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On the Oscillatory and Asymptotic Behavior of Damped Differential Equations with Retarded Argument

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0. Preliminaries

We deal here with the oscillatory and asymptotic behavior of n-th order ($n>1$) retarded differential equations, which contain a damping term involving the $(n-1)$-th derivative of the unknown function. The results are obtained in two steps. In the first part of the paper we consider the simple damped differential equations with retarded argument

$$(*) \qquad [r(t)x^{(n-1)}(t)]' + g(t)\varphi(x[\sigma(t)]) = 0$$

and

$$(**) \qquad [r(t)x^{(n-1)}(t)]' - g(t)\varphi(x[\sigma(t)]) = 0$$

for which the following assumptions are made:

(i) *The function* $\sigma\colon [t_0, \infty) \mapsto \mathbf{R}$ *is continuously differentiable and such that*

$$\sigma(t) \leqq t \qquad \textit{for every} \quad t \geqq t_0$$

$$\sigma'(t) \geqq 0 \qquad \textit{for every} \quad t \geqq t_0$$

$$\lim_{t\to\infty} \sigma(t) = \infty$$

(ii) $g\colon [t_0, \infty) \mapsto [0, \infty)$ *is continuous and not identically zero for all large* t.

(iii) *The function* $\varphi\colon \mathbf{R} \mapsto \mathbf{R}$ *is continuous,* $y \neq 0 \Rightarrow y\varphi(y) > 0$ *and it is strongly superlinear in the sense that it is nondecreasing and*

$$\int^{\infty} \frac{dy}{\varphi(y)} < \infty \ \textit{and} \int^{-\infty} \frac{dy}{\varphi(y)} < \infty$$

Note: Condition (iii) implies that

$$(1) \qquad \lim_{y\to\infty} \frac{\varphi(y)}{y} = \infty = \lim_{y\to-\infty} \frac{\varphi(y)}{y}$$

(iv) $r: [t_0, \infty) \mapsto (0, \infty)$ *is continuous*
For $(*)$ we give some general oscillation results not only for the case where the condition

$$(C_1) \qquad \int^{\infty} \frac{dt}{r(t)} = \infty$$

holds, but also for some cases in which this condition fails. As far as we know, the only result concerning the oscillatory and asymptotic behavior of all solutions of $(*)$ is that of Ševelo and Varech ([6], Theorem 1) in which condition (C_1) is assumed. We also classify all solutions of $(**)$ with respect to their oscillatory character and to their behavior as $t \to \infty$, in the case where (C_1) is assumed.

In the second part we give a comparison lemma, which is a modification of a related lemma due to Staikos and the author ([9], Lemma 1) concerning differential equations without damping terms. This lemma can be used in order to extend the results which are derived in the first part of the paper to more general differential equations. As an application, we give general oscillation results concerning damped differential equations of the form:

$$(***) \qquad [s(t)x^{(n-1)}(t)]' + Q(t, x^{(n-1)}(t)) + p(t)F(x[\sigma(t)]) = 0$$

These results include as special cases the above mentioned result of Ševelo and Varech as well as a result due to Naito ([5], Theorem 1) and, in particular, for $r(t) \equiv 1$, $Q(t, y) \equiv 0$, the related results concerning the retarded differential equation without damping terms

$$x^{(n)}(t) + p(t)F(x[\sigma(t)]) = 0$$

(Cf. [3] and [7]).

In what follows, we consider only such solutions of the equations $(*)$, $(**)$ and $(***)$ which are defined for all large t. The oscillatory character is considered in the usual sense, i.e., a continuous function defined for all large t is called *oscillatory* if it has no last zero, otherwise it is called *nonoscillatory*.

1. Oscillatory and asymptotic behavior of the equations $(*)$ and $(**)$

In order to obtain our results for $(*)$ and $(**)$ we need the following lemmas the first of which is a unified adaptation of two lemmas due to Kiguradze ([1] and [2]).

LEMMA 1. *Let u be a positive ν-times continuously differentiable function on an interval $[a, \infty)$. If $u^{(\nu)}$ is of constant sign and not identically zero for all large t, then there exist a $t_u \geqq a$ and an integer l, $0 \leqq l \leqq \nu$ with $\nu + l$ odd if $u^{(\nu)} \leqq 0$, $\nu + l$ even if $u^{(\nu)} \geqq 0$ and such that for every $t \geqq t_u$*

$$l>0 \Rightarrow u^{(k)}(t)>0 \qquad (k=0,1,\ldots,l-1)$$

and

$$l \leqq \nu-1 \Rightarrow (-1)^{l+k}u^{(k)}(t)>0 \qquad (k=l,l+1,\ldots,\nu-1)$$

LEMMA 2. *Let u be a $(\nu-1)$-times $(\nu>1)$ continuously differentiable function on an interval $[a,\infty)$. Let also $m(t)$ be a positive function on $[a,\infty)$ such that the function $mu^{(\nu-1)}$ is continuously differentiable on $[a,\infty)$. Suppose moreover that for every $t \geqq a$ we have*

$$u(t)>0$$

$$\delta u^{(\nu-1)}(t)>0$$

$$\delta[m(t)u^{(\nu-1)}(t)]' \leqq 0 \quad \textit{and not identically zero for all large } t$$

where $\delta=\pm 1$. Then there exists a constant $K>0$ such that

$$\frac{m(t)}{m^*(t)}\ \frac{|u^{(\nu-1)}(t)|}{u(t/2)}t^{\nu-1} \leqq K \qquad \textit{for all large } t$$

where

$$m^*(t) = \max_{\frac{t}{2}\leqq\vartheta\leqq t} m(\vartheta)$$

PROOF. By Lemma 1 there exist an integer l, $0 \leqq l \leqq \nu-1$, with $l+\nu-1$ even for $\delta=+1$ and $l+\nu-1$ odd for $\delta=-1$ and some $t_u \geqq a$ such that for every $t \geqq t_u$

$$l>0 \Rightarrow u^{(k)}(t)>0 \qquad (k=0,1,\ldots,l-1)$$

(2) and

$$l \leqq \nu-1 \Rightarrow (-1)^{l+k}u^{(k)}(t)>0 \qquad (k=l,l+1,\ldots,\nu-1)$$

Applying Taylor's formula we get

$$u(\vartheta) = u(t_u)+u'(t_u)\frac{\vartheta-t_u}{1!}+\cdots+u^{(l)}(\vartheta^*)\frac{(\vartheta-t_u)^l}{l!}, \quad t_u \leqq \vartheta^* \leqq \vartheta$$

and consequently for every $\vartheta \geqq t_u$ we have

$$u(\vartheta) \geqq \frac{(\vartheta-t_u)^l}{l!}u^{(l)}(\vartheta)$$

Hence there exist $t_1 \geqq t_u$ and $K_1>0$ such that

$$u(t/2) \geqq K_1 t^l u^{(l)}(t/2) \qquad \text{for every} \quad t \geqq t_1 \tag{3}$$

