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Some Results on Hopf Algebras Attached to Group Schemes

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It is known that some roles played by Lie algebras attached to algebraic groups over a field of characteristic zero are played instead by Hopf algebras attached to them in the case of positive characteristic. This is essentially due to the fact that the enveloping algebra of the Lie algebra attached to an algebraic group over a field of positive characteristic is a proper subalgebra of the Hopf algebra attached to it in contrast to the case of characteristic zero, where the Hopf algebra attached to an algebraic group coincides with the enveloping algebra of the Lie algebra attached to it. Hence there arises a motivation to study Hopf algebras attached to group schemes over a field of arbitrary characteristic. In other words if we want to develop an infinitesimal theory of group schemes over a field of arbitrary characteristic, it would be natural to treat rather Hopf algebras than Lie algebras.

The purpose of this paper is to give a theory of Hopf algebras attached to group schemes over an algebraically closed field of arbitrary characteristic, which corresponds to the theory of Lie algebras attached to algebraic groups over a field of characteristic zero developed by C. Chevalley and A. Borel in their books [2] and [1] respectively. In particular we shall show some interesting results on algebraic Hopf subalgebras in connection with adjoint representations of group schemes. Although there are some results on this subject obtained already by J. Dieudonné and M. Takeuchi in their papers [3] and [11] respectively, it seems to the author that their results do not cover the whole which would correspond to the results on Lie algebras in characteristic zero case. For example there is no result on joins of connected group subschemes which are not necessarily reduced.

In § 1 we recall the definition and some properties of group schemes, and then we define Hopf algebras attached to group schemes and other notions necessary in the later sections. The notion and basic properties of h-inverses of Hopf subalgebras by a Hopf algebra homomorphism will be given in § 2. We shall show some basic results on algebraic Hopf subalgebras in § 3. In particular we define the algebraic hull of a Hopf subalgebra of the Hopf algebra attached to

1) This work was completed during the period when the author stayed at Genova by a financial support of *Consiglio Nazionale delle Ricerche* in Italy.

a group scheme corresponding to the algebraic hull of a Lie subalgebra in the case of characteristic zero. In §4 we show the existence of the join and the intersection of connected group subschemes, and we show that the join and the intersection of algebraic Hopf subalgebras are also algebraic. A theory of rational representations of group schemes in a vector space is developed in terms of Hopf algebras in §5. Next we shall show a useful result on adjoint representations of group schemes in §6 which plays very important roles in the following sections. §7 is concerned in normalizers of Hopf subalgebras, formal subgroups and group subschemes. In particular we shall show that the normalizer of any Hopf subalgebra of the Hopf algebra attached to a group scheme is algebraic. Similarly we shall show results on centralizers of them in §8. We study commutators of Hopf subalgebras, formal subgroups and group subschemes in §9. Furthermore the existence of commutators of connected but not necessarily reduced group subschemes is shown. In the last section we shall show how to get most results on algebraic Lie subalgebras of Lie algebras attached to algebraic groups over a field of characteristic zero from the results on algebraic Hopf subalgebras given in the preceding sections, and some new results on algebraic Lie subalgebras will be shown.

Mostly we follow the terminology and the notations from [5] and [7] on scheme theory, from [6] on commutative algebras and from [10] on Hopf algebras.

§1. Preliminaries

Let k be an algebraically closed field of an arbitrary characteristic. In the following we assume that an algebraic scheme X over k means always *a scheme of finite type over* k, and we denote by π_X *the structure morphism of* X *to* Spec(k). Moreover morphisms and fiber products of algebraic schemes over k are always assumed to be k-morphisms and products over k respectively, and we denote by 1_X the identity morphism of X. An algebraic scheme G over k is called *a group scheme over* k if the following conditions are satisfied: (i) There exists a morphism μ of $G\times G$ to G such that $\mu(1_G\times\mu)=\mu(\mu\times 1_G)$. (ii) There exist a morphism γ of G to itself and a morphism ε of Spec(k) to G such that the compositions $\mu(1_G\times\gamma)\Delta_G$ and $\mu(\gamma\times 1_G)\Delta_G$ are equal to $\varepsilon\pi_G$, where Δ_G is the diagonal morphism of G. (ii) Identifying Spec$(k)\times G$ and $G\times$Spec(k) with G canonically, the compositions $\mu(\varepsilon\times 1_G)$ and $\mu(1_G\times\varepsilon)$ are both equal to 1_G. The morphisms μ, ε and γ are called *the multiplication*, *the identity morphism* and *the inverse morphism* of G respectively, and the image e of ε in G is called *the neutral point of* G.

If X and Y are algebraic schemes over k, we denote by Mor(X, Y) the set of morphisms of X to Y. Then if $(G, \mu, \varepsilon, \gamma)$ is a group scheme over k, it can be seen easily that Mor(X, G) is a group under the composition $f*g=\mu(f\times g)\Delta_X$ for f

and g in $\mathrm{Mor}(X, G)$. In particular if we identify the set $G(k)$ of the closed points of G with $\mathrm{Mor}(\mathrm{Spec}(k), G)$, $G(k)$ has a group structure such that the neutral element of $G(k)$ is e and that $\mu(x, y)=x*y$ for x and y in $G(k)$. Let $(G, \mu, \varepsilon, \gamma)$ and $(G', \mu', \varepsilon', \gamma')$ be group schemes over k and let f be a morphism of G to G'. If f satisfies $f\mu=\mu'(f\times f)$, we say that f is *a homomorphism* of G to G'. Then f satisfies necessarily $f\gamma=\gamma' f$ and $f\varepsilon=\varepsilon'$ as seen easily. If x is a closed point of a group scheme $(G, \mu, \varepsilon, \gamma)$ over k, we denote by L_x the morphism $(x\pi_G)*1_G=\mu(x\pi_G\times 1_G)\Delta_G$ and call it *the left translation of G by x*. Similarly we define *the right translation* R_x by $1_G*(x\pi_G)=\mu(1_G\times x\pi_G)\Delta_G$.

We say that a closed subscheme H of a group scheme $(G, \mu, \varepsilon, \gamma)$ over k is *a group subscheme of G* if $\mu|_{H\times H}$ and $\gamma|_H$ decompose through H. It is easy to see that the neutral point e of G is contained in H and $(H, \mu|_{H\times H}, \varepsilon, \gamma|_H)$ is a group scheme over k. Moreover the canonical injection i_H of H into G is a homomorphism. Now denoting by p_i the projections of $G\times G$ to its i-th factor for $i=1, 2$, let S be the morphism of $G\times G$ to $G\times G$ such that $p_1S=p_2$ and $p_2S=p_1$. We say simply S is *the exchange of the factors of* $G\times G$. We put

$$\phi_G = \mu(\mu \times 1_G)(1_G \times 1_G \times \gamma)(1_G \times S)(\Delta_G \times 1_G)$$

and a group subscheme H of G is called *normal* in G if $\phi_G|_{G\times H}$ decomposes through H. Then we have the following

PROPOSITION 1. *Let H be a closed subscheme of a group scheme $(G, \mu, \varepsilon, \gamma)$ over k. Then H is a group (resp. a normal group) subscheme of G if and only if* $\mathrm{Mor}(X, H)$ *is a subgroup (resp. a normal subgroup) of* $\mathrm{Mor}(X, G)$ *for any algebraic scheme X over k.*

This is well known and hence we omit the proof. If (e, k) is the closed subscheme of G with the base space e isomorphic to $\mathrm{Spec}(k)$, (e, k) is a normal group subscheme of G which we call *the neutral group subscheme* of G. It is also known that any connected component of a group scheme G over k is irreducible. In particular the connected component G_0 of G containing e is a normal group subscheme of G.

Let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k, and let $\mathcal{O}$ and $\mathcal{O}'$ be the stalks of G and $G\times G$ at e and $e\times e$ respectively. Then μ and γ give naturally local homomorphisms μ^* and γ^* of $\mathcal{O}$ to $\mathcal{O}'$ and $\mathcal{O}$ respectively. Then the next theorem plays an essential role in the following sections.

THEOREM 1. *Let $(G, \mu, \varepsilon, \gamma)$, $\mathcal{O}$, $\mathcal{O}'$, μ^* and γ^* be as above. Then there is a one to one correspondence between the set of connected group subschemes H of G and that of ideals $\mathfrak{a}$ of $\mathcal{O}$ satisfying $\mu^*(\mathfrak{a})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a})\mathcal{O}'$ and $\gamma^*(\mathfrak{a})=\mathfrak{a}$. If H corresponds to $\mathfrak{a}$ in this way, the stalk of H at e is $\mathcal{O}/\mathfrak{a}$.*

This is Lemma 2 in [15]. We call the ideal $\mathfrak{a}$ corresponding to H *the defining ideal of H in $\mathcal{O}$*. Now let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$, and let $\bar{\mathcal{O}}$ and $\bar{\mathcal{O}}'$ be the $\mathfrak{m}$-adic and $(\mathcal{O}\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathcal{O})\mathcal{O}'$-adic completions of $\mathcal{O}$ and $\mathcal{O}'$ respectively. If λ and σ are the natural continuous extensions of μ^* and γ^* from $\bar{\mathcal{O}}$ to $\bar{\mathcal{O}}'$ and $\bar{\mathcal{O}}$ respectively, $(\bar{\mathcal{O}}, \lambda, \eta, \sigma)$ is a formal group over k in the sense of §5 in [13], where η is the canonical homomorphism of $\bar{\mathcal{O}}$ to $\bar{\mathcal{O}}/\mathfrak{m}=k$. We call this formal group *the formalization of the group scheme G*. Then the set $\mathfrak{H}(G)$ of continuous k-linear maps of $\mathcal{O}$ with the $\mathfrak{m}$-adic topology to k with the discrete topology may be identified with the set $\mathfrak{H}(\bar{\mathcal{O}})$ of continuous k-linear maps of $\bar{\mathcal{O}}$ with the $\mathfrak{m}\bar{\mathcal{O}}$-adic topology to k with the discrete topology. As seen in §5 in [13] $\mathfrak{H}(G)=\mathfrak{H}(\bar{\mathcal{O}})$ has a structure of a Hopf algebra over k whose algebra structure $(\mathfrak{H}(G), m, i)$ comes from the homomorphisms λ and η. The coalgebra structure $(\mathfrak{H}(G), \Delta, \varepsilon)$ is the dual of the algebra structure of $\bar{\mathcal{O}}$ and the antipode c of $\mathfrak{H}(G)$ is the dual of σ. If H is a connected group subscheme of G with the defining ideal $\mathfrak{a}$ in $\mathcal{O}$, the Hopf algebra $\mathfrak{H}(H)$ attached to H may be identified with the Hopf subalgebra of $\mathfrak{H}(G)$ consisting of the elements x in $\mathfrak{H}(G)$ such that x annihilates $\mathfrak{a}$. Then we see easily in a similar argument to the proof of Prop. 4 in [13] that the set of connected group subschemes of G corresponds injectively to a subset of Hopf subalgebras of $\mathfrak{H}(G)$. We understand by *an algebraic Hopf subalgebra* of $\mathfrak{H}(G)$ a Hopf subalgebra corresponding to a connected group subscheme of G in this way.[2)]

Let $(A, \mathfrak{m})$ be a noetherian local ring containing the residue field $k=A/\mathfrak{m}$, and let $(A', \mathfrak{m}')$ be the quotient ring of $A\otimes_k A$ with respect to the maximal ideal $\mathfrak{m}\otimes A+A\otimes\mathfrak{m}$. We denote by a $\bar{A}$ and $\bar{A}'$ the $\mathfrak{m}$-adic and $\mathfrak{m}'$-adic completions of A and A' respectively, and we assume that there are a local homomorphism λ of A to $\bar{A}'$ and an automorphism σ of A such that $(\bar{A}, \bar{\lambda}, \bar{\eta}, \bar{\sigma})$ is a formal group over k, where $\bar{\lambda}$, $\bar{\sigma}$ and $\bar{\eta}$ are the continuous extensions of λ, σ and the canonical map $\eta: A\to k=A/\mathfrak{m}$ to the completions. Then we say that A has *a quasi-bigebra structure (λ, η, σ) over k*. In particular if the image $\lambda(A)$ of λ is contained in $A'\subset\bar{A}'$, we say that A has *a strict quasi-bigebra structure* (λ, η, σ) over k.

