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Certain Functional of Probability Measures on Hilbert Spaces

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§1. Introduction and results

Let E be a real separable Hilbert space with inner product $<\ ,\ >$ and $\mathscr{E}$ the σ-algebra of all Borel subsets of E. We denote by $\mathscr{P}$ the set of all probability measures μ on $(E, \mathscr{E})$ with a finite second moment; $\int \|x\|^2 d\mu(x) < \infty$. For each $\mu \in \mathscr{P}$ there exist a vector m (mean vector) and a bounded linear operator V (covariance operator) with $\int <x, u> d\mu(x) = <m, u>$ and $\int <x-m, u> <x-m, v> d\mu(x) = <Vu, v>$ for all $u, v \in E$. Since the covariance operator is symmetric, non-negative and nuclear, we can find a unique Gaussian measure γ_μ on $(E, \mathscr{E})$ which has the same mean vector and covariance operator as those of μ [4; p. 14 and p. 18]. Let $\mathscr{M}(\mu)$ be the set of all probability measures M on $(E \times E, \mathscr{E} \otimes \mathscr{E})$ with $M(A \times E) = \mu(A)$ and $M(E \times A) = \gamma_\mu(A)$ for all $A \in \mathscr{E}$. We consider a function: $M \to e[\mu; M] = \iint \|x-y\|^2 dM(x, y)$ on $\mathscr{M}(\mu)$, and define a functional e on $\mathscr{P}$ by

$$e[\mu] = \inf_{M \in \mathscr{M}(\mu)} e[\mu; M].$$

The functional e was first introduced by H. Tanaka in the case where E is the one-dimensional space and its basic properties were studied also by himself [5]. H. Murata and H. Tanaka [2] extended the results to the case of multi-dimensional Euclidean spaces.

The purpose of this paper is to show that some of their results can be extended to the case of Hilbert spaces, by the method similar to that of [2] with a slight simplification. That is, we shall prove:

THEOREM 1. *For each $\mu \in \mathscr{P}$ there exists an $M \in \mathscr{M}(\mu)$ with $e[\mu] = e[\mu; M]$ and such a measure M has the form; $M(A \times B) = \gamma_\mu(f^{-1}(A) \cap B)$ for all $A, B \in \mathscr{E}$ with a Borel measurable mapping f from E into itself. Consequently $e[\mu] = \int \|f(y) - y\|^2 d\gamma_\mu(y)$.*

THEOREM 2. *Let μ_1 and μ_2 be measures in $\mathscr{P}$ and $\mu_1 * \mu_2$ their convolution. Then*

$$e[\mu_1 * \mu_2] \leqq e[\mu_1] + e[\mu_2],$$

and the equality holds if and only if both μ_1 *and* μ_2 *are Gaussian.*

Using the results we shall prove also that a sequence of probability distributions of certain stochastic processes $X_n = (X_n(t))_{0 \leqq t < 1}$ converges to a Gaussian measure in $L_2[0, 1)$.

§2. Lemmas

In this section we denote by $\mathscr{C}(E^n)$ the Banach space of all real valued, bounded and continuous functions on E^n with the supremum norm; $\|\varphi\|_\infty = \sup|\varphi(x)|$, and by $\mathscr{C}^*(E^n)$ the topological dual of $\mathscr{C}(E^n)$. Since, for each $M \in \mathscr{M}(\mu)$, the function: $\varphi \to M(\varphi) = \int_{E^2} \varphi dM$ on $\mathscr{C}(E^2)$ is continuous and linear, we consider $\mathscr{M}(\mu)$ as a subset of $\mathscr{C}^*(E^2)$.

LEMMA 1. *For each* $\mu \in \mathscr{P}$ *there exists an* $M \in \mathscr{M}(\mu)$ *with* $e[\mu] = e[\mu; M]$.