Again by Taylor's formula

$$u^{(l)}(t/2) = u^{(l)}(t) - \frac{t}{2}u^{(l+1)}(t) + \cdots + \frac{\delta(-1)^{\nu-l-1}t^{\nu-l-1}}{2^{\nu-l-1}(\nu-l-1)!}u^{(\nu-1)}(t^*),$$

$$\frac{t}{2} \leqq t^* \leqq t,\ t \geqq t_u$$

and consequently, by (2), for every $t \geqq t_u$ we have

$$\begin{aligned} u^{(l)}(t/2) &\geqq \frac{t^{\nu-l-1}}{2^{\nu-l-1}(\nu-l-1)!}\delta u^{(\nu-1)}(t^*) \\ &= \frac{t^{\nu-l-1}}{2^{\nu-l-1}(\nu-l-1)!}\frac{\delta u^{(\nu-1)}(t^*)m(t^*)}{m(t^*)} \\ &\geqq \frac{t^{\nu-l-1}}{2^{\nu-l-1}(\nu-l-1)!}\frac{\delta u^{(\nu-1)}(t)m(t)}{m^*(t)} \\ &= \frac{t^{\nu-l-1}}{2^{\nu-l-1}(\nu-l-1)!}\left|u^{(\nu-1)}(t)\right|\frac{m(t)}{m^*(t)} \end{aligned}$$

Thus

$$u^{(l)}(t/2) \geqq K_2 t^{\nu-l-1}|u^{(\nu-1)}(t)|\frac{m(t)}{m^*(t)} \quad \text{for every} \quad t \geqq t_u \tag{4}$$

where $K_2 = 1/2^{\nu-l-1}(\nu-l-1)!$.

Combining the inequalities (3) and (4) we obtain

$$\frac{m(t)}{m^*(t)}\frac{|u^{(\nu-1)}(t)|}{u(t/2)}t^{\nu-1} \leqq K \quad \text{for every } t \geqq t_1 \tag{5}$$

where $K = 1/K_1K_2$.

Note. If the function m is nondecreasing, then, obviously, (5) takes the form

$$\frac{|u^{(\nu-1)}(t)|}{u(t/2)}t^{\nu-1} \leqq K \quad \text{for every } t \geqq t_1$$

LEMMA 3. *Consider the differential equation* (*) *subject to the conditions* (i)–(iv). *Then we have the following*:

a) *If* (C_1) *holds, then for every nonoscillatory solution* x *of* (*) *we have*

$$x(t)x^{(n-1)}(t) > 0 \quad \textit{for all large} \quad t$$

b) *If for every* $T \geqq t_0$

$$\int^{\infty} R(t, T)dt = \infty, \quad R(t, T) = \left(\int_T^t g(\vartheta)d\vartheta\right)/r(t), \quad t \geqq T \tag{C_2}$$

then for every nonoscillatory solution x *of* (*) *with* $\lim_{t\to\infty} x(t) \neq 0$ *we have*

$$x(t)x^{(n-1)}(t)>0 \qquad \textit{for all large} \quad t$$

(c) *If*

$$(C_3) \qquad \int^{\infty}\frac{dt}{r(t)} < \infty \textit{ and for some } k > 1$$

$$\int^{\infty}\sigma^{n-2}(t)g(t)h^{k}(t)dt = \infty,\ h(t) = \int_{t}^{\infty}\frac{d\vartheta}{r(\vartheta)}$$

then for every nonoscillatory solution x of (*) *with* $\lim_{t\to\infty} x(t)\neq 0$ *we have*

$$x(t)x^{(n-1)}(t)>0 \qquad \textit{for all large} \quad t$$

PROOF. Let x be a nonoscillatory solution of (*). Without loss of generality we suppose that $x(t)>0$ for every $t\geqq t_0$, since the substitution $x=-u$ transforms (*) into an equation of the same form subject to similar assumptions. Next, by (i), we choose some $t_1\geqq t_0$ such that

$$x[\sigma(t)]>0 \qquad \text{for every} \quad t\geqq t_1$$

Thus, in all cases a)–c) we have

$$(6) \qquad [r(t)x^{(n-1)}(t)]' \leqq 0 \qquad \text{for every} \quad t\geqq t_1$$

Moreover, since $g(t)$ is not identically zero for all large t, the same holds for $[r(t)x^{(n-1)}(t)]'$ and consequently the function $r(t)x^{(n-1)}(t)$ is positive or negative for all large t. Thus, since $r(t)>0$ for every $t\geqq t_0$, we must have $x^{(n-1)}(t)>0$ or $x^{(n-1)}(t)<0$ for all large t.

We shall prove that the assumption

$$x^{(n-1)}(t)<0 \qquad \text{for all large} \quad t$$

leads to a contradiction in all cases a)–c), provided that in cases b) and c) we have $\lim_{t\to\infty} x(t)\neq 0$. To do this we suppose that for some $t_2\geqq t_1$ we have

$$(7) \qquad x^{(n-1)}(t)<0 \qquad \text{for every} \quad t\geqq t_2$$

By (6), integrating from t_2 to $t\geqq t_2$ we get

$$r(t)x^{(n-1)}(t) \leqq r(t_2)x^{(n-1)}(t_2)$$

and consequently

$$-x^{(n-1}(t) \geqq -r(t_2)x^{(n-1)}(t_2)\frac{1}{r(t)} \qquad \text{for every} \quad t\geqq t_2$$

Integrating again from t_2 to $t\geqq t_2$, we obtain

$$-x^{(n-2)}(t)+x^{(n-2)}(t_2) \geqq -r(t_2)x^{(n-1)}(t_2)\int_{t_2}^{t} \frac{d\vartheta}{r(\vartheta)}$$

and consequently condition (C_1) implies

$$\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$$

which contradicts the positivity of x. This contradiction proves a).