§2. h-inverses by Hopf algebra homomoprhisms

In the following we understand by a Hopf algebra $(B, m, i, \Delta, \varepsilon, c)$ a Hopf algebra B over k with an antipode c whose algebra and coalgebra structures are given by (B, m, i) and (B, Δ, ε) respectively. A Hopf algebra $(B, m, i, \Delta, \varepsilon, c)$ is called *colocal*[3)] if (B, Δ, ε) is cocommutative and has only one minimal sub-

2) In [14] and [15] we called such a Hopf subalgebra *algebraic in wider sense.*

3) In [10] a colocal coalgebra is called irreducible.

coalgebra. Then the unique minimal subcoalgebra of B is $i(k)$ and $i(1)$ is the unique grouplike element of B which we denote by 1. A colocal Hopf algebra B is called *of finite type* if the space $\mathfrak{L}(B)$ of primitive elements in B is finite dimensional. It is well known that the dual space B^* of a colocal Hopf algebra B of finite type over k is a formal group over k whose Hopf algebra $\mathfrak{H}(B^*)$ is canonically isomorphic to B as Hopf algebras. Conversely if A is a formal group over k, the Hopf algebra $\mathfrak{H}(A)$ attached to A is a colocal Hopf algebra of finite type over k and the dual space $\mathfrak{H}(A)^*$ of $\mathfrak{H}(A)$ is isomorphic to A as formal groups over k. Thus there is a one to one correspondence between the set of isomorphism classes of colocal Hopf algebras of finite type over k and that of formal groups over k.

Let $(B, m, i, \Delta, \varepsilon, c)$ be a colocal Hopf algebra of finite type over k, and let $(A, \lambda, \eta, \sigma)$ be its dual formal group over k. Then B has *an A-module structure* as follows: if a and x are in A and B respectively such that $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$, we put $a\cdot x=\sum_{(x)} <x_{(2)}, a>x_{(1)}$, where we denote by $<x, a>$ the image of x in k by the linear map a. It is easy to see that this composition gives an A-module structure of B and that *a subspace C of B is a subcoalgebra of B if and only if C is an A-submodule of B.* (cf. C.3 in [12], pp. 177–178).

Now we want to give the definition of h-inverses by Hopf algebra homomorphisms which are generalizations of h-kernels. For this purpose we need the following

PROPOSITION 2. *Let $(B, m, i, \Delta, \varepsilon, c)$ and $(B', m', i', \Delta', \varepsilon', c')$ be colocal Hopf algebras over k. Let f be a Hopf algebra homomorphism of B to B' and D' a Hopf subalgebra of B'. Then there exists a Hopf subalgebra D of B satisfying the following conditions*:

(i) $f(D)$ is contained in D'.

(ii) If D_1 is a subcoalgebra of B such that $f(D_1)\subset D'$, then D_1 is contained in D.

PROOF. Put $D=\{x\in B|(1_B\otimes f)\Delta(x)-x\otimes 1\in B\otimes_k D'^{\circ}\}$, where D'° is the kernel of the linear map $\varepsilon'|_{D'}$. Since f and Δ are k-algebra homomorphisms, it is easy to see that D is a subalgebra of B. To see that D is a subcoalgebra of B, it is sufficient to show that D is an A-submodule of B, where $A=B^*$ is the dual algebra of the coalgebra B. If A' is the dual algebra B'^* of B', $B\otimes_k B'$ is an $A\otimes_k A'$-module defined by $(a\otimes a')\cdot(x\otimes x')=a\cdot x\otimes a'\cdot x'$ for a in A, a' in A', x in B and x' in B'. Since ε' is the unit of the algebra A', we see

$$\begin{aligned}(a\otimes\varepsilon')\cdot(1_B\otimes f)\Delta(x) &= (a\otimes\varepsilon')\cdot(\sum_{(x)} x_{(1)}\otimes f(x_{(2)}))\\ &= (1_B\otimes f)(\sum_{(x)} a\cdot x_{(1)}\otimes x_{(2)})\\ &= (1_B\otimes f)\Delta(a\cdot x)\end{aligned}$$

for a in A and x in B by the cocommutativity of B and the equality $\Delta(a\cdot x)=\sum_{(x)} x_{(1)}\otimes(a\cdot x_{(2)})$ (cf. p. 177 in [12]), where $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. Therefore if x is in D, we have

$$(1_B\otimes f)\Delta(a\cdot x)-a\cdot x\otimes 1=(a\otimes\varepsilon')\cdot\{(1_B\otimes f)\Delta(x)-x\otimes 1\}$$
$$\in(a\otimes\varepsilon')\cdot(B\otimes_k D'^{\circ})\subset B\otimes_k D'^{\circ}.$$

This means that $a\cdot x$ is contained in D if x is in D, and hence D is a subcoalgebra of B. Moreover we see

$$(*)\qquad (1_B\otimes f)\Delta c(x)=(1_B\otimes f)(c\otimes c)\Delta(x)=(c\otimes c')(1_B\otimes f)\Delta(x)$$

for any x in B. Since we have $c'(D'^{\circ})\subset D'^{\circ}$, we see $c(D)\subset D$ by $(*)$ and $c(1)=1$. Therefore D is a Hopf subalgebra of B. Now let x be an element in D. Then we see, by the definition of D, $(f\otimes f)\Delta(x)-f(x)\otimes 1\in B'\otimes D'^{\circ}$, and hence, using the equality $(f\otimes f)\Delta=\Delta' f$, $\Delta'(f(x))-f(x)\otimes 1\in B'\times D'^{\circ}$. Then we have $1\otimes f(x)-\varepsilon'(f(x))\otimes 1\in k\otimes D'^{\circ}$. This means that $f(x)$ is contained in D', and therefore we see $f(D)\subset D'$. Finally let D_1 be a subcoalgebra of B such that $f(D_1)\subset D'$, and let x be an element of D_1°, where D_1° is the intersection $B^{\circ}\cap D_1$. Then we see $\Delta(x)-x\otimes 1-1\otimes x\in D_1^{\circ}\otimes D_1^{\circ}$ (cf. p. $181\sim 182$ in [12]) and hence $(1_B\otimes f)\Delta(x)-x\otimes 1\in D_1\otimes D'^{\circ}\subset B\otimes D'^{\circ}$, because we have $f(x)\in f(D_1^{\circ})\subset D'^{\circ}$ from $f(D_1)\subset D'$. Therefore we have $D_1^{\circ}\subset D$ by the definition of D and also $D_1=k\oplus D_1^{\circ}\subset D$ identifying $i(k)$ with k. q.e.d.

Let B, B', D' and f be as above. Then the Hopf subalgebra D obtained in Prop. 2 is called *the h-inverse of D' by f* and is denoted by $h\text{-}f^{-1}(D')$. In particular if D' is the smallest coalgebra $i'(k)=B'_0$ which is also a Hopf subalgebra of B', $h\text{-}f^{-1}(B'_0)$ is called *the h-kernel of f* and is denoted by h-ker$f$.

Let $(A_1, \lambda_1, \eta_1, \sigma_1)$ and $(A_2, \lambda_2, \eta_2, \sigma_2)$ be formal groups over k with the maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Then a local homomorphism ϕ of A_1 to A_2 is called a formal group homomorphism if the diagram

$$(**)\qquad \begin{array}{ccc} A_1 & \xrightarrow{\ \phi\ } & A_2 \\ {\scriptstyle\lambda_1}\downarrow & & \downarrow{\scriptstyle\lambda_2} \\ \bar{A}'_2 & \xrightarrow[\overline{\phi\times\phi}]{} & \bar{A}'_2 \end{array}$$

is commutative, where $\bar{A}'_i$ is the $(\mathfrak{m}_i\otimes A_i+A_i\otimes\mathfrak{m}_i)$-adic completion of $A_i\otimes A_i$ and $\overline{\phi\otimes\phi}$ is the continuous extension of $\phi\otimes\phi$: $A_1\otimes A_1\to A_2\otimes A_2$.

LEMMA 1. *Let $(A_1, \lambda_1, \eta_1, \sigma_1)$ and $(A_2, \lambda_2, \eta_2, \sigma_2)$ be as above, and let $(B_j, m_j, i_j, \Delta_j, \varepsilon_j, c_j)$ be the Hopf algebra $\mathfrak{H}(A_j)$ for $j=1, 2$. Let ϕ be a local homomorphism of A_1 to A_2 and ϕ^* the transpose of ϕ. Then we have the follow-*

ings:
(*i*) ϕ *is a formal group homomorphism if and only if* $\bar{\phi}^*=\phi^*|_{B_2}$ *is a bigebra homomorphism of* B_2 *to* B_1.
(*ii*) *If* ϕ *is a formal group homomorphism, we have* $\eta_1=\eta_2\phi$ *and* $\phi\sigma_1=\sigma_2\phi$.

PROOF. Since $\bar{A}'_j$ is the dual algebra of the coalgebra $B_j\otimes B_j$ for $j=1, 2$, we see easily that $\bar{\phi}^*\otimes\bar{\phi}^*$ is the restriction of the transpose of $\overline{\phi\otimes\phi}$ to $B_2\otimes B_2$. Therefore if ϕ is a formal group homomorphism, we have $m_1(\bar{\phi}^*\otimes\bar{\phi}^*)=\bar{\phi}^*m_2$ from (**) and $m_j=\lambda_j^*|_{B_j\otimes B_j}$ for $j=1, 2$. On the other hand since we see $\eta_1=\eta_2\phi$ by the definitions of η_1 and η_2, we have $i_1=\bar{\phi}^*i_2$. This means that $\bar{\phi}^*$ is an algebra homomorphism of B_2 to B_1. Similarly we see that $\bar{\phi}^*$ is a coalgebra homomorphism, because ϕ is a local homomorphism of A_1 to A_2. A similar argument shows the converse. Now assume that ϕ is a formal group homomorphism. Then since $\bar{\phi}^*$ is a bigebra homomorphism as seen in the above, we have $\bar{\phi}^*\sigma_2^*=\sigma_1^*\bar{\phi}^*$ as seen easily. This means $\sigma_2\phi=\phi\sigma_1$. q.e.d.

