1} \sum_{r=1}^{\infty} \sum_{n=2^{r-1}}^{2^r-1} n^{q/p-1} d_n(M)^q \\
&= 2^{4q/p+2q-1} \sum_{n=1}^{\infty} n^{q/p-1} \beta_n(T)^q.
\end{aligned}$$

Hence

$$\nu^{Q}_{p,q}(T) \leqq 2^{4/p-1/q+2} b_{p,q}(T).$$

Thus the proof is completed.

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