PROOF. We shall prove first that $\mathscr{M}(\mu)$ is a weakly compact subset of $\mathscr{C}^*(E^2)$. Let U^0 be the closed unit ball in $\mathscr{C}^*(E^2)$, which is known to be weakly compact. Since $\mathscr{M}(\mu)$ is contained in U^0, it is enough to show that $\mathscr{M}(\mu)$ is weakly closed. Let M_0 be an element in the weak closure of $\mathscr{M}(\mu)$. Then there is a net $(M_\lambda)_{\lambda \in \Lambda}$ in $\mathscr{M}(\mu)$ which converges weakly to M_0. It is easily seen that M_0 is linear and positive, and satisfies $M_0(1) = 1$. For a given $\varepsilon > 0$, we can find a compact subset F of E with $\mu(F) \geqq 1 - \varepsilon/2$ and $\gamma_\mu(F) \geqq 1 - \varepsilon/2$. Let $K = F \times F$. Then $M_\lambda(K) \geqq \mu(F) - \gamma_\mu(F^c) \geqq 1 - \varepsilon$ for all $\lambda \in \Lambda$. Therefore if a function φ in $\mathscr{C}(E^2)$ vanishes on K, $|M_0(\varphi)| = \lim_\lambda |M_\lambda(\varphi)| \leqq \varepsilon \|\varphi\|_\infty$, which implies that M_0 is a Baire (hence Borel) probability measure on E^2. For any $\varphi \in \mathscr{C}(E)$, since $\varphi \circ \pi_i \in \mathscr{C}(E^2)$ $(i = 1, 2)$[1], we have $M_0(\varphi \circ \pi_1) = \lim_\lambda M_\lambda(\varphi \circ \pi_1) = \mu(\varphi)$ and similarly, $M_0(\varphi \circ \pi_2) = \gamma_\mu(\varphi)$, which shows that M_0 belongs to $\mathscr{M}(\mu)$. Thus $\mathscr{M}(\mu)$ is weakly closed. Now let $\varphi_n(x, y) = \inf(n, \|x - y\|^2)$ and $\Phi_n(M) = \int_{E^2} \varphi_n dM$ for each $M \in \mathscr{M}(\mu)$. Then Φ_n are continuous on $\mathscr{M}(\mu)$ and $\Phi_n \uparrow e[\mu; \cdot]$ as $n \to \infty$. Therefore $e[\mu; \cdot]$ is lower semicontinuous on $\mathscr{M}(\mu)$ and hence, there is an $M \in \mathscr{M}(\mu)$ with $e[\mu] = e[\mu; M]$.

From now on we use $\mathscr{M}_0(\mu)$ to denote the set of M, in $\mathscr{M}(\mu)$ and with $e[\mu] = e[\mu; M]$. We set $\Gamma = \{(x, y, x', y') \in E^4 : \langle x - x', y - y' \rangle \geqq 0\}$ and $\mathscr{M}_\Gamma(\mu) = \{M \in \mathscr{M}(\mu) : M \otimes M(\Gamma) = 1\}$, where "$\otimes$" denotes the direct product of measures.

LEMMA 2. $\mathscr{M}_0(\mu) \subset \mathscr{M}_\Gamma(\mu)$ *for all* $\mu \in \mathscr{P}$.

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1) Throughout this paper, π_1 and π_2 denote the first and the second coordinate mappings of E^2 onto E.