PROPOSITION 3. *Let* A_1 *and* A_2 *be formal groups over* k, *and let* ϕ *be a formal group homomorphism of* A_1 *to* A_2. *Let* $A_1/\mathfrak{a}_1$ *and* $A_2/\mathfrak{a}_2$ *be formal subgroups of* A_1 *and* A_2 *respectively. Then we have the followings*:
(*i*) $A_1/\phi^{-1}(\mathfrak{a}_2)$ *is a formal subgroup of* A_1. *If* D_2 *is the Hopf subalgebra of* $\mathfrak{H}(A_2)$ *corresponding to* $A_2/\mathfrak{a}_2$, $A_1/\phi^{-1}(\mathfrak{a}_2)$ *corresponds to the Hopf subalgebra* $\phi^*(D_2)$ *of* $\mathfrak{H}(A_1)$.
(*ii*) $A_2/\phi(\mathfrak{a}_1)A_2$ *is a formal subgroup of* A_2. *If* D_1 *is the Hopf subalgebra of* $\mathfrak{H}(A_1)$ *corresponding to* A_1/a_1, $A_2/\phi(a_1)A_2$ *corresponds to the h-inverse h-*$\bar{\phi}^{*-1}(D_1)$ *of* D_1 *in* $\mathfrak{H}(A_2)$ *by* $\bar{\phi}^*=\phi^*|_{\mathfrak{H}(A_2)}$.

PROOF. (i) If a is any element in A_1, we have the following: $a\in\phi^{-1}(\mathfrak{a}_2)$ $\Leftrightarrow\phi(a)\in\mathfrak{a}_2\Leftrightarrow<x, \phi(a)>=0$ for any x in $D_2\Leftrightarrow<\phi^*(x), a>=0$ for any x in $D_2\Leftrightarrow a\in(\phi^*(D_2))^{\perp}$, where $V^{\perp}$ means the null space in A_1 of V in $\mathfrak{H}(A_1)$ with respect to the inner product $<\ ,\ >$ of $\mathfrak{H}(A_1)$ and A_1. This means that $\phi^{-1}(\mathfrak{a}_2)$ is the null space $(\phi^*(D_2))^{\perp}$ of $\phi^*(D_2)$ in A_1. Then $\phi^*(D_2)$ is a Hopf subalgebra of $\mathfrak{H}(A_1)$ and $A_1/\phi^{-1}(\mathfrak{a}_2)$ is the formal subgroup of A_1 corresponding to $\phi^*(D_2)$.
(ii) From the commutative diagram (**) we see easily that $\lambda_2(\phi(\mathfrak{a}_1)A_2)$ is contained in $(\phi(\mathfrak{a}_1)A_2\otimes A_2+A_2\otimes\phi(\mathfrak{a}_1)A_2)\bar{A}'_2$. Moreover since $\phi\sigma_1=\sigma_2\phi$ by Lemma 1, (ii), we see $\sigma_2(\phi(\mathfrak{a}_1)A_2)=\phi(\sigma_1(\mathfrak{a}_1))A_2\subset\phi(\mathfrak{a}_1)A_2$. Therefore we see easily from the definition that $A_2/\phi(\mathfrak{a}_1)A_2$ is a formal subgroup of A_2. Denote by D' the Hopf subalgebra of $\mathfrak{H}(A_2)$ corresponding to the formal subgroup $A_2/\phi(\mathfrak{a}_1)A_2$. If x is any element of D', we see $<\phi^*(x), \mathfrak{a}_1>=<x, \phi(\mathfrak{a}_1)>\subset<x, \phi(\mathfrak{a}_1)A_2>=\{0\}$. This means $\phi^*(D')\subset D_1$. Moreover let D'' be any subcoalgebra of $\mathfrak{H}(A_2)$ such that $\phi^*(D'')\subset D_1$. Let x be any element of D'' and put $\Delta_2(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$, where Δ_2 is the comultiplication of $\mathfrak{H}(A_2)$, and $x_{(1)}$ and $x_{(2)}$ are in D''. Let a_1 and a_2 be any elements of A_1 and A_2 respectively. Then we see

$$<x, \phi(a_1)a_2> = <\Delta_2(x), \phi(a_1)\otimes a_2> = \sum_{(x)} <x_{(1)}, \phi(a_1)><x_{(2)}, a_2>$$

$$= \sum_{(x)} <\phi^*(x_{(1)}), a_1><x_{(2)}, a_2> = 0,$$

because $\phi^*(D'')$ is contained in D_1. This means $D'' \subset D' = (\phi(\mathfrak{a}_1)A_2)^\perp$. Therefore D' is the h-inverse of D_1 by $\bar{\phi}^*$ from Prop. 2. q.e.d.

COROLLARY. *Let A_1, A_2, ϕ and $\bar{\phi}^*$ be as in Prop. 3. Then the image $\phi^*(\mathfrak{H}(A_2))$ in $\mathfrak{H}(A_1)$ corresponds to the formal subgroup $A_1/\phi^{-1}(0)$ of A_1 and the h-kernel of $\bar{\phi}^*$ corresponds to the formal subgroup $A_2/\phi(\mathfrak{m}_1)A_2$ of A_2, where $\mathfrak{m}_1$ is the maximal ideal of A_1.*

Let A_1, A_2, $\mathfrak{a}_1$, $\mathfrak{a}_2$ and ϕ be as in Prop. 3. Then the formal subgroup $A_1/\phi^{-1}(\mathfrak{a}_2)$ of A_1 is called *the image of the formal subgroup $A_2/\mathfrak{a}_2$ of A_2 by ϕ*, and the formal subgroup $A_2/\phi(\mathfrak{a}_1)A_2$ of A_2 is called *the inverse imageof the formal subgroup $A_1/\mathfrak{a}_1$ of A_1 by ϕ*. In particular $A_1/\phi^{-1}(0)$ and $A_2/\phi(\mathfrak{m}_1)A_2$ are called *the image* and *the kernel* of the formal group homomorphism ϕ respectively, where $\mathfrak{m}_1$ is the maximal ideal of A_1.

§3. Algebraic Hopf subalgebras

First we need the following elementary lemma.

LEMMA 2. *Let V be a vector space over k, and let U, W and T be subspaces of V such that $W \supset T$. Then we have*

$$(W\otimes V + V\otimes T) \cap (U\otimes U) = (U \cap W)\otimes U + U\otimes(U \cap T).$$

PROOF. Let $\{x_\sigma | \sigma \in S_1\}$ be a basis for $T \cap U$ over k, and let $\{x_\tau | \tau \in S_2\}$ and $\{x_\lambda | \lambda \in S_3\}$ be subsets of T and $W \cap U$ such that $\{x_\sigma\} \cup \{x_\tau\}$ and $\{x_\sigma\} \cup \{x_\lambda\}$ are bases for T and $W \cap U$ over k respectively. Then $\{x_\sigma\} \cup \{x_\tau\} \cup \{x_\lambda\}$ is a linearly independent subset of W over k and hence there is a subset $\{x_\nu | \nu \in S_4\}$ of W such that $\{x_\sigma\} \cup \{x_\tau\} \cup \{x_\lambda\} \cup \{x_\nu\}$ is a basis for W over k. Similarly there exists a subset $\{x_\mu | \mu \in S_5\}$ of U such that $\{x_\sigma\} \cup \{x_\lambda\} \cup \{x_\mu\}$ is a basis for U over k. Then we see as above that $\{x_\sigma\} \cup \{x_\lambda\} \cup \{x_\mu\} \cup \{x_\tau\} \cup \{x_\nu\}$ is a linearly independent subset of V over k and hence there exists a subset $\{x_\pi | \pi \in S_6\}$ of V such that $\{x_\sigma\} \cup \{x_\lambda\} \cup \{x_\mu\} \cup \{x_\tau\} \cup \{x_\nu\} \cup \{x_\pi\}$ is a basis for V over k. If y is an element of $W\otimes V + V\otimes T$, we can express y uniquely as follows:

$$y = \sum_{(\xi,\eta)}' \alpha_{\xi\eta} x_\xi \otimes y_\eta, \quad \alpha_{\xi\eta} \in k \text{ and } \alpha_{\xi\eta} = 0 \text{ for almost all } (\xi, \eta),$$

where $\sum'$ runs over all (ξ, η) which are contained in $S_i \times S_j$ for $1 \leq i \leq 4$ and $1 \leq j \leq 6$ or for $1 \leq i \leq 6$ and $1 \leq j \leq 2$. Similarly if y is in $U\otimes U$, we can express

y uniquely as follows:

$$y = \sum_{(\xi,\eta)}'' \beta_{\xi\eta} x_\xi \otimes y_\eta, \quad \beta_{\xi\eta} \in k \text{ and } \beta_{\xi\eta} = 0 \quad \text{for almost all} \quad (\xi, \eta),$$

where $\sum''$ runs over all (ξ, η) which are contained in $S_i \times S_j$ for $i, j = 1, 3, 5$. Therefore if y belongs to $(W \otimes V + V \otimes T) \cap (U \otimes U)$, we see

$$y = \sum_{(\xi,\eta)}''' \gamma_{\xi\eta} x_\xi \otimes y_\eta, \quad \gamma_{\xi\eta} \in k \text{ and } \gamma_{\xi\eta} = 0 \quad \text{for almost all} \quad (\xi, \eta),$$

where $\sum'''$ runs over all (ξ, η) which are contained in $S_i \times S_j$ for $i = 1, 3$ and $j = 1, 3, 5$ or for $i = 1, 3, 5$ and $j = 1$. This means that the left hand side of our equality is contained in the right hand side. The inverse inclusion is clear. q.e.d.

LEMMA 3. *Let $(A, \mathfrak{m})$ be a local ring with a quasi-bigebra structure over k. Then the canonical homomorphism ϕ of $A \otimes_k A$ to the quotient ring $(A \otimes_k A)_{A \otimes \mathfrak{m} + \mathfrak{m} \otimes A}$ is injective.*

PROOF. If $\bar{A}$ is the $\mathfrak{m}$-adic completion of A, $\bar{A}$ is a formal group over k. If the characteristic of k is zero, A is an integral domain as well known. In particular A has no non-trivial zero-divisor and hence ϕ is injective. If the characteristic of k is $p > 0$, $\bar{A}$ is isomorphic to the tensor product of a formal power series ring and an artinian local ring of the form $k[T_1, \ldots, T_n]/(T_n^{p^{e_n}}, \ldots, T_n^{p^{e_1}})$ by Prop. 2 in [14]. Therefore we can see easily that the zero ideal of $\bar{A} \otimes_k \bar{A}$ is primary and the nilradical of $\bar{A} \otimes_k \bar{A}$ is contained in $\bar{A} \otimes \mathfrak{m}\bar{A} + \mathfrak{m}\bar{A} \otimes \bar{A}$. Hence the zero ideal of $A \otimes_k A$ is also primary and its nilradical $\mathfrak{n}$ is contained in $(\bar{A} \otimes \mathfrak{m}\bar{A} + \mathfrak{m}\bar{A} \otimes \bar{A}) \cap (A \otimes_k A)$, which coincides with $\mathfrak{m} \otimes A + A \otimes \mathfrak{m}$ by Lemma 2. This means that $A - (\mathfrak{m} \otimes A + A \otimes \mathfrak{m})$ does not contain any zero-divisor of $A \otimes_k A$, and hence ϕ is injective. q.e.d.