To prove b) we remark that the assumption $\lim_{t\to\infty} x(t)\neq 0$ implies the existence of a constant $L>0$ such that

$$\varphi(x[\sigma(t)]) \geqq L \qquad \text{for every} \quad t \geqq t_2$$

This, by (*), leads to the inequality

$$[r(t)x^{(n-1)}(t)]'+g(t)L \leqq 0 \qquad \text{for every} \quad t \geqq t_2 \tag{8}$$

By (8), integrating from t_2 to $t \geqq t_2$ we get

$$r(t)x^{(n-1)}(t)-r(t_2)x^{(n-1)}(t_2)+L\int_{t_2}^{t} g(\vartheta)d\vartheta \leqq 0$$

and consequently

$$-x^{(n-1)}(t) \geqq L\left(\int_{t_2}^{t} g(\vartheta)d\vartheta\right)/r(t) \qquad \text{for every} \quad t \geqq t_2$$

Using this inequality and condition (C_2) we obtain again the contradiction

$$\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$$

To prove c) we rewrite (*) as follows:

$$[r(t)x^{(n-1)}(t)]'+g(t)\frac{\varphi(x[\sigma(t)])}{x[\sigma(t)/2]}x[\sigma(t)/2] = 0, \qquad t \geqq t_2 \tag{9}$$

and we remark that (1) and $\lim_{t\to\infty} x(t)\neq 0$ imply the existence of some $t_3 \geqq t_2$ and of a positive constant L_1 such that

$$\frac{\varphi(x[\sigma(t)])}{x[\sigma(t)/2]} \geqq L_1 \qquad \text{for every} \quad t \geqq t_3$$

By this inequality, (9) leads to

$$[r(t)x^{(n-1)}(t)]'+L_1 g(t)x[\sigma(t)/2] \leqq 0, \qquad t \geqq t_3 \tag{10}$$

Applying Lemma 2 with $\nu=n-1$ and $m(t)=1$, by (10) and (7) we derive the inequality

$$(11)\qquad [r(t)x^{(n-1)}(t)]' + KL_1 g(t)\sigma^{n-2}(t)x^{(n-2)}(t) \leqq 0, \qquad t \geqq t_3$$

By (11), $x^{(n-2)}(t)$ is obviously a positive solution of the linear second order ordinary differential equation

$$(12)\qquad [r(t)y']' + \frac{KL_1\sigma^{n-2}(t)g(t)x^{(n-2)}(t) + \gamma(t)}{x^{(n-2)}(t)}y = 0, \qquad t \geqq t_3$$

where $\gamma(t) = -[r(t)x^{(n-1)}(t)]' - KL_1\sigma^{n-2}(t)g(t)x^{(n-2)}(t)$, $t \geqq t_3$. Since, by (11),

$$\gamma(t) \geqq 0 \qquad \text{for every} \quad t \geqq t_3$$

the functions r and g_x, where

$$g_x(t) = KL_1\sigma^{n-2}(t)g(t) + \frac{\gamma(t)}{x^{(n-2)}(t)}$$

are obviously subject to the conditions

$$\int^{\infty} \frac{dt}{r(t)} < \infty \quad \text{and for some} \quad k > 1, \quad \int^{\infty} g_x(t)h^k(t)dt = \infty$$

Thus, applying a result due to Moore ([4], Theorem 2) we conclude that all solutions of (12) are oscillatory. But this is a contradiction, since $x^{(n-2)}$ is a nonoscillatory solution of (12). This contradiction proves c).

THEOREM 1. *Consider the differential equation* (*) *subject to the conditions* (i)–(iv) *and*

$$(C_4)\qquad \int^{\infty} g(t)\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta = \infty \qquad \textit{for every} \quad T \geqq t_0$$

where $r^*(t) = \max_{\frac{t}{2} \leqq \vartheta \leqq t} r(\vartheta)$.

Then:

α) *under condition* (C_1) *every solution of* (*) *is for n even oscillatory and for n odd either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with its first* $n-2$ *derivatives.*

β) *under condition* (C_2) *or* (C_3) *every solution of* (*) *is either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with its first* $n-2$ *derivatives.*

Note: In the case where the function r is nondecreasing, condition (C_4) can be replaced by

$$(C_4)^* \qquad \int^{\infty} \frac{\sigma^{n-1}(t)}{r(t)} g(t)dt = \infty$$

PROOF OF THE THEOREM. Let x be a nonoscillatory solution of $(*)$ with $\lim_{t\to\infty} x(t)\neq 0$. As in the proof of Lemma 3, we assume, without loss of generality, that $t_1 \geqq t_0$ is chosen so that

$$x[\sigma(t)]>0 \qquad \text{for every} \quad t \geqq t_1$$

This, by $(*)$ and (ii), (iii) implies that

$$[r(t)x^{(n-1)}(t)]' \leqq 0 \qquad \text{for every} \quad t \geqq t_1$$

where this function is not identically zero for all large t.

Now, under one of the conditions (C_1)–(C_3) we have, by Lemma 3,

$$x^{(n-1)}(t)>0 \qquad \text{for all large} \quad t$$

Without loss of generality we assume that

$$x^{(n-1)}(t)>0 \qquad \text{for every} \quad t \geqq t_1$$

By Lemma 1 there exists some $t_2 \geqq t_1$ such that

$$x'(t)>0 \quad \text{or} \quad x'(t)<0 \qquad \text{for every} \quad t \geqq t_2$$

and consequently we have to examine the following two cases:

Case 1. $x'>0$ *on* $[t_2, \infty)$. Let z be the function defined by the formula

$$z(t) = -[r(t)x^{(n-1)}(t)]\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]\varphi(x[\sigma(\vartheta)])}d\vartheta,\ t \geqq t_2 \tag{13}$$

We obviously have

$$z(t) \leqq 0 \qquad \text{for every} \quad t \geqq t_2 \tag{14}$$

By (13), for every $t\geqq t_2$, we get

$$\begin{aligned} z'(t) &= -[r(t)x^{(n-1)}(t)]'\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]\varphi(x[\sigma(\vartheta)])}d\vartheta \\ &\qquad\qquad -\frac{r(t)x^{(n-1)}(t)\sigma^{n-2}(t)\,\sigma'(t)}{r^*[\sigma(t)]\varphi(x[\sigma(t)])} \\ &= g(t)\varphi(x[\sigma(t)])\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]\varphi(x[\sigma(\vartheta)])}d\vartheta \\ &\qquad\qquad -\frac{\sigma^{n-2}(t)}{r^*[\sigma(t)]}\,\frac{r(t)x^{(n-1)}(t)}{x'[\sigma(t)/2]}\,\frac{x'[\sigma(t)/2]\sigma'(t)}{\varphi(x[\sigma(t)])} \end{aligned}$$

Since the functions φ and x are nondecreasing and the function $r(t)x^{(n-1)}(t)$ is nonincreasing, we obtain

$$z'(t) \geqq g(t)\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta$$

$$-2\frac{x^{(n-1)}[\sigma(t)]}{x'[\sigma(t]}\frac{r[\sigma(t)]}{r^*[\sigma(t)]}\sigma^{n-2}(t)\frac{[x[\sigma(t)/2]]'}{\varphi(x[\sigma(t)/2])}$$

for every $t \geqq t_2$. Thus applying Lemma 2 with $u=x'$, $m=r$, $v=n-1$ and $\sigma(t)$ in place of t, we have

$$z'(t) \geqq g(t)\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta - 2K\frac{[x[\sigma(t)/2]]'}{\varphi(x[\sigma(t)/2])}$$

for every $t \geqq t_3$, where $t_3 \geqq t_2$ is chosen properly. By this inequality, integrating from t_3 to $t \geqq t_3$ and taking into account (iii) and (C_4) we obtain $\lim_{t\to\infty} z(t)=\infty$, which contradicts (14).