PROOF. To avoid the trivial case we assume $\int \|x-m\|^2 d\mu(x)>0$, where m is the mean vector of μ. Under the assumption any $M\in\mathscr{M}(\mu)$ has no atomic point. Suppose that there is an $M\in\mathscr{M}_0(\mu)$ with $M\otimes M(\Gamma)<1$. There then exist $\mathscr{E}^2$-measurable sets A and B with $A\times B\subset\Gamma^c$ and $M(A)M(B)=M\otimes M(A\times B)>0$. We choose two compact sets K and K' with $K\subset A$ and $K'\subset B$ and $M(K)M(K')>0$. The function: $(x, y, x', y')\to<x-x', y-y'>$ is continuous and strictly negative on $K\times K'$ and hence, there is a constant $\delta>0$ with $<x-x', y-y'>\leqq-\delta$ for all $(x, y, x', y')\in K\times K'$. This implies also that K and K' are disjoint. Now we consider an ergodic automorphism T acting on $(E^2, \mathscr{E}^2, M)$ (such an automorphism always exists, for the space $(E^2, \mathscr{E}^2, M)$ is isomorphic mod 0 to the interval $[0, 1)$ with the Lebesgue measure). By ergodicity there is an integer $n\geqq 0$ with $M(K\cap T^{-n}K')>0$. We set $C=K\cap T^{-n}K'$, $C'=T^nC$ and $D=(C\cup C')^c$. The sets C and C' are disjoint since they are subsets of K and K' respectively. We define a mapping S from E^2 into itself by

$$S(z)=\begin{cases}(\pi_1T^nz, \pi_2z) & \text{for } z\in C,\\ (\pi_1T^{-n}z, \pi_2z) & \text{for } z\in C',\\ (\pi_1z, \pi_2z) & \text{for } z\in D.\end{cases}$$

Let $M_S(A)=M(S^{-1}(A))$ for all $A\in\mathscr{E}^2$. Then it is easy to see that M_S belongs to $\mathscr{M}(\mu)$ and that

$$\begin{aligned}&e[\mu; M]-e[\mu; M_S]\\ &=\int_{E^2}\|\pi_1-\pi_2\|^2\,dM-\int_{E^2}\|\pi_1-\pi_2\|^2dM_S\\ &=-2\int_C<\pi_1z-\pi_1T^nz, \pi_2z-\pi_2T^nz>dM(z)\\ &\geqq\delta M(C)>0,\end{aligned}$$

because $(z, T^nz)\in C\times C'\subset K\times K'$ whenever $z\in C$. This contradicts the assumption that M belongs to $\mathscr{M}_0(\mu)$.

Let $M\in\mathscr{M}(\mu)$. A Markov kernel P_M, defined on $\mathscr{E}\times E$, is called a regular conditional probability of M (with respect to γ_μ) if $M(A\times B)=\int_B P_M(A, y)d\gamma_\mu(y)$ for all $A, B\in\mathscr{E}$.

LEMMA 3. *Let $M\in\mathscr{M}_\Gamma(\mu)$. Then there exist a set N in $\mathscr{E}$ and a regular conditional probability P_M of M such that $\gamma_\mu(N)=0$ and, for any $y, y'\in N^c$*

$$P_M(\ , y)\otimes P_M(\ , y')(\Gamma(y, y'))=1,$$

where $\Gamma(y, y')=\{(x, x'): (x, y, x', y')\in\Gamma\}$.

PROOF. Let $(G_n)_{n\geq 1}$ be a countable open base of E. For each $n\geq 1$ we define a partition $\mathscr{B}_n=\{B_{nj}: 0\leq j\leq 2^n-1\}$ of E as follows; if $j=k_1 2^{n-1}+k_2 2^{n-2}+\cdots+k_n$ is the binary expansion of j, $B_{nj}=G_1^{k_1}\cap G_2^{k_2}\cap\cdots\cap G_n^{k_n}$, where G^1 and G^0 are understood to be G and G^c respectively. We denote by $\mathscr{E}_n$ the σ-algebra generated by $\mathscr{B}_n$. Then the algebra $\mathscr{B}=\cup_{n\geq 1}\mathscr{E}_n$ contains at most countable number of elements, and generates the σ-algebra $\mathscr{E}$. For a fixed $A\in\mathscr{E}$ we define functions $Q_M^n(A,\ )$ on E by

$$Q_M^n(A, y)=\begin{cases} M(A\times B_{nj})/\gamma_\mu(B_{nj}) & \text{if}\quad y\in B_{nj}\quad\text{and}\quad \gamma_\mu(B_{nj})>0\\ 0 & \text{if}\quad y\in B_{nj}\quad\text{and}\quad \gamma_\mu(B_{nj})=0.\end{cases}$$