PROPOSITION 4. *Let $(A, \mathfrak{m})$ be a local ring with a strict quasi-bigebra structure (λ, η, σ) over k. Let $\bar{A}$ and $\bar{A}/\bar{\mathfrak{a}}$ be the $\mathfrak{m}$-adic completion of A and a formal subgroup of it respectively. Then if we put $\mathfrak{a} = \bar{\mathfrak{a}} \cap A$, we have $\lambda(\mathfrak{a}) \subset (\mathfrak{a} \otimes A + A \otimes \mathfrak{a})A'$ and $\sigma(\mathfrak{a}) = \mathfrak{a}$, where $A' = (A \otimes_k A)_{\mathfrak{m} \otimes A + A \otimes \mathfrak{m}}$.*

PROOF. Let $(\bar{\lambda}, \bar{\eta}, \bar{\sigma})$ be the quasi-bigebra structure of the formal group $\bar{A}$ defined by (λ, η, σ) and let $\bar{A}'$ be the $(\mathfrak{m} \otimes A + A \otimes \mathfrak{m})A'$-adic completion of A'. By Lemma 3 we may consider $A \otimes_k A$, $\bar{A} \otimes_k \bar{A}$ and A' as subrings of $\bar{A}'$. By our assumptions we have $\lambda(A) \subset A'$ and $\bar{\lambda}(\bar{\mathfrak{a}}) \subset (\bar{\mathfrak{a}} \otimes \bar{A} + \bar{A} \otimes \bar{\mathfrak{a}})\bar{A}'$ and hence we see $\lambda(\mathfrak{a}) \subset (\bar{\mathfrak{a}} \otimes \bar{A} + \bar{A} \otimes \bar{\mathfrak{a}})\bar{A}' \cap A'$. Therefore if x is an element of $\lambda(\mathfrak{a})$, there is an element s in $A \otimes_k A - (\mathfrak{m} \otimes A + A \otimes \mathfrak{m})$ such that sx is in $A \otimes_k A$. This means that sx is in $(\bar{\mathfrak{a}} \otimes \bar{A} + \bar{A} \otimes \bar{\mathfrak{a}})\bar{A}' \cap (A \otimes_k A) = (\bar{\mathfrak{a}} \otimes \bar{A} + \bar{A} \otimes \bar{\mathfrak{a}})\bar{A}' \cap (\bar{A} \otimes_k \bar{A}) \cap (A \otimes_k A) = (\bar{\mathfrak{a}} \otimes \bar{A} + \bar{A} \otimes \bar{\mathfrak{a}}) \cap (A \otimes_k A)$, because $\bar{\mathfrak{a}} \otimes \bar{A} + \bar{A} \otimes \bar{\mathfrak{a}}$ is a primary ideal of $\bar{A} \otimes_k \bar{A}$ contained in $\mathfrak{m}\bar{A} \otimes \bar{A} + \bar{A} \otimes \mathfrak{m}\bar{A}$ as seen easily in a similar way to the proof of Lemma

3. But the right hand side of the above equality coincides with $\mathfrak{a}\otimes A+A\otimes\mathfrak{a}$ by Lemma 2. Therefore sx is in $\mathfrak{a}\otimes A+A\otimes\mathfrak{a}$ and hence x is in $(\mathfrak{a}\otimes A+A\otimes\mathfrak{a})A'$. On the other hand we see $\sigma(\mathfrak{a})\subset A\cap\bar{\sigma}(\bar{\mathfrak{a}})=A\cap\bar{\mathfrak{a}}=\mathfrak{a}$ and hence $\sigma(\mathfrak{a})=\mathfrak{a}$. q.e.d.

PROPOSITION 5. *Let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k and $\mathfrak{H}(G)$ the Hopf algebra attached to G. If D is a Hopf subalgebra of $\mathfrak{H}(G)$, there is the least algebraic Hopf subalgebra C of $\mathfrak{H}(G)$ such that $C\supset D$.*

PROOF. Let $\bar{\mathfrak{a}}$ be the ideal of the formalization A of G corresponding to D, i.e., $\bar{\mathfrak{a}}$ is the null space $D^{\perp}$ of D in A. If $\mathcal{O}$ is the stalk of G at e which we consider as a subring of A, we put $\mathfrak{a}=\bar{\mathfrak{a}}\cap\mathcal{O}$. Then by Prop. 4 $\mathfrak{a}$ satisfies $\mu^*(\mathfrak{a})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a})\mathcal{O}'$ and $\gamma^*(\mathfrak{a})=\mathfrak{a}$, because $\mathcal{O}$ has the quasi-bigebra structure $(\mu^*, \varepsilon^*, \gamma^*)$. But this means by Th. 1 that there exists a unique connected group subscheme H of G having $\mathfrak{a}$ as the defining ideal in $\mathcal{O}$. We put $C=\mathfrak{H}(H)$. If C' is any algebraic Hopf subalgebra of $\mathfrak{H}(G)$ containing D, we denote by $A/\bar{\mathfrak{a}}'$ the formal subgroup of A corresponding to C'. Then we see $\bar{\mathfrak{a}}\supset\bar{\mathfrak{a}}'$. Since C' is algebraic, there exists an ideal $\mathfrak{a}'$ of $\mathcal{O}$ such that $\bar{\mathfrak{a}}'=\mathfrak{a}'A$. Then we see $\mathfrak{a}'=\bar{\mathfrak{a}}'\cap\mathcal{O}$, and hence $\mathfrak{a}=\bar{\mathfrak{a}}\cap\mathcal{O}\supset\bar{\mathfrak{a}}'\cap\mathcal{O}=\mathfrak{a}'$. If H' is the connected group subscheme of G defined by $\mathfrak{a}'$, we see H is a group subscheme of H', and hence C' contains C. This means that C is the least algebraic Hopf subalgebra of $\mathfrak{H}(G)$ containing D. q.e.d.

Let G, $\mathfrak{H}(G)$ and D be as above, and let C be the unique least algebraic Hopf subalgebra of $\mathfrak{H}(G)$ containing D. Then C is called *the algebraic hull of D in* $\mathfrak{H}(G)$ and is denoted by $\mathscr{A}(D)$. D is algebraic if and only if $D=\mathscr{A}(D)$.

Let $(G_1, \mu_1, \varepsilon_1, \gamma_1)$ and $(G_2, \mu_2, \varepsilon_2, \gamma_2)$ be group schemes over k and let f be a homomorphism of G_1 to G_2. Denoting by $\mathcal{O}_i$ the stalk of G_i at the neutral point e_i of G_i for $i=1, 2$, let f^* be the comorphism of $\mathcal{O}_2$ to $\mathcal{O}_1$ defined by f. If A_i is the formalization of G_i for $i=1, 2$, we denote by $\bar{f}_*$ the continuous extension of f^* from A_2 to A_1. It is easy to see that $\bar{f}^*$ is a formal group homomorphism. Then there is a unique Hopf algebra homomorphism f_* of $\mathfrak{H}(A_1)=\mathfrak{H}(G_1)$ to $\mathfrak{H}(A_2)=\mathfrak{H}(G_2)$ such that $\bar{f}^*$ is the transpose of f_* by Lemma 1. We say $\bar{f}^*$ and f_* to be *the formal comorphism* and *the tangential homomorphism* of f respectively.

LEMMA 4. *Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be noetherian local rings, and ϕ a local homomorphism of R_1 to R_2. Denoting by $\bar{R}_i$ the $\mathfrak{m}_i$-adic completion of R_i for $i=1, 2$, let $\bar{\phi}$ be the continuous extension of ϕ from $\bar{R}_1$ to $\bar{R}_2$. Then we have the followings*:

(*i*) *If $\mathfrak{a}_2$ is an ideal of R_2, $\phi^{-1}(\mathfrak{a}_2)$ is dense in $\bar{\phi}^{-1}(\mathfrak{a}_2\bar{R}_2)$.*

(*ii*) *If $\mathfrak{a}_1$ is an ideal of R_1, $\phi(\mathfrak{a}_1)R_2$ is dense in $\bar{\phi}(\mathfrak{a}_1\bar{R}_1)\bar{R}_2$.*

PROOF. (i) Since ϕ gives an injective local homomorphism ϕ' of $R_1/\phi^{-1}(\mathfrak{a}_2)$ to $R_2/\mathfrak{a}_2$ naturally, we have a local homomorphism $\bar{\phi}'$ of the completion $\overline{R_1/\phi^{-1}(\mathfrak{a}_2)}$ of $R_1/\phi^{-1}(\mathfrak{a}_2)$ to the completion $\overline{R_2/\mathfrak{a}_2}$ of $R_2/\mathfrak{a}_2$. Then it is easy to see that $\bar{\phi}'$ is also injective. On the other hand we have $\overline{R_1/\phi^{-1}(\mathfrak{a}_2)} = \bar{R}_1/\phi^{-1}(\mathfrak{a}_2)\bar{R}_1$ and $\overline{R_2/\mathfrak{a}_2} = \bar{R}_2/\mathfrak{a}_2\bar{R}_2$. This means $\bar{\phi}^{-1}(\mathfrak{a}_2\bar{R}_2) = \phi^{-1}(\mathfrak{a}_2)\bar{R}_1$ and we see that $\phi^{-1}(\mathfrak{a}_2)$ is dense in $\bar{\phi}^{-1}(a_2\bar{R}_2)$.
(ii) This is a direct consequence of the fact that R_2 is dense in $\bar{R}_2$. q.e.d.

PROPOSITION 6. *Let G_1, G_2, $\mathfrak{H}(G_1)$, $\mathfrak{H}(G_2)$, f and f_* be as above. Let D_1 and D_2 be algebraic Hopf subalgebras of $\mathfrak{H}(G_1)$ and $\mathfrak{H}(G_2)$ respectively. Then the image $f_*(D_1)$ of D_1 and the h-inverse h-$f_*^{-1}(D_2)$ of D_2 are algebraic.*

PROOF. Let A_i be the formalization of G_i for $i=1, 2$, and let $\bar{f}^*$ be the formal comorphism of f. If $A_i/\bar{\mathfrak{a}}_i$ is the formal subgroup of A_i corresponding to D_i for $i=1, 2$, then $f_*(D_1)$ (resp. h-$f_*^{-1}(D_2)$) corresponds to $A_2/\bar{f}^{*-1}(\bar{\mathfrak{a}}_1)$ (resp. $A_1/\bar{f}^*(\bar{\mathfrak{a}}_2)A_1$) by Prop. 3. Now put $\mathfrak{a}_1 = \bar{\mathfrak{a}}_1 \cap \mathcal{O}_1$ and $\mathfrak{a}_2 = \bar{\mathfrak{a}}_2 \cap \mathcal{O}_2$ denoting by $\mathcal{O}_i$ the stalk of G_i at the neutral point e_i for $i=1, 2$. Since D_1 and D_2 are algebraic, we have $\mathfrak{a}_1 A_1 = \bar{\mathfrak{a}}_1$ and $\mathfrak{a}_2 A_2 = \bar{\mathfrak{a}}_2$. By Lemma 4 $f^{*-1}(\mathfrak{a}_1)$ is dense in $\bar{f}^{*-1}(\mathfrak{a}_1 A_1)$ and hence $f^{*-1}(\mathfrak{a}_1)$ is equal to $\bar{f}^{*-1}(\mathfrak{a}_1 A_1) \cap \mathcal{O}_2 = \bar{f}^{*-1}(\bar{\mathfrak{a}}_1) \cap \mathcal{O}_2$ as easily seen. Moreover we have $f^{*-1}(\mathfrak{a}_1)A_2 = \bar{f}^{*-1}(\bar{\mathfrak{a}}_1)$. Similarly we see $f^*(\mathfrak{a}_2)\mathcal{O}_1 = \bar{f}^*(\bar{\mathfrak{a}}_2)A_1 \cap \mathcal{O}_1$ and $(f^*(\bar{\mathfrak{a}}_2)\mathcal{O}_1)A_1 = \bar{f}^*(\bar{\mathfrak{a}}_2)$. Therefore we see by Prop. 4 and Th. 1 $f^{*-1}(\mathfrak{a}_1)$ (resp. $f^*(\mathfrak{a}_2)\mathcal{O}_1$) is the defining ideal of a group subscheme of G_2 in $\mathcal{O}_2$ (resp. of G_1 in $\mathcal{O}_1$) whose formalization is $A_2/\bar{f}^{*-1}(\bar{\mathfrak{a}}_1)$ (resp. $A_1/\bar{f}^*(\mathfrak{a}_2)A_1$). This means that $f_*(D_1)$ and h-$f_*^{-1}(D_2)$ are algebraic. q.e.d.