Case 2. $x'<0$ *on* $[t_2, \infty)$. In this case we consider the function w defined by the formula

$$w(t) = -[r(t)x^{(n-1)}(t)]\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta, \quad t \geqq t_2 \tag{15}$$

We obviously have

$$w(t) \leqq 0 \qquad \text{for every} \quad t \geqq t_2 \tag{16}$$

By (15) for every $t \geqq t_2$, we get

$$\begin{aligned} w'(t) = {} & -[r(t)x^{(n-1)}(t)]'\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta \\ & - \frac{x^{(n-1)}(t)r(t)}{r^*[\sigma(t)]}\sigma^{n-2}(t)\sigma'(t) \\ \geqq {} & g(t)\varphi(x[\sigma(t)])\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta + \\ & +2\frac{x^{(n-1)}[\sigma(t)]}{|x'[\sigma(t)/2]|}\frac{r[\sigma(t)]}{r^*[\sigma(t)]}\sigma^{n-2}(t)[x[\sigma(t)/2]]' \end{aligned}$$

Moreover, since $\lim_{t\to\infty} x(t) \neq 0$, there exists a positive constant c such that

$$\varphi(x[\sigma(t)]) \geqq c \qquad \text{for every} \quad t \geqq t_2$$

Thus, by applying Lemma 2 with $u=-x'=|x'|$, $m=r$, $v=n-1$ and $\sigma(t)$ in place of t, we finally obtain

$$w'(t) \geqq cg(t)\int_{t_2}^{t} \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*[\sigma(\vartheta)]}d\vartheta + 2K[x[\sigma(t)/2]]'$$

for every $t \geqq t_3$, where $t_3 \geqq t_2$ is chosen properly. This last inequality, by (C_4) and the fact that the solution x is bounded, leads to $\lim_{t\to\infty} w(t) = \infty$, which contradicts (16).

We have proved by now, that for every nonoscillatory solution x of $(*)$ $\lim_{t\to\infty} x(t) = 0$ and consequently $x(t)x'(t) < 0$ for all large t. If condition (C_1) is satisfied, then $x(t)x^{(n-1)}(t) > 0$ for all large t and consequently n must be odd. Moreover, as it is easy to see, $\lim_{t\to\infty} x(t) = 0$ implies that $\lim_{t\to\infty} x^{(i)}(t) = 0$ for all $i = 0, 1, \ldots, n-2$.

REMARK. The following examples show that in the case where the condition (C_1) fails, while one of the conditions (C_2) or (C_3) is satisfied, we may have nonoscillatory solutions x of $(*)$ with $\lim_{t\to\infty} x(t) = 0$ and $x(t)x^{(n-1)}(t) < 0$ for all large t. The same examples also show that condition (C_2) may hold in cases where (C_1) and (C_3) fail as well as that condition (C_3) may hold in cases where (C_2) fails.

EXAMPLE 1. Consider the differential equation

$$[t^2 x']' + (1/t^2 \sin^2 1/t) x^3 = 0, \qquad t > 1 \tag{17}$$

This equation admits the positive solution $x(t) = \sin 1/t$, $t > 1$ for which we have $x'(t) = -1/t^2 \cos 1/t < 0$. We observe that condition (C_2) is satisfied since for every $T > 1$ we have

$$\lim_{t\to\infty} \frac{\int_T^t (1/\vartheta^2 \sin^2 1/\theta)\, d\vartheta}{t} = 1$$

while the conditions (C_1) and (C_3) fail. It is also easy to see that condition (C_4), and in particular $(C_4)^*$, is satisfied and consequently, by Theorem 1, every solution x of (17) is oscillatory or such that $\lim_{t\to\infty} x(t) = 0$.

EXAMPLE 2. The differential equation

$$[t^5 x''']' + 6t^2 x^2 \operatorname{sgn} x = 0, \qquad t > 0 \tag{18}$$

admits $x(t) = 1/t$, $t > 0$ as a solution for which $x''' = -6/t^4 < 0$. It is easy to verify that conditions (C_1) and (C_2) fail, while condition (C_3) is satisfied for $k = 5/4$. Since moreover condition $(C_4)^*$ is also satisfied, by Theorem 1, every solution x of (18) is oscillatory or such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0$$

THEOREM 2. *Consider the differential equation* $(**)$ *subject to the condi-*

tions (i)–(iv), (C_1), (C_4) *and*

(C_5) *for every* $c \neq 0$

$$\int^{\infty} g(t)\varphi\left[c\frac{\sigma^{n-1}(t)}{r^*[\sigma(t)]}\right]dt = \pm\infty$$

Then every solution x *of* $(**)$ *satisfies exactly one of the following*:

(α) x *is oscillatory*

(β) x *and its first* $n-2$ *derivatives tend monotonically to zero as* $t\to\infty$

(γ) *It holds*

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} x^{(i)}(t) = \infty \quad (i = 0, 1,\dots, n-2)$$

or

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t) = -\infty \quad \textit{and} \quad \lim_{t\to\infty} x^{(i)}(t) = -\infty \quad (i = 0, 1,\dots, n-2)$$

Moreover, (β) *occurs only in the case of even* n.