Then $(Q_M^n(A,\ ), \mathscr{E}_n)_{n\geq 1}$ is a martingale on the probability space $(E, \mathscr{E}, \gamma_\mu)$ and hence, there is a set N_A with $\gamma_\mu(N_A)=0$ such that, for each $y\notin N_A$, $Q_M(A, y)=\lim Q_M^n(A, y)$ exists. Now let $N=\cup_{A\in\mathscr{B}}N_A$. Then $\gamma_\mu(N)=0$ and, for each $y\notin N$, $Q_M(\ , y)$ is a finitely additive probability measure on $\mathscr{B}$. Using the injection: $x\to(I_{G_n}(x))_{n\geq 1}$ of E into $\{0, 1\}^{\aleph_0}$, we can extend each $Q_M(\ , y)$ to a probability measure on $\mathscr{E}$, where I_G denotes the indicator of a set G (for the details, see [3]). We define probability measures $P_M(\ , y)$ by; $P_M(\ , y)=Q_M(\ , y)$ for $y\notin N$ and $P_M(\ , y)=\mu_0$ for $y\in N$, where μ_0 is an arbitrary probability measure on E. Then P_M is a regular conditional probability of M. We remark here that, for each $y\notin N$, the sequence of measures $(Q_M^n(\ , y))_{n\geq 1}$ converges weakly to $P_M(\ , y)$. In fact, for an $\varepsilon>0$ and an open set G in E, we can find $A\in\mathscr{B}$ with $A\subset G$ and $P_M(G, y)-\varepsilon\leq P_M(A, y)$, for $\mathscr{B}$ contains the open base $(G_n)_{n\geq 1}$. Hence $P_M(G, y)-\varepsilon\leq\lim Q_M^n(A, y)\leq\liminf Q_M^n(G, y)$, and so, $P_M(G, y)\leq\liminf Q_M^n(G, y)$, which implies that $(Q_M^n(\ , y))$ converges weakly to $P_M(\ , y)$. By the assumption $Q_M^n(\ , y)\otimes Q_M^n(\ , y')(\Gamma(y, y'))=1$ for almost all (y, y') with respect to $\gamma_\mu\otimes\gamma_\mu$, however, for any $A\in\mathscr{E}$, since the function $Q_M^n(A,\ )$ is equal to a constant on each B_{nj}, the equality holds for all $(y, y')\in E^2$. The sequence $(Q_M^n(\ , y))_{n\geq 1}$ converges weakly to $P_M(\ , y)$ for any $y\notin N$ and the set $\Gamma(y, y')$ is closed, tending n to infinity, we have $P_M(\ , y)\otimes P_M(\ , y')(\Gamma(y, y'))=1$ for all $y, y'\notin N$.

§3. Proofs of the theorems

In this section we assume that the covariance operators of measures in $\mathscr{P}$ are non-singular since the other case is reduced easily to this case. For each $\mu\in\mathscr{P}$ we denote by $\mathscr{F}(\mu)$ the set of all Borel measurable mappings f from E into itself with $\gamma_\mu(f^{-1}(A))=\mu(A)$ for all $A\in\mathscr{E}$. For each $f\in\mathscr{F}(\mu)$ there exists exactly one probability measure M_f in $\mathscr{M}(\mu)$ with $M_f(A\times B)=\gamma_\mu(f^{-1}(A)\cap B)$. We denote by $\mathscr{M}_F(\mu)$ the set of all M_f with $f\in\mathscr{F}(\mu)$.