COROLLARY. *Let G_1, G_2, $\mathcal{O}_1$, $\mathcal{O}_2$, f and f^* be as above. Then if $\mathfrak{a}_i$ is the defining ideal of a group subscheme of G_i in $\mathcal{O}_i$ for $i=1, 2$, $f^*(\mathfrak{a}_2)\mathcal{O}_1$ (resp. $f^{*-1}(\mathfrak{a}_1)$) is the defining ideal of a group subscheme of G_1 in $\mathcal{O}_1$ (resp. of G_2 in $\mathcal{O}_2$).*

We shall terminate this section by giving the notions of direct images and inverse images of group subschemes by a group homomorphism of a group scheme to another, and we shall restate the above proposition and the corollary in terms of them. For this purpose we need the next lemmas.

LEMMA 5. *Let G_1, G_2 and f be as above. If x is a closed point of G, we have $fL_x = L_{f(x)}f$ and $fR_x = R_{f(x)}f$.*

LEMMA 6. *Let G_1, G_2 and f be as above. Then the image of the base space of a group subscheme of G_1 in G_2 by f is a closed subset of G_2.*

These lemmas are well known and the proofs are not so difficult. Therefore we omit them.

PROPOSITION 7. *Let G_1, G_2, $\mathcal{O}_1$, $\mathcal{O}_2$, f and f^* be as above, and let H_1 be a group subscheme of G_1 with the defining ideal $\mathfrak{a}_1$ in $\mathcal{O}_1$. Then there exists a unique group subscheme H_2 of G_2 such that the defining ideal of H_2 in $\mathcal{O}_2$ is $f^{*-1}(\mathfrak{a}_1)$ and that we have $f(H_1)=H_2$ as sets. Moreover $f|_{H_1}$ decomposes through H_2.*

PROOF. By Cor. to Prop. 6 and Th. 1 there is a unique connected group subscheme N_2 of G_2 with the defining ideal $f^{*-1}(\mathfrak{a}_1)$ in $\mathcal{O}_2$. Let $W=\mathrm{Spec}(C)$ be any affine open subset of G_2 containing e_2 and let $\mathfrak{b}_C$ be the defining ideal of the closed subscheme $W \cap N_2$ of W. Then we see easily $L_a^*(\mathfrak{b}_C\mathcal{O}_{G_2,a})=R_a^*(\mathfrak{b}_C\mathcal{O}_{G_2,a})=f^{*-1}(\mathfrak{a}_1)$ for any closed point a in $W \cap N_2$ and $\mathfrak{b}_C\mathcal{O}_{G_2,b}=\mathcal{O}_{G_2,b}$ for any closed point b in $W-N_2$. On the other hand $f(H_1)$ is a closed subset of G_2 which is equal to $N_2 \cup L_{a_1}(N_2)\cdots L_{a_s}(N_2)$ as sets by Lemma 6. Then using W and $\mathfrak{b}_C$, we can see, in the same way as the proof of Lemma 2 in [15], that there exists a coherent sheaf $\mathfrak{c}$ of the ideals of $\mathcal{O}_{G_2}$ such that the closed subset of G_2 defined by $\mathfrak{c}$ is $f(H_1)$ and that we have $f^{*-1}(\mathfrak{a}_1)=\mathfrak{c}_{e_2}=L_y^*(\mathfrak{c}_y)$ for any closed point y in $f(H_1)$. Since we have $fR_x=R_{f(x)}f$ and $fL_x=L_{f(x)}f$ for any closed point x of G_1 by Lemma 5, we see $R_x^*f_x^*=f^*R_{f(x)}^*$ and $L_x^*f_x^*=f^*L_{f(x)}^*$, where f_x^* is the comorphism of $\mathcal{O}_{G_2,f(x)}$ to $\mathcal{O}_{G_1,x}$ defined by f. Then we have $R_{f(x)}^{*-1}f^{*-1}(\mathfrak{a}_1)=f_x^{*-1}R_x^{*-1}(\mathfrak{a}_1)=f_x^{*-1}L_x^{*-1}(\mathfrak{a}_1)=L_{f(x)}^{*-1}f^{*-1}(\mathfrak{a}_1)=\mathfrak{c}_{f(x)}$ for any closed point x in H_1, because we have $R_x^{*-1}(\mathfrak{a}_1)=R_{x^{-1}}^*(\mathfrak{a}_1)=L_{x^{-1}}^*(\mathfrak{a}_1)=L_x^{*-1}(\mathfrak{a}_1)$ and $f^{*-1}(\mathfrak{a}_1)=L_{f(x)}^*(\mathfrak{c}_{f(x)})$. This means $f^{*-1}(\mathfrak{a}_1)=\mathfrak{c}_{e_2}=R_y^*(\mathfrak{c}_y)$ for any closed point y in $f(H_1)$. The same argument as as the proof of Lemma 2 in [15] shows that $H_2=(f(H_1), \mathcal{O}_{G_2}/\mathfrak{c})$ is a group subscheme of G_2 satisfying our condition. The uniqueness of H_2 is clear. Since $L_x^{*-1}(\mathfrak{a}_1)$ is the stalk of the coherent sheaf of the ideals of $\mathcal{O}_{G_1}$ defining H_1 and we have $f^{*-1}L_x^{*-1}(\mathfrak{a}_1)=\mathfrak{c}_{f(x)}$, it is clear that $f|_{H_1}$ decomposes through H_2. q.e.d.

PROPOSITION 8. *Let G_1, G_2, $\mathcal{O}_1$, $\mathcal{O}_2$, f and f^* be as above and let H_2 be a group subscheme of G_2 with the defining ideal $\mathfrak{a}_2$ in $\mathcal{O}_2$. Then there exists a unique group subscheme H_1 of G_1 such that the defining ideal of H_1 in $\mathcal{O}_1$ is $f^*(\mathfrak{a}_2)\mathcal{O}_1$ and that $f^{-1}(H_2)=H_1$ as sets. Moreover $f|_{H_1}$ decomposes through H_2.*

PROOF. It is easy to see that $f^{-1}(H_2)$ is a closed subset of G_1 and that the set of closed points of $f^{-1}(H_2)$ is a group. On the other hand there exists a unique connected group subscheme N_1 of G_1 with the defining ideal $f^*(\mathfrak{a}_2)\mathcal{O}_1$ in $\mathcal{O}_1$ by Cor. to Prop. 6 and Th. 1. Then $f^{-1}(H_2)$ is a finite disjoint union $\bigcup_{i=1}^{s} L_{x_i}(N_1)$ as seen easily, where x_i is a closed point of $f^{-1}(H_2)$ for each i. A similar argument to the proof of Prop. 7 shows that there exists a unique group subscheme H_1 of G_1 with the underlying space $f^{-1}(H_2)$ and the connected component N_1 containing the neutral point e_1. It is easy to see that H_1 satisfies our conditions. q.e.d.

Let G_1, G_2 and f be as above. If H_1 is a group subscheme of G_1, we call the group subscheme H_2 of G_2 obtained in Prop. 7 *the direct image of* H_1 *by* f. In particular we call the direct image of G_1 by f *the image of* f. On the other hand if H_2 is a group subscheme of G_2, we call the group subscheme H_1 of G_1 obtained in Prop. 8 *the inverse image of* H_2 *by* f. In particular we call the inverse image of the trivial group subscheme (e_2, Spec(k)) by f *the kernel of* f.

PROPOSITION 9. *Let* G_1, G_2, f, H_1 *and* H_2 *be as above. Then we have the followings*:

(i) *The direct image of* H_1 *by* f *is the smallest group subscheme* H'_2 *of* G_2 *such that* $f|_{H_1}$ *decomposes through* H'_2. *If* D_1 *is the Hopf subalgebra of* $\mathfrak{H}(G_1)$ *corresponding to* H_1, $f_*(D_1)$ *is the Hopf subalgebra of* $\mathfrak{H}(G_2)$ *corresponding to the direct image of* H_1.

(ii) *The inverse image of* H_2 *by* f *is the largest group subscheme* H'_1 *of* G_1 *such that* $f|_{H'_1}$ *decomposes through* H_2. *If* D_2 *is the Hopf subalgebra of* $\mathfrak{H}(G_2)$ *corresponding to* H_2, h-$f_*^{-1}(D_2)$ *is the Hopf subalgebra of* $\mathfrak{H}(G_1)$ *corresponding to the inverse image of* H_2.

PROOF. This is a direct consequence of Prop. 7, 8 and 3. q.e.d.

§4. Joins and intersections of group subschemes

First we assume that k is an algebraically closed field of a positive characteristic p. Let X be an algebraic scheme over k and let x be a point of X. If $\mathcal{O}_x$ is the stalk of X at x, we denote by F_x the Frobenius endomorphism of $\mathcal{O}_x$, i.e., $F_x(a)=a^p$ for any a in $\mathcal{O}_x$. If we put $\ker F_x^i=\mathfrak{a}_i$, we have

$$(0) = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_i \subset \mathfrak{a}_{i+1} \cdots .$$

Since $\mathcal{O}_x$ is noetherian, there is an integer N such that $\mathfrak{a}_n=\mathfrak{a}_N$ for any $n\geq N$. Then we say that X *has the exponent not larger than* N *at* x and we denote this by $\exp_x X\leq N$. In other words we have $\exp_x X\leq N$ if and only if we have $\mathfrak{n}^{p^N}=0$, where $\mathfrak{n}$ is the nilradical of $\mathcal{O}_x$. In particular if X is a group scheme G over k, then the stalk of G at any closed point x is isomorphic to that of G at the neutral point e of G. Hence we say that G *has the exponent not larger than* N if $\exp_e G \leq N$, and then we denote this by $\exp G\leq N$.