PROOF. Let x be a nonoscillatory solution of $(**)$ with $\lim_{t\to\infty} x(t)\neq 0$. As in the proof of Theorem 1, we assume, without loss of generality, that for some $t_1 \geqq t_0$ it holds

$$x[\sigma(t)] > 0 \qquad \text{for every} \quad t \geqq t_1$$

Using $(**)$, (ii) and (iii), it is easy to see that for some $t_2 \geqq t_1$ we have $x^{(n-1)}>0$ or $x^{(n-1)}<0$ on $[t_2, \infty)$. Thus, we have the following two cases:

Case 1. $x^{(n-1)}>0$ *on* $[t_2, \infty)$. By $[r(t)x^{(n-1)}(t)]' \geqq 0$, $t \geqq t_2$, we get $r(t)x^{(n-1)}(t) \geqq r(t_2)x^{(n-1)}(t_2)$ and consequently

$$x^{(n-1)}(t) \geqq r(t_2)x^{(n-1)}(t_2)\frac{1}{r(t)} \qquad \text{for every} \quad t \geqq t_2$$

This, by (C_1), implies that $\lim_{t\to\infty} x^{(n-2)}(t) = \infty$ and hence

$$\lim_{t\to\infty} x^{(i)}(t) = \infty \qquad (i = 0, 1,\dots, n-2)$$

Taking $t_3 \geqq t_2$ such that

$$x^{(i)}(t) > 0 \qquad \text{for every} \quad t \geqq t_3$$

and applying Taylor's formula we obtain

$$x(t) = x(t/2) + \frac{x'(t/2)}{1!}\frac{t}{2} + \cdots + \frac{x^{(n-1)}(t^*)}{(n-1)!}\frac{t^{n-1}}{2^{n-1}}$$

for some t^*, $t/2 \leqq t^* \leqq t$, and every $t \geqq 2t_3 = t_4$. Thus,

$$x(t) \geqq \frac{t^{n-1}}{2^{n-1}(n-1)!} \frac{x^{(n-1)}(t^*)r(t^*)}{r(t^*)} \geqq \frac{x^{(n-1)}(t_3)r(t_3)}{2^{n-1}(n-1)!} \frac{t^{n-1}}{r^*(t)} \qquad \text{for every} \quad t \geqq t_4$$

and consequently there exists some $t_5 \geqq t_4$ such that

$$x[\sigma(t)] \geqq c\frac{\sigma^{n-1}(t)}{r^*[\sigma(t)]} \qquad \text{for every} \quad t \geqq t_5 \tag{19}$$

where $c = x^{(n-1)}(t_3)r(t_3)/2^{n-1}(n-1)!$.

Now, from equation (**) integrating from t_5 to $t \geqq t_5$ and using (19) and (C_5) it is easy to see that

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t) = \infty$$

Hence the solution x satisfies (γ).

Case 2. $x^{(n-1)} < 0$ *on* $[t_2, \infty)$. By considering the functions $z_1 = -z$ and $w_1 = -w$, respectively, in place of the functions z and w of the proof of Theorem 1 and using Lemma 2, we obtain the desired contradictions.

The proof of the theorem is now obvious.

2. Further oscillation results

LEMMA 4. (*Comparison principle*). *Let the differential equations*

$$[s(t)x^{(n-1)}(t)]' + Q(t, x^{(n-1)}(t)) + F(t, x<\tau_0(t)>,\dots, x^{(n-1)}<\tau_{n-1}(t)>) = 0, \qquad t \geqq t_0 \tag{E}$$

and

$$[r(t)y^{(n-1)}(t)]' + g(t)G(t, y<\sigma_0(t)>,\dots, y^{(n-1)}<\sigma_{n-1}(t)>) = 0 \tag{$E_{g,r}$}$$

where

$$x<\tau_i(t)> \equiv (x[\tau_{i1}(t)],\dots, x[\tau_{i\mu_i}(t)]),$$

$$x<\sigma_i(t)> \equiv (x[\sigma_{i1}(t)],\dots, x[\sigma_{i\nu_i}(t)])$$

μ_i, ν_i *are positive integers* $(i = 0, 1,\dots, n-1)$, *and* g, r *belong to certain function classes* $\mathscr{G}$, $\mathscr{R}$. *Let also that for any* $T \geqq t_0$, $g_{z,T}$ *and* $r_{z,T}$ *denote the functions defined by*

$$g_{z,T}(t) = \frac{F(t, z<\tau_0(t)>, \ldots, z^{(n-1)}<\tau_{n-1}(t)>)}{G(t, z<\sigma_0(t)>, \ldots, z^{(n-1)}<\sigma_{n-1}(t)>)} \cdot$$

$$\cdot \exp\Big(\int_T^t \frac{Q(\vartheta, z^{(n-1)}(\vartheta)}{s(\vartheta)z^{(n-1)}(\vartheta)} d\vartheta\Big), \qquad t \geqq T$$

$$r_{z,T}(t) = s(t) \exp\Big(\int_T^t \frac{Q(\vartheta, z^{(n-1)}(\vartheta))}{s(\vartheta)z^{(n-1)}(\vartheta)} d\vartheta\Big), \qquad t \geqq T$$

If P is a propositional function with domain a function class $\mathscr{E}$ and

$$\mathscr{S} = \{x \in \mathscr{E} : x \text{ is a solution of } (E)\},$$

$$\mathscr{S}_{g,r} = \{x \in \mathscr{E} : x \text{ is a solution of } (E_{g,r})\}$$

then

$$(\forall g \in \mathscr{G})(\forall r \in \mathscr{R})(\forall y \in \mathscr{S}_{g,r})P(y)$$

and

$$(\forall x \in \mathscr{S}) \sim P(x) \Longrightarrow (\exists T \geqq t_0) g_{x,T} \in \mathscr{G} \quad \text{and} \quad r_{x,T} \in \mathscr{R}$$

imply

$$(\forall x \in \mathscr{S})P(x)$$

PROOF. If the conclusion is false, then for some $z \in \mathscr{S}$ we have $\sim P(z)$ and consequently for some $T \geqq t_0$, $g_{z,T} \in \mathscr{G}$ and $r_{z,T} \in \mathscr{R}$. Thus

$$(\forall y \in \mathscr{S}_{g_{z,T}, r_{z,T}})P(y)$$

But z is obviously a solution of the differential equation $(E_{g_{z,T}, r_{z,T}})$, i.e., $z \in \mathscr{S}_{g,r}$, and consequently $P(z)$is true, which is a contradiction.

Next we give applications of Lemma 4 in order to extend Theorem 1 to differential equations of the form (***). It is obvious that parallel arguments can be used in order to extend Theorem 2 to differential equations of the form

$$[s(t)x^{(n-1)}(t)]' + Q(t, x^{(n-1)}(t)) - p(t)F(x[\sigma(t)]) = 0$$

In addition to (i) we suppose that

(I) $p: [t_0, \infty) \mapsto (0, \infty)$ *is continuous*

(II) $F: \times \mathbf{R} \mapsto \mathbf{R}$ *is continuous, nondecreasing and such that*

$$y \neq 0 \Longrightarrow yF(y) > 0$$

(III) $s: [t_0, \infty) \mapsto (0, \infty)$ *is continuous*

(IV) $Q: [t_0, \infty) \times \mathbf{R} \mapsto \mathbf{R}$ *is continuous and such that*

$$(\forall t \geqq t_0) y \neq 0 \Longrightarrow yQ(t, y) > 0$$

(V) *There exists a continuous function* $q_1 : [t_0, \infty) \mapsto [0, \infty)$ *such that*

$$(\forall t \geqq t_0)(\forall y \neq 0) \frac{Q(t, y)}{y} \leqq q_1(t)$$

To obtain our results, we also need the following lemma.