First we shall prove Theorem 1. To this end it is enough to show that

$\mathscr{M}_\Gamma(\mu)\subset\mathscr{M}_F(\mu)$, for, by Lemma 1 and Lemma 2, $\mathscr{M}_0(\mu)\neq\emptyset$ and $\mathscr{M}_0(\mu)\subset\mathscr{M}_F(\mu)$. Now let $M\in\mathscr{M}_\Gamma(\mu)$. By Lemma 3, there exist a set $N\in\mathscr{E}$ and a regular conditional probability P_M of M with $\gamma_\mu(N)=0$ and $P_M(\ ,y)\otimes P_M(\ ,y')(\Gamma(y,y'))=1$ for all $y, y'\notin N$. We denote by $S(y)$ the support of $P_M(\ ,y)$; the smallest closed set with full measure. For each $y, y'\notin N$, since $\Gamma(y,y')^c$ is open and $P_M(\ ,y)\otimes P_M(\ ,y')(\Gamma(y,y')^c)=0$, $S(y)\times S(y')\subset\Gamma(y,y')$, that is, $<x-x', y-y'>\geqq 0$ for all $(x,x')\in S(y)\times S(y')$. Let V be the covariance operator of μ and $(e_n)_{n\geqq 1}$ an orthonormal basis of E consisting of eigen vectors of V. For each $n\geqq 1$, we set $\gamma_n=\gamma_\mu \mathrm{pr}_{[e_n]}^{-1}$ and $\gamma_n^\perp=\gamma_\mu(\mathrm{pr}_{[e_n]^\perp})^{-1}$.[2)] Notice that γ_n is equivalent (mutually absolutely continuous) to the one-dimensional Lebesgue measure. A family $(C(\eta))_{\eta\in R^1}$ of non-empty subsets of R^1 is said to be *monotone* if there is a set N_0 with Lebesgue measure zero such that if $\eta, \eta'\notin N_0$ then $(\xi-\xi')(\eta-\eta')\geqq 0$ for all $(\xi,\xi')\in C(\eta)\times C(\eta')$. It is known that if $(C(\eta))_{\eta\in R^1}$ is monotone, $\mathrm{Card}\,(C(\eta))=1$ for all $\eta\notin N_0$ (see [2], the following statements are also the same as those of [2]). For each $z\in[e_n]^\perp$, we set

$$S_n^z(\eta)=\{<x, e_n>: x\in S(\eta e_n+z)\},$$

$$N_n^z=\{\eta\in R^1: \eta e_n+z\in N\}.$$

Assume that $\gamma_n(N_n^z)=0$ for some $z\in[e_n]^\perp$. Then the Lebesgue measure of N_n^z is equal to zero and if $\eta, \eta'\notin N_n^z$ then, for any $(<x, e_n>, <x', e_n>)\in S_n^z(\eta)\times S_n^z(\eta')$,

$$\begin{aligned}&(<x, e_n>-<x', e_n>)(\eta-\eta')\\&\quad=<x-x', (\eta e_n+z)-(\eta' e_n+z)>\geqq 0,\end{aligned}$$

which implies that $(S_n^z(\eta))_{\eta\in R^1}$ is monotone. Thus $\mathrm{Card}\,(S_n^z(\eta))=1$ for all $\eta\notin N_n^z$. Now let

$$B_n=\{z\in[e_n]^\perp: \gamma_n(N_n^z)=0\},$$

$$D_n=(\mathrm{pr}_{[e_n]^\perp})^{-1}(B_n^c)$$

and $D=\bigcup_{n\geqq 1}D_n$. Since $\gamma_\mu(N)=0$, using the Fubini theorem, we can easily prove that $\gamma_\mu(D_n)=\gamma_n^\perp(B_n^c)=0$ for all $n\geqq 1$ and hence, $\gamma_\mu(D)=0$. Now let $y\notin D\cup N$ and let $y=\eta_n e_n+z_n$ for each $n\geqq 1$, where $z_n\in[e_n]^\perp$. Then $\gamma_n(N_n^{z_n})=0$ and $\eta_n\notin N_n^{z_n}$ for all $n\geqq 1$, and so, $\mathrm{Card}(\{<x, e_n>: x\in S(y)\})=1$ for all $n\geqq 1$, which implies that $\mathrm{Card}(S(y))=1$ for all $y\notin D\cup N$. We define a mapping f of E into itself by