LEMMA 7. *Let* X *and* Y *be algebraic schemes over* k, *and let* f *be a morphism of* X *to* Y *such that the comorphism* f_x^* *of the stalk* $\mathcal{O}_{f(x)}$ *of* Y *at* $f(x)$ *to the stalk* $\mathcal{O}_x$ *of* X *at* x *defined by* f *is an injection. Then if we have* $\exp_x X\leq N$, *the inequality* $\exp_{f(x)} Y \leq N$ *holds.*

PROOF. If F and F' are the Frobenius endomorphisms of $\mathcal{O}_x$ and $\mathcal{O}_{f(x)}$ respectively, we see easily $f_x^* F'^i = F^i f_x^*$ for any $i \geq 0$. Therefore if we put $\mathfrak{a}_i = \ker F^i$ and $\mathfrak{b}_i = \ker F'^i$ for any $i \geq 0$, $\mathfrak{b}_i$ is equal to $f_x^{*-1}(\mathfrak{a}_i)$ by our assumption on f_x^*. This means that we have $\mathfrak{b}_N = \mathfrak{b}_n$ if $\mathfrak{a}_N = \mathfrak{a}_n$ for $n \geq N$. q.e.d.

LEMMA 8. *Let* X_i *be an algebraic scheme over* k *for* $1 \leq i \leq n$. *If* x_i *is a closed point of* X_i *such that* $\exp_{x_i} X_i \leq N$ *for each* i, *we have* $\exp_{x_1 \times \cdots \times x_n} X_1 \times \cdots \times X_n \leq N$.

PROOF. It is sufficient to show the case of $n=2$. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the nilradicals of the stalks $\mathcal{O}_1$ and $\mathcal{O}_2$ of X_1 and X_2 at X_1 and X_2 respectively. Then the nilradical of $\mathcal{O}_1 \otimes_k \mathcal{O}_2$ is $\mathfrak{n}_1 \otimes \mathcal{O}_2 + \mathcal{O}_1 \otimes \mathfrak{n}_2$. In fact $\mathcal{O}_1/\mathfrak{n}_1$ and $\mathcal{O}_2/\mathfrak{n}_2$ have no nilpotent elements except zero. This means that $\mathcal{O}_1/\mathfrak{n}_1 \otimes_k \mathcal{O}_2/\mathfrak{n}_2 = \mathcal{O}_1 \otimes_k \mathcal{O}_2/(\mathfrak{n}_1 \otimes \mathcal{O}_2 + \mathcal{O}_1 \otimes \mathfrak{n}_2)$ is reduced, because k is algebraically closed (cf. Matsumura [6], (27, E), Lemma 2). On the other hand the stalk $\mathcal{O}$ of $X_1 \times X_2$ at $x_1 \times x_2$ is the quotient ring of $\mathcal{O}_1 \otimes_k \mathcal{O}_2$ with respect to a maximal ideal of $\mathcal{O}_1 \otimes_k \mathcal{O}_2$. Therefore the nilradical $\mathfrak{n}$ of $\mathcal{O}$ is generated by the image of $\mathfrak{n}_1 \otimes \mathcal{O}_2 + \mathcal{O}_1 \otimes \mathfrak{n}_2$ in $\mathcal{O}$. Since we have $\mathfrak{n}_1^{p^N} = \mathfrak{n}_2^{p^N} = 0$, we see easily $\mathfrak{n}^{p^N} = 0$. This means $\exp_{x_1 \times x_2} X_1 \times X_2 \leq N$. q.e.d.

LEMMA 9. *Let* A *be a noetherian ring and* $\mathfrak{p}$ *a prime ideal of* A. *Then if* n *is a positive integer, we have the minimal condition on the set of* $\mathfrak{p}$*-primary ideals* $\mathfrak{q}$ *of* A *such that* $\mathfrak{p}^n \subset \mathfrak{q}$.

PROOF. If $\mathfrak{p}^n \subset \mathfrak{q}$, we see $\mathfrak{p}^n A_\mathfrak{p} \subset \mathfrak{q} A_\mathfrak{p}$. Further we know that $\mathfrak{q} \subset \mathfrak{q}'$. If and only if $\mathfrak{q} A_\mathfrak{p} \subset \mathfrak{q}' A_\mathfrak{p}$ for any $\mathfrak{p}$-primary ideals $\mathfrak{q}$ and $\mathfrak{q}'$. Therefore it is sufficient to show that the minimal condition on the set of ideals of $A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}$ holds. Since $A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}$ is an artinian local ring, our assertion is true. q.e.d.

In the following let k be an algebraically closed field of any characteristic and let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k. Denote by $\mathcal{O}_n$ the stalk of $G \times \cdots \times G$ (n *times*) at the neutral point $e \times \cdots \times e$ for any $n > 0$. In particular put $\mathcal{O} = \mathcal{O}_1$. Then we denote by Δ_n the comorphism of $\mathcal{O} = \mathcal{O}_1$ to $\mathcal{O}_n$ defined by the multiplication μ_n of $G \times \cdots \times G$ (n *times*) to G for $n \geq 2$.

LEMMA 10. *Let* G, $\mathcal{O}$, $\mathcal{O}_n$ *and* Δ_n *be as above, and let* $\mathfrak{a}_i$ *be an ideal of* $\mathcal{O}$ *for* $1 \leq i \leq n$ *such that the ideal*

$$\mathfrak{b}_j = \mathfrak{a}_1 \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} + \mathcal{O} \otimes \mathfrak{a}_2 \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} + \cdots + \mathcal{O} \otimes \cdots \otimes \mathcal{O} \otimes \mathfrak{a}_j$$

of $\mathcal{O} \otimes \cdots \otimes \mathcal{O}$ (j *times*) *is primary for each* $j = 1, 2, \ldots, n$. *Then if we put* $\mathfrak{c}_j = \Delta_j^{-1}(\mathfrak{b}_j \mathcal{O}_j)$ *for* $2 \leq j \leq n$, *we have* $\mathfrak{c}_j \subset \mathfrak{c}_{j-1}$ *for* $3 \leq j \leq n$ *and* $\mathfrak{c}_j \subset \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_j$

for $2 \leq j \leq n$. Moreover $\mathfrak{c}_j$ is a primary ideal of $\mathcal{O}$ for $2 \leq j \leq n$.

PROOF. First we show the case of $j=2$. Let p_1 and p_2 be the comorphisms of $\mathcal{O}_2$ to $\mathcal{O}$ defined by the morphisms:

$$G \cong G \times \operatorname{Spec}(k) \xrightarrow{1_G \times e} G \times G \quad \text{and} \quad G \cong \operatorname{Spec}(k) \times G \xrightarrow{e \times 1_G} G \times G$$

respectively. Then we see $p_1 \Delta_2 = p_2 \Delta_2 = 1_{\mathcal{O}}$, and we have $\mathfrak{a}_1 = p_1(\mathfrak{b}_2 \mathcal{O}_2)$ and $\mathfrak{a}_2 = p_2(\mathfrak{b}_2 \mathcal{O}_2)$ as seen easily. By the definition of $\mathfrak{c}_2$ we see $\mathfrak{a}_1 = p_1(\mathfrak{b}_2 \mathcal{O}_2) \supset p_1 \Delta_2 \Delta_2^{-1}(\mathfrak{b}_2 \mathcal{O}_2) = \mathfrak{c}_2$. Similarly we see $\mathfrak{a}_2 \supset \mathfrak{c}_2$. Therefore $\mathfrak{c}_2$ is contained in $\mathfrak{a}_1 \cap \mathfrak{a}_2$. Now we assume $j \geq 3$. Let g and h be the natural homomorphisms $\mathcal{O} \otimes \mathcal{O} \to \mathcal{O}_2$ and $\mathcal{O}_{j-1} \otimes \mathcal{O} \to \mathcal{O}_j$ obtained from localizations respectively, and let ϕ_j be the unique homomorphism of $\mathcal{O}_2$ to $\mathcal{O}_j$ satisfying $\phi_j g = h(\Delta_{j-1} \otimes 1_{\mathcal{O}})$. Then we see easily $\Delta_j = \phi_j \Delta_2$. On the other hand we see also easily that g and h are injective. Therefore $\phi_j^{-1}(\mathfrak{b}_j \mathcal{O}_j)$ coincides with $(\Delta_{j-1} \otimes 1_{\mathcal{O}})^{-1}(h^{-1}(\mathfrak{b}_j \mathcal{O}_j))\mathcal{O}_2 = (\mathfrak{c}_{j-1} \otimes \mathcal{O} + \mathcal{O} \otimes \mathfrak{a}_j)\mathcal{O}_2$, because $\mathfrak{b}_j(\mathcal{O}_{j-1} \otimes \mathcal{O})$ is primary in $\mathcal{O}_{j-1} \otimes \mathcal{O}$ by our assumption and the injectivity of g. This means $\mathfrak{c}_j = \Delta_j^{-1}(\mathfrak{b}_j \mathcal{O}_j) = \Delta_2^{-1}((\mathfrak{c}_{j-1} \otimes \mathcal{O} + \mathcal{O} \otimes \mathfrak{a}_j)\mathcal{O}_2)$ by the equality $\Delta_j = \phi_j \Delta_2$. On the other hand $\Delta_2^{-1}((\mathfrak{c}_{j-1} \otimes \mathcal{O} + \mathcal{O} \otimes \mathfrak{a}_j)\mathcal{O}_2)$ is contained in $\mathfrak{c}_{j-1}$ and $\mathfrak{a}_j$ as seen in the same way as above. This means $\mathfrak{c}_j \subset \mathfrak{c}_{j-1}$ and $\mathfrak{c}_j \subset \mathfrak{a}_j$. By induction on j we see easily $\mathfrak{c}_j \subset \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_j$ for $j = 2, \ldots, n$. Now since $\mathfrak{b}_j$ is a primary ideal of $\mathcal{O} \otimes \cdots \otimes \mathcal{O}$ (*j times*), so is $\mathfrak{b}_j \mathcal{O}_j$, and hence we see $\mathfrak{c}_j$ is a primary ideal of $\mathcal{O}$. q.e.d.

LEMMA 11. *Let G, $\mathcal{O}$, $\mathcal{O}_n$ and Δ_n be as above, and let H_1 and H_2 be connected group subschemes of G defined by ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in $\mathcal{O}$ respectively. Denoting by E the set $\{(i_1, \ldots, i_n) \mid i_j = 1 \text{ or } 2;\ n \geq 2\}$, put, for each $(i_1, \ldots, i_n)$ in E,*

$$\mathfrak{c}_{i_1 \cdots i_n} = \Delta_n^{-1}((\mathfrak{a}_{i_1} \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} + \cdots + \mathcal{O} \otimes \cdots \otimes \mathcal{O} \otimes \mathfrak{a}_{i_n})\mathcal{O}_n).$$

Then there exists an element $(j_1, \ldots, j_m)$ in E such that $\mathfrak{c}_{j_1 \cdots j_m}$ is contained in $\mathfrak{c}_{i_1 \cdots i_n}$ for any $(i_1, \ldots, i_n)$ in E.