LEMMA 5. *Let u be as in Lemma 2 with* $\delta = +1$. *Then there exists a positive constant K such that*

$$u(t) \leqq K t^{\nu-2} \int_a^t \frac{d\vartheta}{m(\vartheta)} \equiv K R_\nu(t) \qquad \text{for all large } t \tag{20}$$

PROOF. It is obvious that $\lim_{t\to\infty} \frac{u(t)}{t^{\nu-2}}$ exists in $\mathbf{R}^*$ ($\mathbf{R}^*$ is the extended real line). Thus we have the following two cases:

Case 1. $\lim_{t\to\infty} \frac{u(t)}{t^{\nu-2}} < \infty$. In this case there exists a $K_1 > 0$ such that

$$u(t) \leqq K_1 t^{\nu-2} \qquad \text{for every} \quad t \geqq a$$

Since moreover

$$\int_a^t \frac{d\vartheta}{m(\vartheta)} \geqq \int_a^{2a} \frac{d\vartheta}{m(\vartheta)} \equiv K_2 \qquad \text{for every} \quad t \geqq 2a$$

it is obvious that

$$u(t) \leqq K t^{\nu-2} \int_a^t \frac{d\vartheta}{m(\vartheta)} \qquad \text{for every} \quad t \geqq 2a$$

where $K = K_1/K_2$.

Case 2. $\lim_{t\to\infty} \frac{u(t)}{t^{\nu-2}} = \infty$. In this case we obviously have $\lim_{t\to\infty} u^{(\nu-2)}(t) = \infty$. But, we also have $0 \leqq \lim_{t\to\infty} m(t) x^{(\nu-1)}(t) < \infty$ and consequently there exists $L_1 > 0$ such that

$$u^{(\nu-1)}(t) < \frac{L_1}{m(t)} \qquad \text{for evry} \quad t \geqq a$$

Thus

$$u^{(\nu-2)}(t) - u^{(\nu-2)}(a) \leqq L_1 \int_a^t \frac{d\vartheta}{m(\vartheta)} \qquad \text{for every} \quad t \geqq a \tag{21}$$

which implies that

$$\int^{\infty} \frac{d\vartheta}{m(\vartheta)} = \infty \tag{22}$$

If now $t_1 \geqq a$ is such that $u^{(i)}(t) > 0$ for every $t \geqq t_1$ $(i=0, 1, \dots, \nu-1)$ (cf. Lemma 1), then by Taylor's formula we get

$$u(t) \leqq u(t_1) + \frac{u'(t_1)}{1!}(t-t_1) + \cdots + \frac{u^{(\nu-2)}(t)}{(\nu-2)!}(t-t_1)^{\nu-2} \qquad \text{for every} \quad t \geqq t_1$$

and consequently, using (21),

$$u(t) \leqq u(t_1) + \cdots + \frac{u^{(\nu-2)}(a)}{(\nu-2)!}(t-t_1)^{\nu-2} + \frac{L_1}{(\nu-2)!}(t-t_1)^{\nu-2} \int_a^t \frac{d\vartheta}{m(\vartheta)}, \; t \geqq t_1$$

which, by (22), easily leads to (20).

We introduce, now, the following conditions in which $q_2(t)$ denotes the (non-negative) function defined by

$$q_2(t) = \inf_{y \neq 0} Q(t, y)/y, \qquad t \geqq t_0$$

and for any $T \geqq t_0$

$$r_i(t, T) = s(t) \exp\left(\int_T^t \frac{q_i(\vartheta)}{s(\vartheta)} d\vartheta\right), \; r_i^*(t, T) = \max_{\frac{t}{2} \leqq \vartheta \leqq t} r_i(\vartheta, T) \qquad (i = 1, 2)$$

(H_1) *for every* $T \geqq t_0$

$$\int^{\infty} \frac{dt}{r_1(t, T)} = \infty$$

(H_2) *for every* $T \geqq t_0$

$$\int^{\infty} \frac{\int_T^t p(\vartheta) d\vartheta}{r_1(t, T)} dt = \infty$$

(H_3) *for every* $T \geqq t_0$

$$\int^{\infty} \frac{dt}{r_2(t, T)} < \infty \text{ and for some } k > 1, \; \int^{\infty} \sigma^{n-2}(t) p(t) h^k(t, T) dt = \infty,$$

where

$$h(t, T) = \int_t^{\infty} \frac{d\vartheta}{r_1(\vartheta, T)}$$

(H_4) *there exists a continuous and nondecreasing function* $f\colon \mathbf{R}\mapsto\mathbf{R}$ *such that*

$$y \neq 0 \Longrightarrow f(y) > 0$$

$$\int^{\infty} \frac{dy}{F(y)f(y)} < \infty, \int^{-\infty} \frac{dy}{F(y)f(y)} < \infty$$

and for every $T \geqq t_0$ *and any* c *with* $|c|$ *sufficiently large*

$$\int^{\infty} \frac{p(t)r_2(t, T)}{s(t)f(cR_n(\sigma(t), T))} \left(\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r_1^*[\sigma(\vartheta)]} d\vartheta \right) dt = \infty$$

where $R_n(t, T) = t^{n-2} \int_T^t \frac{d\vartheta}{r_2(\vartheta, T)}$.

THEOREM 3. *Consider the differential equation* (***) *subject to the conditions* (i), (I)–(V) *and* (H_4). *Then*

a) *under condition* (H_1) *every solution of* (***) *is for n even oscillatory and for n odd either oscillatory on tending monotonically to zero as* $t\to\infty$ *together with its first* $n-2$ *derivatives.*

b) *under* (H_2) *or* (H_3) *every solution of* (***) *is either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with its first* $n-2$ *derivatives.*

Proof. Consider the differential equations (***) and

$$[r(t)y^{(n-1)}(t)]' + g(t)f(x[\sigma(t)])F(x[\sigma(t)]) = 0 \tag{23}$$

in place of (E) and $(E_{g,r})$ respectively (cf. Lemma 4). Let $\mathscr{E}$ be the class of all functions x defined on an interval $[t_x, \infty)$ and let P be such that

$$P(x)\colon x \text{ is oscillatory or } \lim_{t\to\infty} x(t) = 0$$

Furthermore, let $\mathscr{G}$ be the class of all nonnegative functions g defined on a half-line $[t_g, \infty)$ and $\mathscr{R}$ the class of all positive functions r defined on a half-line $[t_r, \infty)$ such that