2) For each $A\subset E$, $[A]$ denotes the closed linear subspace generated by A, and $\mathrm{pr}_{[A]}$ is the orthogonal projection to $[A]$. If $(X, \mathscr{X}, \nu)$ is a measure space and f is a measurable mapping of X into $(Y, \mathscr{Y})$, we use νf^{-1} to denote the measure defined by $\nu(f^{-1}(B))$ $(B\in\mathscr{Y})$.

$$f(y)=\begin{cases} x_y & \text{for} \quad y \notin D \cup N, \quad \text{where} \quad S(y)=\{x_y\}, \\ 0 & \text{for} \quad y \in D \cup N, \end{cases}$$

then for each $y \notin D \cup N$, $P_M(\ ,y)=\varepsilon_{f(y)}$, the unit distribution at $f(y)$. Thus $M=M_f$ with $f \in \mathscr{F}(\mu)$.

Next we shall prove Theorem 2. Let $\mu_1, \mu_2 \in \mathscr{P}$. For any $M_i \in \mathscr{M}(\mu_i)$ $(i=1, 2)$, since $M_1 * M_2 \in \mathscr{M}(\mu_1 * \mu_2)$,

$$\begin{aligned} e[\mu_1 * \mu_2] &\leqq e[\mu_1 * \mu_2;\ M_1 * M_2] \\ &= e[\mu_1;\ M_1]+e[\mu_2;\ M_2], \end{aligned}$$

it follows that $e[\mu_1 * \mu_2] \leqq e[\mu_1]+e[\mu_2]$. Now assume that the equality holds. By Theorem 1, there exist $f_i \in \mathscr{F}(\mu_i)$ with $e[\mu_i]=e[\mu_i;\ M_{f_i}]$ $(i=1, 2)$. The relation;

$$\begin{aligned} e[\mu_1 * \mu_2;\ M_{f_1} * M_{f_2}] &= e[\mu_1;\ M_{f_1}]+e[\mu_2;\ M_{f_2}] \\ &= e[\mu_1]+e[\mu_2] = e[\mu_1 * \mu_2], \end{aligned}$$

implies that $M_{f_1} * M_{f_2} \in \mathscr{M}_0(\mu_1 * \mu_2)$ and hence, by Theorem 1, there is an $f \in \mathscr{F}(\mu_1 * \mu_2)$ with $M_{f_1} * M_{f_2}=M_f$. Thus

$$\begin{aligned} &\gamma_{\mu_1} \otimes \gamma_{\mu_2}(\{(y,\ y'): f_1(y)+f_2(y') \in A,\ y+y' \in B\}) \\ &=\gamma_{\mu_1} \otimes \gamma_{\mu_2}(\{(y,\ y'): f(y+y') \in A,\ y+y' \in B\}) \end{aligned}$$

for all $A, B \in \mathscr{E}$. Consequently $f(y+y')=f_1(y)+f_2(y')$ for almost all (y, y') with respect to $\gamma_{\mu_1} \otimes \gamma_{\mu_2}$. Let $(e_n)_{n \geqq 1}$ be an orthonormal basis of E consisting of eigen vectors of the covariance operator of $\mu_1 * \mu_2$ and let $L_n=[e_1, e_2, \ldots, e_n]$. We denote by $\mathscr{L}_n$ the Borel σ-algebra of L_n and set $\mathscr{B}_n=\mathrm{pr}_{L_n}^{-1}(\mathscr{L}_n)$. It is clear that $(\mathscr{B}_n)_{n \geqq 1}$ is increasing and generates $\mathscr{E}$. We denote by $\gamma^{(n)}$ and $\gamma_i^{(n)}$ [resp. $\sigma^{(n)}$ and $\sigma_i^{(n)}$] the probability measures on L_n [resp. $L_n^{\perp}$] induced by pr_{L_n} [resp. $\mathrm{pr}_{L_n^{\perp}}$] from γ and γ_{μ_i} respectively $(i=1, 2)$, where γ is a short for $\gamma_{\mu_1} * \gamma_{\mu_2}$. We consider the Bochner integrals;