PROOF. Since $\mathcal{O}/\mathfrak{a}_i$ is the stalk of the group scheme H_i at e, the completion A_i of $\mathcal{O}/\mathfrak{a}_i$ with respect to the maximal ideal of $\mathcal{O}/\mathfrak{a}_i$ is a formal group over k for each $i = 1, 2$. Therefore we see easily by Prop. 2 in [14] that the zero ideal of $A_{i_1} \otimes_k \cdots \otimes_k A_{i_n}$ is primary for any $(i_1, \ldots, i_n)$ in E, and hence the subring $\mathcal{O}/\mathfrak{a}_{i_1} \otimes_k \cdots \otimes_k \mathcal{O}/\mathfrak{a}_{i_n}$ has the same property. Applying Lemma 10 we see that $\mathfrak{c}_{i_1 \cdots i_n}$ is a primary ideal of $\mathcal{O}$ satisfying $\mathfrak{c}_{i_1 \cdots i_n} \subset \mathfrak{a}_{i_1} \cap \cdots \cap \mathfrak{a}_{i_n}$ and $\mathfrak{c}_{i_1 \cdots i_n} \subset \mathfrak{c}_{i_1 \cdots i_{n-1}}$ for any $(i_1, \ldots, i_n)$ in E. Now since the radical $\operatorname{rad}(\mathfrak{c}_{i_1 \cdots i_n})$ of $\mathfrak{c}_{i_1 \cdots i_n}$ is prime for any $(i_1, \ldots, i_n)$ in E, there exists a minimal element $\mathfrak{p}$ in the set $\{\operatorname{rad}(\mathfrak{c}_{i_1 \cdots i_n}) \mid (i_1, \ldots, i_n) \in E\}$.

First we assume that the characteristic of k is $p > 0$, and suppose $\exp H_1 \leq N$ and $\exp H_2 \leq N$. Then by Lemma 8 we have $\exp H_{i_1} \times \cdots \times H_{i_n} \leq N$ for any $(i_1, \ldots, i_n)$ in E, and hence we see easily $\mathfrak{p}^{p^N} \subset \mathfrak{c}_{i_1 \cdots i_n}$ for $(i_1, \ldots, i_n)$ in E such that $\operatorname{rad}(\mathfrak{c}_{i_1 \cdots i_n})$

$=\mathfrak{p}$ in the same way as the proof of Lemma 7. This means that there exists a minimal ideal $\mathfrak{c}_{j_1\cdots j_m}$ in the set $\{\mathfrak{c}_{i_1\cdots i_n}|\mathrm{rad}(\mathfrak{c}_{i_1\cdots i_n})=\mathfrak{p}\}$ by Lemma 9. For any $(i_1,\ldots, i_n)$ in E we see $\mathfrak{c}_{j_1\cdots j_m}\supset\mathfrak{c}_{j_1\cdots j_m i_1\cdots i_n}$ by Lemma 10 and hence $\mathfrak{p}=\mathrm{rad}(\mathfrak{c}_{j_1\cdots j_m})=\mathrm{rad}\,\mathfrak{c}_{j_1\cdots j_m i_1\cdots i_n}$ by the minimality of $\mathfrak{p}$. This means $\mathfrak{c}_{j_1\cdots j_m}=\mathfrak{c}_{j_1\cdots j_m i_1\cdots i_n}$ from the definition of $\mathfrak{c}_{j_1\cdots j_m}$. On the other hand we can see $\mathfrak{c}_{i_1\cdots i_n}\supset\mathfrak{c}_{j i_1\cdots j_n}$ for $j=1, 2$ in the same way as the proof of Lemma 10. Repeating this we see $\mathfrak{c}_{i_1\cdots i_n}\supset\mathfrak{c}_{j_1\cdots j_m i_1\cdots i_n}=\mathfrak{c}_{j_1\cdots j_m}$. Therefore $\mathfrak{c}_{j_1\cdots j_m}$ is the smallest ideal in the set $\{\mathfrak{c}_{i_1\cdots i_n}|(i_1,\ldots, i_n)\in E\}$.

In the case of characteristic zero, H_1 and H_2 are both reduced as well known, i.e., $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are prime. This means that $\mathfrak{c}_{i_1\cdots i_n}$ is prime for any $(i_1,\ldots, i_n)$ in E as seen easily. Therefore $\mathfrak{p}=\mathfrak{c}_{j_1\cdots j_m}$ is the minimal ideal in the set $\{\mathfrak{c}_{i_1\cdots i_n}|(i_1,\ldots, i_n)\in E\}$. q.e.d.

LEMMA 12. *Let $\mathcal{O}$, $\mathcal{O}_n$, $\mathfrak{a}_i$ $(i=1, 2,\ldots, n)$ and $\mathfrak{b}_n$ be as in Lemma 10. Then the continuous dual coalgebra C_n of the residue ring $\mathcal{O}_n/\mathfrak{b}_n\mathcal{O}_n$ is canonically isomorphic to $(\mathcal{O}/\mathfrak{a}_1)^c\otimes_k\cdots\otimes_k(\mathcal{O}/\mathfrak{a}_n)^c$, where $(\mathcal{O}/\mathfrak{a}_i)^c$ is the continuous dual coalgebra of $\mathcal{O}/\mathfrak{a}_i$ for $i=1, 2,\ldots, n$.*

PROOF. If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$, $\mathfrak{m}_n=(\mathfrak{m}\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O}+\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes\mathfrak{m})\mathcal{O}_n$ is that of $\mathcal{O}_n$. Since the continuous dual coalgebra of $\mathcal{O}_n$ is $\varinjlim_t(\mathcal{O}_n/\mathfrak{m}_n^t)^*$, C_n coincides with

$$\begin{aligned}&\varinjlim_t(\mathcal{O}_n/((\mathfrak{a}_1+\mathfrak{m}^t)\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O}+\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes(\mathfrak{a}_n+\mathfrak{m}^t))\mathcal{O}_n)^*\\ &=\varinjlim_t(\mathcal{O}/\mathfrak{a}_1+\mathfrak{m}^t)^*\otimes\cdots\otimes(\mathcal{O}/\mathfrak{a}_n+\mathfrak{m}^t)^*=(\mathcal{O}/\mathfrak{a}_1)^c\otimes\cdots\otimes(\mathcal{O}/\mathfrak{a}_n)^c.\end{aligned}$$

q.e.d.

Let B be a Hopf algebra over k, and let D and E be Hopf subalgebras of B. Then if C is the subalgebra of B generated by D and E, C is also a Hopf subalgebra of B as seen easily. C is the smallest Hopf subalgebra of B containing D and E. We denote C by $J(D, E)$ and call it *the join of D and E*. Similarly if $D_1,\ldots, D_n$ are Hopf subalgebras of B, we can define the join $J(D_1,\ldots, D_n)$ of $D_1,\ldots, D_n$. On the other hand if we put $I(D, E)=D\cap E$, $I(D, E)$ is a Hopf subalgebra of B by Lemma 1 in [13]. We call $I(D, E)$ *the intersection of D and E*. Similarly we can define $I(D_1,\ldots, D_n)$.

THEOREM 2. *Let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k, and let D_1 and D_2 be algebraic Hopf subalgebras of the Hopf algebra $\mathfrak{H}(G)$ attached to G. Then the join $J(D_1, D_2)$ of D_1 and D_2 is also algebraic.*

PROOF. Let H_1 and H_2 be the connected group subschemes of G such that $D_1=\mathfrak{H}(H_1)$ and $D_2=\mathfrak{H}(H_2)$, and denote by $\mathfrak{a}_1$ and $\mathfrak{a}_2$ the defining ideals of H_1 and H_2 in the stalk $\mathcal{O}$ of G at e respectively. Let $\mathcal{O}_n$, Δ_n, E and $\mathfrak{c}_{i_1\cdots i_n}$ be as in

Lemma 11, and let $\mathfrak{c}=\mathfrak{c}_{j_1\cdots j_m}$ be the smallest ideal in the set $\{\mathfrak{c}_{i_1\cdots i_n}|(i_1,\ldots, i_n) \in E\}$ (cf. Lemma 11). If ψ_m is the homomorphism of $\mathcal{O}_2$ to $\mathcal{O}_{2m}$ obtained naturally from the homomorphism $\Delta_m \otimes \Delta_m$ of $\mathcal{O}\otimes_k \mathcal{O}$ to $\mathcal{O}_m \otimes_k \mathcal{O}_m$ by localizations, we see easily $\Delta_{2m}=\psi_m \Delta_2$ by the associativity of μ. Then a similar argument to the proof of Lemma 10 gives

$$\begin{aligned}\psi_m^{-1}((\mathfrak{a}_{j_1} \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} + \cdots + \mathcal{O} \otimes \cdots \otimes \mathcal{O} \otimes \mathfrak{a}_{j_m} \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} \\ + \mathcal{O} \otimes \cdots \otimes \mathcal{O} \otimes a_{j_1} \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} + \cdots + \mathcal{O} \otimes \cdots \otimes \mathcal{O} \otimes \mathfrak{a}_{j_m})\mathcal{O}_{2m}) \\ = (\mathfrak{c} \otimes \mathcal{O} + \mathcal{O} \otimes \mathfrak{c})\mathcal{O}_2,\end{aligned}$$

and hence we see $\mathfrak{c}_{j_1\cdots j_m j_1 \cdots j_m}=\Delta_2^{-1}((\mathfrak{c}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{c})\mathcal{O}_2)$. By Lemma 10 we see $\mathfrak{c}_{j_1\cdots j_m j_1\cdots j_m} \subset \mathfrak{c}_{j_1\cdots j_m}=\mathfrak{c}$, and hence $\mathfrak{c}_{j_1\cdots j_m j_1\cdots j_m}=\mathfrak{c}$ from the minimality of $\mathfrak{c}$. This means $\Delta_2(\mathfrak{c}) \subset (\mathfrak{c}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{c})\mathcal{O}_2$. On the other hand let γ^* be the comorphism of $\mathcal{O}$ to itself defined by γ, and let σ_n be the automorphism of $G\times\cdots\times G$ (*n times*) such that σ_n maps the i-th factor to the $(n-i+1)$-th factor for each $i=1, 2,\ldots, n$. If we denote by σ_n^* the comorphism of $\mathcal{O}_n$ to itself defined by σ_n, we see easily $\Delta_n\gamma^*=\sigma_n^*(\gamma\times\cdots\times\gamma)^*\Delta_n$. In fact we have $\mu_n(\gamma\times\cdots\times\gamma)\sigma_n=\gamma\cdot\mu_n$ as seen easily if μ_n is the multiplication of $G\times\cdots\times G$ to G. This means $\gamma^{*-1}(\mathfrak{c}_{i_1\cdots i_n}) = \mathfrak{c}_{i_n\cdots i_1}$ for any $(i_1,\ldots, i_n)$ in E, because we have $\gamma^*(\mathfrak{a}_i)=\mathfrak{a}_i$ for $i=1, 2$. In particular we see $\gamma^{*-1}(\mathfrak{c})=\gamma^{*-1}(\mathfrak{c}_{j_1\cdots j_m})=\mathfrak{c}_{j_m\cdots j_1}\supset\mathfrak{c}$ and $\gamma^{*-1}(\mathfrak{c})\subset\gamma^{*-1}(\mathfrak{c}_{j_m\cdots j_1}) =\mathfrak{c}_{j_1\cdots j_m}=\mathfrak{c}$. This means $\gamma^*(\mathfrak{c})=\mathfrak{c}$. Therefore $\mathfrak{c}$ is the defining ideal of a connected group subscheme H of G in $\mathcal{O}$ by Th. 1.