$(\forall r \in \mathscr{R})(\forall g \in \mathscr{G})$ conditions (C_1) and (C_4) are satisfied

(respectively:

$(\forall r \in \mathscr{R})(\forall g \in \mathscr{G})$ conditions (C_2) and (C_4) are satisfied,

respectively:

$(\forall r \in \mathscr{R})(\forall g \in \mathscr{G})$ conditions (C_3) and (C_4) are satisfied)

By Theorem 1, it is obvious that for any $g \in \mathscr{G}$ and $r \in \mathscr{R}$ and every solution

y of (23), $P(y)$ is satisfied. Moreover, if x is a solution of (***) for which $P(x)$ is not true, i.e. x is nonoscillatory and $\lim_{t\to\infty} x(t)\neq 0$, then we have

$$x(t)x^{(n-1)}(t) > 0 \quad \text{or} \quad x(t)x^{(n-1)}(t) < 0 \qquad \text{for all large} \quad t.$$

To prove this, we suppose, without loss of generality, that for some $t_1 \geqq t_0$ we have

$$x[\sigma(t)] > 0 \qquad \text{for every} \quad t \geqq T$$

If now $t^* \geqq T$ is a root of $x^{(n-1)}(t)$, then from equation (***) we get

$$[s(t)x^{(n-1)}(t)]'_{t=t^*} < 0$$

and consequently there exists a maximal interval (t^*_-, t^*_+) containing t^* for which we have

$$[s(t)x^{(n-1)}(t)]' < 0 \qquad \text{for every} \quad t \in (t^*_-, t^*_+) \tag{24}$$

Thus, by $s(t^*)x^{(n-1)}(t^*)=0$ and (24) we must have

$$s(t)x^{(n-1)}(t) < 0 \qquad \text{for every} \quad t \in (t^*, t^*_+)$$

which, again by (24), implies that

$$\lim_{t\to t^*_+} s(t)x^{(n-1)}(t) < 0$$

By this last relation, taking into account the definition of t^*_+, it is easy to see that $t^*_+ = \infty$. Hence $s(t)x^{(n-1)}(t)$ and consequently $x^{(n-1)}(t)$ is of constant sign for all large t.

Without loss of generality we suppose that

$$x^{(n-1)} > 0 \quad \text{or} \quad x^{(n-1)} < 0 \quad \text{on} \quad [T, \infty)$$

Next we consider the functions

$$r_{x,T}(t) = s(t)\exp\left(\int_T^t \frac{Q(\vartheta, x^{(n-1)}(\vartheta))}{s(\vartheta)x^{(n-1)}(\vartheta)}\, d\vartheta\right), \; t \geqq T$$

$$g_{x,T}(t) = \frac{p(t)}{f(x[\sigma(t)])}\exp\left(\int_T^t \frac{Q(\vartheta, x^{(n-1)}(\vartheta))}{s(\vartheta)x^{(n-1)}(\vartheta)}\, d\vartheta\right) \equiv \frac{p(t)r_{x,T}(t)}{f(x[\sigma(t)])s(t)}, \; t \geqq T$$

and the equation

$$[r_{x,T}(t)y^{(n-1)}(t)]' + g_{x,T}(t)f(y[\sigma(t)])F(y[\sigma(t)]) = 0, \quad t \geqq T \tag{$E_{g_{x,T}, r_{x,T}}$}$$

It is easy to check, by conditions (H_1)–(H_3), that $r_{x,T} \in \mathscr{R}$. Hence, by applying Lemma 3 to $(E_{g_{x,T},r_{x,T}})$, we must have

$$x^{(n-1)}(t) > 0 \qquad \text{for every} \quad t \geqq T$$

Since moreover

$$[r_{x,T}(t)x^{(n-1)}(t)]' < 0 \qquad \text{for every} \quad t \geqq T$$

by Lemma 5, we have that there exist $K>0$ and $T_1 \geqq T$ with

$$x[\sigma(t)] \leqq K\sigma^{n-2}(t)\int_T^t \frac{d\vartheta}{r_{x,T}(\vartheta)} \qquad \text{for every} \quad t \geqq T_1$$

Thus, for every $t \geqq T_1$

$$g_{x,T}(t)\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*_{x,T}[\sigma(\vartheta)]}d\vartheta \geqq \frac{p(t)r_2(t,\, T)}{f(KR_n[\sigma(t),\, T])}\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*_1[\sigma(\vartheta)]}d\vartheta$$

which by (H_4) implies that $g_{x,T} \in \mathscr{G}$.

Now, applying Lemma 4, we conclude that all solutions of (***) are oscillatory or tending to zero as $t \to \infty$.

The proof of the theorem is completed as that of Theorem 1.

REMARK 1. In the case where $Q(t,\, x^{(n-1)}(t)) \equiv 0$ it is obvious that condition (I) can be relaxed to

(I)* $p\colon [t_0,\, \infty) \mapsto [0,\, \infty)$ *is continuous and not identically zero for all large* t.

Also, in the same case, we can take $q_1 \equiv 0 \equiv q_2$, which implies that

$$r_1(t,\, T) = r_2(t,\, T) = s(t)$$

and consequently the conditions (H_1)–(H_4) take the forms

$(H_1)^*$
$$\int^\infty \frac{dt}{s(t)} = \infty$$

$(H_2)^*$ *for every* $T \geqq t_0$

$$\int^\infty \frac{\int_T^t p(\vartheta)d\vartheta}{s(\vartheta)}dt = \infty$$

$(H_3)^*$ *for every* $T \geqq t_0$

$$\int^\infty \frac{dt}{s(t)} < \infty \quad \text{and for some} \quad k > 1,\ \int^\infty \sigma^{n-2}(t)p(t)h^k(t)dt = \infty$$

where

$$h(t) = \int_t^\infty \frac{d\vartheta}{s(\vartheta)}$$

$(H_4)^*$ *there exists a continuous and nondecreasing function* $f\colon \mathbf{R}\to\mathbf{R}$ *such that*

$$y \neq 0 \Longrightarrow f(y) > 0,$$

(25) $$\int^\infty \frac{dy}{F(y)f(y)} < \infty, \int^{-\infty} \frac{dy}{F(y)f(y)} < \infty$$

and for every $T \geqq t_0$ *and any* c *with* $|c|$ *sufficiently large*

(26) $$\int^\infty \frac{p(t)}{f(cR_n[\sigma(t), T]} \left(\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{s^*[\sigma(\vartheta)]} d\vartheta \right) dt = \infty$$

where

$$R_n(t, T) = t^{n-2} \int_T^t \frac{d\vartheta}{s(\vartheta)}$$

Thus we have the following:

COROLLARY 1. *Consider the differential equation*

(27) $$[s(t)x^{(n-1)}(t)]' + p(t)F(x[\sigma(t)]) = 0$$

subject to the conditions (i), (I)*, (II), (III) *and* $(H_4)^*$.