$$f_n(y_n) = \int_{L_n^{\perp}} f(y_n+z_n)\, d\sigma^{(n)}(z_n)$$

$$f_{ni}(y_n) = \int_{L_n^{\perp}} f_i(y_n+z_n)\, d\sigma_i^{(n)}(z_n) \qquad (i=1,\ 2),$$

where $y_n \in L_n$. Then $f_n(y_n+y_n')=f_{n1}(y_n)+f_{n2}(y_n')$ for almost all (y_n, y_n') with respect to $\gamma_1^{(n)} \otimes \gamma_2^{(n)}$. Therefore f_n is equal almost everywhere to an affine transformation from L_n into E and hence γf_n^{-1} is a Gaussian measure on E. On the other hand, since $\int_E \|f(y)\|^2 d\gamma(y)<\infty$ and $E[f|\mathscr{B}_n]=f_n$ (as E-valued random variables on $(E, \mathscr{E}, \gamma)$), by a martingale theorem, due to Chatterji [1], $\int_E \|f_n(y)-f(y)\|^2 d\gamma(y) \to 0$ as $n\to\infty$. Therefore $\mu_1 * \mu_2 = \gamma f^{-1}$ is Gaussian and hence, $e[\mu_1 * \mu_2]=0$. Consequently $e[\mu_i]=0$ $(i=1, 2)$, which implies both μ_1 and μ_2 are Gaussian. Thus the proof of Theorem 2 is completed.

§4. An application

Let $(\Omega, \mathscr{B}, P)$ be a probability space. A measurable stochastic process $X(\omega)=(X(t, \omega))_{0\leqq t<1}$ is considered as an $L_2[0, 1)$-valued random variable if $E[\int_0^1 (X(t))^2 dt]<\infty$. We denote by μ_X the probability distribution (in $L_2[0, 1)$) of X. In this section we use $e[X]$ to denote $e[\mu_X]$.

Now we consider a family of real random variables $\{\xi_{nj}: j=1, 2,\dots, 2^n-1, n=1, 2,\dots\}$ with the following properties; (i) all ξ_{nj} have the same distribution, (ii) for each $n\geqq 1$, $(\xi_{nj})_{1\leqq j<2^n}$ is independent and (iii) $E\xi_{nj}^4=c<\infty$, $E\xi_{nj}^2=1$ and $E\xi_{nj}=0$. Using the family we define a sequence of processes $(X_n)_{n\geqq 1}$ by

$$X_n(t, \omega) = 2^{-n/2}\sum_{1\leqq j<2^n} S_{nj}(\omega) f_{nj}(t)$$

for $(t, \omega)\in[0, 1)\times\Omega$, where $S_{nj}=\sum_{1\leqq i\leqq j}\xi_{ni}$ and f_{nj} are the indicators of intervals $[j2^{-n}, (j+1)2^{-n})$. It is known that the sequence $(X_n)_{n\geqq 1}$ of stochastic processes converges in law to a Brownian motion. The purpose of this section is to prove this in a wider space $L_2[0, 1)$ by making use of the functional e. Since $E\left[\int_0^1 (X_n(t))^2 dt\right]=(1-2^{-n})/2<\infty$, X_n are $L_2[0, 1)$-valued random variables, and $\mu_n=\mu_{X_n}$ have the mean vectors 0 and the covariance operators V_n, the integral operators with kernel $v_n(s, t)=([s2^{-n}]\wedge[t2^{-n}])/2^n$. We prove that $(\mu_n)_{n\geqq 1}$ converges to the Gaussian measure with mean vector 0 and covariance operator V, with kernel $v(s, t)=s\wedge t$. Using the random variables $S_{nj}^0=\sum_{1\leqq i\leqq j}\xi_{n+1,2i}$ and $S_{nj}^1=\sum_{1\leqq i\leqq j}\xi_{n+1,2i-1}$ $(1\leqq j<2^n)$, we define the processes X_n^0, X_n^1 and Z_n by

