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On an Infinite-Dimensional Lie Algebra Satisfying the Maximal Condition for Subalgebras

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R. K. Amayo and I. Stewart have asked the following among "some open questions" at the end of their book [1]: Do there exist Lie algebras satisfying the maximal condition for subalgebras that are not finite-dimensional? The purpose of this paper is to give the affirmative answer to this question.

They have shown in [1, p. 177] that the Lie algebra W over a field $\mathfrak{k}$ of characteristic 0 with basis $\{w(1), w(2),\dots\}$ and multiplication

$$[w(i), w(j)] = (i-j)w(i+j)$$

satisfies the maximal condition for subideals. We shall show that the same Lie algebra W actually satisfies the maximal condition for subalgebras.

We first show the following

LEMMA. *Let S be a subset of* $\mathbf{N}$ *satisfying the condition: If $s, t \in S$ and $s \neq t$, $s+t \in S$. Then there exist the different elements $s_1, s_2,\dots, s_r$ of S such that*

(i) *s_1 is the smallest element of S,*

(ii) $S=\{s_1\} \cup \{s_2+ns_1 \mid n=0, 1, 2,\dots\} \cup \cdots \cup \{s_r+ns_1 \mid n=0, 1, 2,\dots\}$.

PROOF. We define recursively subsets S_i of S and integers s_i for integers $i \geq 1$ as follows: Define s_1 as the smallest element of S and put $S_1=\{s_1\}$. Let $i \geq 1$ and assume that S_i, s_i are already defined and $S_i \neq S$. Let s_{i+1} be the smallest element of $S \backslash S_i$ and put $S_{i+1}=S_i \cup \{s_{i+1}+ns_1 \mid n=0, 1, 2,\dots\}$. Then $\{s \in S \mid s \leq s_{i+1}\} \subseteq S_{i+1}$ and, for $T_{i+1}=\{s \in S \mid s \geq s_{i+1}\}$, if $s \in T_{i+1}$ and t is the smallest element of T_{i+1} such that $t>s$ then $t-s \leq s_1-i+1$. Therefore the construction terminates after a finite number of steps. Thus there exists an integer r such that $S=S_r$.

We now show the following

THEOREM. *W satisfies the maximal condition for subalgebras.*

PROOF. For any element x of W, let $m(x)$ be the integer m such that

$$x = \sum_{i=1}^{m} \alpha_i w(i), \qquad \alpha_m \neq 0.$$

Let H be any subalgebra of W and let S be the set of all $m(x)$ for $x \in H$. If $s, t \in S$

and $s \neq t$, then

$$s = m(x),\ t = m(y) \qquad \text{for some } x,\ y \in H.$$

and therefore

$$s+t = m([x, y]) \in S.$$

Hence there exist the elements $s_1, s_2, \dots, s_r$ of S satisfying the conditions (i), (ii) in the lemma. For $i=1, 2, \dots, r$, we take an element z_i of H such that $m(z_i)=s_i$. We assert that any element x of H belongs to $<z_1, z_2, \dots, z_r>$.

Let us define recursively elements x_i of H and integers p_i for integers $i \geq 0$ as follows. Put $x_0 = x$ and $p_0 = m(x)$. Assume that x_i and $p_i = m(x_i)$ are already defined and that $x_i \notin <z_1, z_2, \dots, z_r>$. If $p_i = s_1$, then $m(x_i - \beta z_1) < s_1$ for some $\beta \in \mathfrak{k}$. Since $x_i - \beta z_1 \in H$, we have $x_i - \beta z_1 = 0$ by the minimality of s_1. This contradicts our assumption. Therefore

$$p_i = s_{\mu(i)} + n_i s_1, \qquad \mu(i) \neq 1.$$

Then there exists a γ_i in $\mathfrak{k}$ such that

$$m(x_i - \gamma_i [z_{\mu(i),\, n_i}\, z_1]) < p_i.$$

We now define x_{i+1} and p_{i+1} by

$$x_{i+1} = x_i - \gamma_i [z_{\mu(i),\, n_i}\, z_1] \quad \text{and} \quad p_{i+1} = m(x_{i+1}).$$

Since $p_{i+1} < p_i$, the recursive construction terminates after a finite number of steps. This shows that $x_n \in <z_1, z_2, \dots, z_r>$ for some n. It follows that $x \in <z_1, z_2, \dots, z_r>$.

Thus we conclude that $H = <z_1, z_2, \dots, z_r>$. Consequently every subalgebra of W is finitely generated. It is now immediate that W satisfies the maximal condition for subalgebras.

We denote, as usual, by Max, Min and Min$-\lhd$ respectively the classes of Lie algebras satisfying the maximal condition for subalgebras, the minimal condition for subalgebras and for ideals. Then we have the following

COROLLARY. Max $\nsubseteq$ Min *and* Max $\nsubseteq$ Min$-\lhd$.

PROOF. Let I_n be the subspace of W spanned by all $w(i)$ with $i \geq n$. Then $I_1 \geq I_2 \geq \cdots$ is a strictly descending series of ideals of W. Therefore $W \notin$ Min$-\lhd$ and a priori $W \notin$ Min.

Reference

[1] R. K. Amayo and I. Stewart, Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

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