Then

a) *under condition* $(H_1)^*$ *all solutions of* (27) *are for* n *even oscillatory, while for* n *odd are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-2$ *derivatives.*

b) *under* $(H_2)^*$ *or* $(H_3)^*$ *every solution of* (27) *is either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with its first* $n-2$ *derivatives.*

Let us now consider in particular the case where the function ρ:

(28) $$\rho(t) = \frac{s(t)}{t}, \ t \geqq t_1, \ t_1 > \max\{t_0, 0\}$$

is nonincreasing, when we obviously have that the function s satisfies $(H_1)^*$. Since, for some ϑ_1, $\frac{\vartheta}{2} \leqq \vartheta_1 \leqq \vartheta$ and every $t \geqq T$

$$\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{s^*[\sigma(\vartheta)]} d\vartheta = \int_{\sigma(T)}^{\sigma(t)} \frac{\vartheta^{n-2}}{s^*(\vartheta)} d\vartheta = \int_{\sigma(T)}^{\sigma(t)} \frac{\vartheta^{n-2}}{s(\vartheta_1)} d\vartheta$$

holds, by (28), we obtain

$$\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{s^*[\sigma(\vartheta)]}d\vartheta \geqq \int_{\sigma(t)/2}^{\sigma(t)} \frac{\vartheta^{n-3}\vartheta_1}{s(\vartheta_1)}d\vartheta \geqq \frac{1}{4}\frac{\sigma(t)}{s[\sigma(t)/4]}\cdot\int_{\sigma(t)/2}^{\sigma(t)} \vartheta^{n-3}d\vartheta =$$

$$=\frac{2^{n-2}-1}{2^n}\cdot\frac{\sigma^{n-1}(t)}{s[\sigma(t)/4]}$$

Thus (26) can be replaced by:
"for every $T \geqq t_0$ and any c with $|c| \geqq 1$

$$\int^{\infty} \frac{p(t)\sigma^{n-1}(t)}{f(cR_n[\sigma(t), T])s[\sigma(t)/4]}dt = \infty" \tag{29}$$

Moreover, supposing that for any β: $0<\beta\leqq 1$ the function

$$f(z) = \begin{cases} \dfrac{1}{\rho(R_n^{-1}(\beta|z|))}, & \text{for } |z| \geqq t_1 \\ \dfrac{1}{\rho(R_n^{-1}(\beta t_1))}, & \text{for } |z| < t_1 \end{cases}$$

satisfies (25) it is easy to see, by putting $c=\frac{1}{4\beta}$, that (29) can be replaced by:

$$\int^{\infty} \sigma^{n-2}(t)p(t)dt = \infty$$

Thus we obtain the following result, which is due to Ševelo and Varech ([6] Th. 1)

COROLLARY 2. *Consider the differential equation* (27) *subject to the conditions* (i), (I)*, (II), (III),

(VI) *the function* $\rho(t)=\frac{s(t)}{t}$, $t>\max\{t_0, 0\}$ *is nondecreasing*

and

(VIII) *for any* β: $0<\beta\leqq 1$

$$\int^{\infty} \frac{\rho(R_n^{-1}(\beta z))}{F(z)}dz < \infty, \quad \int^{-\infty} \frac{\rho(R_n^{-1}(-\beta z))}{F(z)}dz < \infty$$

Then, under the condition

$$\int^{\infty} \sigma^{n-2}(t)p(t)dt = \infty$$

all solutions of (27) *are for n even oscillatory, while for n odd are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $n-2$ *derivatives.*

REMARK 2. If $s(t)\equiv 1$, then the function $r_{x,T}$, which is defined in the proof of Theorem 3, is obviously nondecreasing. Hence,

$$g_{x,T}(t)\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*_{x,T}[\sigma(\vartheta)]} = \frac{p(t)r_{x,T}(t)}{f(x[\sigma(t)])}\int_T^t \frac{\sigma^{n-2}(\vartheta)\sigma'(\vartheta)}{r^*_{x,T}[\sigma(\vartheta)]}d\vartheta$$

$$\geqq \frac{p(t)}{f(x[\sigma(t)])}\int_T^t \sigma^{n-2}(\vartheta)\sigma'(\vartheta)d\vartheta, \qquad t\geqq T$$

and consequently (H_4) can be replaced by:

$(H_4)_*$ *there exists a continuous and nondecreasing function* f: $\mathbf{R}\mapsto\mathbf{R}$ *such that*

$$y\neq 0 \Longrightarrow f(y)>0,$$

$$\int^{\infty}\frac{dy}{F(y)f(y)}<\infty, \int^{-\infty}\frac{dy}{F(y)f(y)}<\infty$$

and for every $T\geqq t_0$ *and any* c *with* $|c|$ *sufficiently large*

$$\int^{\infty}\frac{p(t)g^{n-1}(t)}{f(cR_n[\sigma(t), T])}dt=\infty$$

Thus, we can easily derive the following theorem in which, $(H_1)_*$, $(H_2)_*$, $(H_3)_*$ denote the conditions (H_1), (H_2), (H_3) respectively for $s(t)\equiv 1$.

THEOREM 4. *Consider the differential equation*

$$x^{(n)}(t)+Q(t, x^{(n-1)}(t))+p(t)F(x[\sigma(t)])=0 \tag{30}$$

subject to the conditions (i), (I), (II), (IV), (V) *and* $(H_4)_*$. *Then*

a) *under condition* $(H_1)_*$ *all solutions of* (30) *are for n even oscillatory, while for n odd are either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with their first* $(n-1)$ *derivatives*

b) *under* $(H_2)_*$ *or* $(H_3)_*$ *every solution of* (30) *is either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with its first* $n-1$ *derivatives.*

This theorem extends and improves a recent result due to Naito ([5] Th. 1) in several directions.

REMARK 3. We notice that we can also obtain, by using Theorem 1 and applying Lemma 4, oscillation results similar to those in [8] for differential equations with retarded arguments of the form

$$[s(t)x^{(n-1)}(t)]'+Q(t, x^{(n-1)}(t))+p(t)F(x[\sigma_0(t)], x[\sigma_1(t)],\dots, x[\sigma_\mu(t)])=0$$

We omit the details.

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