$$X_n^0 = 2^{-n/2}\sum_{1\leqq j<2^n} S_{nj}^0 f_{nj},$$

$$X_n^1 = 2^{-n/2}\sum_{1\leqq j<2^n} S_{nj}^1 \tilde{f}_{nj} \qquad (\tilde{f}_{nj}=f_{n+1,2j-1}+f_{n+1,2j}),$$

$$Z_n = 2^{-(n+1)/2}\xi_{n+1,2^{n+1}-1} f_{n+1,2^{n+1}-1},$$

respectively. By our assumptions, for each n, (X_n^0, X_n^1, Z_n) is independent and X_n, X_n^0 and X_n^1 have the same distribution as random variables taking values in the (2^n-1)-dimensional Euclidean space. Since $X_{n+1}=2^{-1/2}(X_n^0+X_n^1)+Z_n$ and $e[Z_n]\leqq 4^{-n}$, we have $e[X_{n+1}]\leqq e[X_n]+4^{-n}$ for all $n\geqq 1$. Therefore the limit $\alpha=\lim_n e[X_n]$ exists. Let $g_0=1$ and $g_{mk}=2^{(m-1)/2}(f_{m,k-1}-f_{mk})$ for $k=1, 3,\dots, 2^m-1$ and $m=1, 2,\dots$. It is known that $\{g_0, g_{mk}: \text{odd } k<2^m, m\geqq 1\}$ is an orthonormal basis in $L_2[0, 1)$ (the Haar functions). For any m_0, since

$$\sup_n \sum_{\substack{\text{odd}\, k<2^m \\ m\geqq m_0}} E[<X_n, g_{mk}>^2] \leqq 2^{-(m_0-1)},$$

$(\mu_n)_{n\geqq 1}$ is relatively, weakly compact [3; p. 154]. Hence we can choose a subsequence $(\mu_{n_i})_{i\geqq 1}$ of $(\mu_n)_{n\geqq 1}$ that converges weakly to a probability measure μ on $L_2[0, 1)$. From (iii) it follows that $\int \|x\|^4 d\mu_n(x) = E[\|X_n\|^4] \leqq 2^{-n}c + 3(1-2^{-n})$ and $\int \|x\|^4 d\gamma_{\mu_n}(x) \leqq 3$, and hence we have

$$\begin{aligned} &\lim_m \sup_n \int_{\{\|X\|\geqq m\}} \|x\|^2 d\mu_n(x) \\ &= \lim_m \sup_n \int_{\{\|X\|\geqq m\}} \|x\|^2 d\gamma_{\mu_n}(x) = 0. \end{aligned}$$

Using the relations, we can prove that $e[\mu] = \lim_i e[X_{n_i}]$. Let μ_0 and μ_1 be the limit distributions of $(X^0_{n_i}/\sqrt{2})_{i\geqq 1}$ and $(X^1_{n_i}/\sqrt{2})_{i\geqq 1}$ respectively. Then $\mu = \mu_0 * \mu_1$. On the other hand, since

$$\begin{aligned} \alpha &= e[\mu] \leqq (e[\mu_0] + e[\mu_1])/2 \\ &= \lim_i (e[X^0_{n_i}/\sqrt{2}] + e[X^1_{n_i}/\sqrt{2} + Z_n]) \\ &\leqq \lim_i (e[X^0_{n_i}/\sqrt{2}] + e[X^1_{n_i}/\sqrt{2}] + e[Z_{n_i}]) \\ &= \alpha/2 + \alpha/2 = \alpha, \end{aligned}$$

we have $e[\mu] = e[\mu_0] + e[\mu_1]$, which implies, by Theorem 2, that μ is Gaussian.

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