Now since $\mathfrak{c}$ is contained in $\mathfrak{a}_1 \cap \mathfrak{a}_2$ by Lemma 10, we see $\mathfrak{H}(H)$ contains both $D_1=\mathfrak{H}(H_1)$ and $D_2=\mathfrak{H}(H_2)$. On the other hand if $\mathfrak{b}'_{j_1\cdots j_m}$ is $(\mathfrak{a}_{j_1}\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O} +\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes\mathfrak{a}_{j_m})\mathcal{O}_m$, we see that Δ_m gives an injection of $\mathcal{O}/\mathfrak{c}$ into $\mathcal{O}_m/\mathfrak{b}'_{j_1\cdots j_m}$. Therefore the transpose Δ_m^* of Δ_m maps the continuous dual coalgebra $C_{j_1\cdots j_m}$ of $\mathcal{O}_m/\mathfrak{b}'_{j_1\cdots j_m}$ onto the continuous dual coalgebra $\mathfrak{H}(H)$ of $\mathcal{O}/c$. Moreover since Δ_m^* gives the multiplication of m elements in $\mathfrak{H}(G)$, we see, by Lemma 12, that $\mathfrak{H}(H)$ is contained in the algebra generated by D_1 and D_2. This means $\mathfrak{H}(H)=J(D_1, D_2)$. In other words $J(D_1, D_2)$ is algebraic. q.e.d.

COROLLARY. *In the above theorem we assume that the characteristic of k is $p>0$. Then if H, H_1 and H_2 are connected group subscheme of G such that $\mathfrak{H}(H)=J(\mathfrak{H}(H_1), \mathfrak{H}(H_2))$, we have* $\exp H\leq\max(\exp H_1, \exp H_2)$. *In any characteristic the connected group subscheme H of G corresponding to $J(\mathfrak{H}(H_1), \mathfrak{H}(H_2))$ is reduced, if H_1 and H_2 are reduced and connected group subschemes of G.*

PROOF. The first assertion is shown already in the proof of Lemma 11 and Th. 2. On the other hand a group scheme G over k is reduced if and only if $\exp G=0$ if the characteristic of k is $p>0$. Therefore in this case the last assertion follows from the first one. If the characteristic of k is zero, any group scheme over k is reduced, and hence our assertion is true. q.e.d.

Let G be a group scheme over k, and let $H_1,\dots, H_n$ be connected group subschemes of G. Then there exists a unique connected group subscheme H of G such that $\mathfrak{H}(H)=J(\mathfrak{H}(H_1),\dots, \mathfrak{H}(H_n))$ by Th. 2. It is easy to see that H is the smallest connected group subscheme of G containing each H_i as a group subscheme of it for each $i=1, 2,\dots, n$. We call H *the join of* $H_1,\dots, H_n$ and denote it by $J(H_1,\dots, H_n)$.[4] By Cor. to Th. 2 we have $\exp J(H_1,\dots, H_n)\leq N$ in a positive characteristic case if $\exp H_i\leq N$ for each $i=1,\dots, n$.

PROPOSITION 10. *Let G and $H_1,\dots, H_n$ be as above. Then the intersection $I(\mathfrak{H}(H_1),\dots, \mathfrak{H}(H_n))$ of $\mathfrak{H}(H_1),\dots, \mathfrak{H}(H_n)$ is algebraic.*

PROOF. It is sufficient to show our assertion in the case of $n=2$. Let the notations be as in Lemma 11. Then we see that $\Delta_2(\mathfrak{a}_1+\mathfrak{a}_2)=\Delta_2(\mathfrak{a}_1)+\Delta_2(\mathfrak{a}_2)$ is contained in the ideal of $\mathcal{O}_2$ generated by $\mathfrak{a}_1\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a}_1+\mathfrak{a}_2\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a}_2$ $=(\mathfrak{a}_1+\mathfrak{a}_2)\otimes\mathcal{O}+\mathcal{O}\otimes(\mathfrak{a}_1+\mathfrak{a}_2)$ by Th. 1. Similarly we see $\gamma^*(\mathfrak{a}_1+\mathfrak{a}_2)=\mathfrak{a}_1+\mathfrak{a}_2$. This means by Th. 1 that $\mathfrak{a}_1+\mathfrak{a}_2$ is the defining ideal of a connected group subscheme H of G. Now if A is the formalization of G, we see $(\mathfrak{a}_1+\mathfrak{a}_2)A=\mathfrak{a}_1A$ $+\mathfrak{a}_2A$. Therefore we have $\mathfrak{H}(H_1)\cap\mathfrak{H}(H_2)=(\mathfrak{H}(H_1)^{\perp}+\mathfrak{H}(H_2)^{\perp})^{\perp}=(\mathfrak{a}_1A+\mathfrak{a}_2A)^{\perp}$ as seen easily. Since we have $(\mathfrak{a}_1A+\mathfrak{a}_2A)\cap\mathcal{O}=\mathfrak{a}_1+\mathfrak{a}_2$, we see $\mathfrak{H}(H_1)\cap\mathfrak{H}(H_2)$ $=\mathfrak{H}(H)$, and hence $\mathfrak{H}(H_1)\cap\mathfrak{H}(H_2)$ is algebraic. q.e.d.

Let G and $H_1,\dots, H_n$ be as above. Then there exists a unique connected group subscheme H of G such that $\mathfrak{H}(H)=I(\mathfrak{H}(H_1),\dots, \mathfrak{H}(H_n))$ by the above proposition and Th. 1. We call H *the intersection* of $H_1,\dots, H_n$ and denote it by $I(H_1,\dots, H_n)$. We have $\mathfrak{H}(I(H_1,\dots, H_n))=I(\mathfrak{H}(H_1),\dots, \mathfrak{H}(H_n))=\bigcap_{i=1}^{n}\mathfrak{H}(H_i)$. More generally let $\{H_\lambda|\lambda\in\Lambda\}$ be a family of an arbitrary number of connected group subschemes H_λ of G, and let $\mathfrak{a}_\lambda$ be the defining ideal of H_λ in the stalk $\mathcal{O}$ of G at the neutral point e of G for each λ in Λ. Then there exists the largest ideal $\mathfrak{a}_{\lambda_1}+\cdots+\mathfrak{a}_{\lambda_n}$ of $\mathcal{O}$ in the family of the ideals $\mathfrak{a}_{\lambda'_1}+\cdots+\mathfrak{a}_{\lambda'_m}$ ($\lambda'_j\in\Lambda$), because $\mathcal{O}$ is noetherian. It is easy to see that the connected group subscheme $H=I(H_{\lambda_1},$ $\dots, H_{\lambda_n})$ is the largest connected group subscheme of G which is a group subscheme of any H_λ ($\lambda\in\Lambda$). We call H *the intersection* of H_λ ($\lambda\in\Lambda$) and denote it by $\underset{\lambda\in\Lambda}{I}(H_\lambda)$. Then we have $\mathfrak{H}(\underset{\lambda\in\Lambda}{I}(H_\lambda))=\bigcap_{\lambda\in\Lambda}\mathfrak{H}(H_\lambda)$. As for the join of an arbitrary number of connected group subschemes we have the following

PROPOSITION 11. *Let G be a group scheme over k, and let $\{H_\lambda|\lambda\in\Lambda\}$ be a family of connected group subschemes H_λ of G. If the characteristic of k*

4) If H_i is reduced for any i, $J(H_i,\dots, H_n)$ coincides with *the group closure* of $\bigcup_i H_i$ in the sense of [1]. See (2.2) in [1].

is $p>0$, *we assume that* $\exp H_\lambda \leq N$ *for any* $\lambda \in \Lambda$. *Then the subalgebra D of* $\mathfrak{H}(G)$ *generated by all* $\mathfrak{H}(H_\lambda)$ $(\lambda \in \Lambda)$ *is an algebraic Hopf subalgebra of* $\mathfrak{H}(G)$. *In particular D is generated by a finite number of* $\mathfrak{H}(H_{\lambda_1}),\dots,\mathfrak{H}(H_{\lambda_m})$ *as algebras.*

PROOF. Let $\mathcal{O}$, $\mathcal{O}_n$ and Δ_n be as in Lemma 10, and let $\mathfrak{a}_\lambda$ be the defining ideal of H_λ in $\mathcal{O}$. If we put $\mathfrak{c}_{\lambda_1\cdots\lambda_n}=\Delta_n^{-1}((\mathfrak{a}_{\lambda_1}\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O}+\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes\mathfrak{a}_{\lambda_n})\mathcal{O}_n)$, we see, in the same way as the proof of Lemma 11, that there is the smallest ideal $\mathfrak{c}=\mathfrak{c}_{\lambda_1\cdots\lambda_m}$ in the family of ideals $\mathfrak{c}_{\lambda'_1\cdots\lambda'_n}$ $(\lambda'_i \in \Lambda)$. Thus we need here the assumption $\exp H_\lambda \leq N$ for any $\lambda \in \Lambda$. Moreover we see that $\mathfrak{c}$ is the defining ideal of a connected group subscheme H of G in the same way as the proof of Th. 2. Since $\mathfrak{a}_\lambda$ contains $\mathfrak{c}$ for any λ in Λ as seen easily from Lemma 10, $\mathfrak{H}(H)$ contains $\mathfrak{H}(H_\lambda)$ for any λ in Λ. On the other hand $\mathfrak{H}(H)$ is the image of $\mathfrak{H}(H_{\lambda_1})\otimes\cdots\otimes\mathfrak{H}(H_{\lambda_m})$ by the multiplication Δ_m^* of $\mathfrak{H}(G)$ as seen in the same way as the proof of Th. 2. This means that $\mathfrak{H}(H)$ coincides with D. q.e.d.

Let G, $\{H_\lambda | \lambda \in \Lambda\}$ and D be as in Prop. 11. Then the unique group subscheme H of G satisfying $D=\mathfrak{H}(H)$ is called *the join of* H_λ $(\lambda \in \Lambda)$ and is denoted by $\underset{\lambda\in\Lambda}{J}(H_\lambda)$.

COROLLARY. *Let G be a group scheme over k, and let D be a Hopf subalgebra of* $\mathfrak{H}(G)$. *Then the followings are equivalent:*
(*i*) *D is algebraic.*
(*ii*) *D is generated by a finite number of algebraic Hopf subalgebras of* $\mathfrak{H}(G)$ *as algebras over k.*
(*iii*) *In the case of characteristic zero, D is generated by any number of algebraic Hopf subalgebras of* $\mathfrak{H}(G)$ *as algebras over k.*

In a positive characteristic case, D is generated by any number of algebraic Hopf subalgebras $\mathfrak{H}(H_\lambda)$ *of* $\mathfrak{H}(G)$ *as algebras over k such that* $\exp H_\lambda \leq N$ *for any* λ.

LEMMA 13. *Assume that the characteristic of k is* $p>0$, *and let G be a group scheme over k. Then we have the followings:*
(*i*) *Any finite dimensional Hopf subalgebra D of* $\mathfrak{H}(G)$ *is algebraic.*
(*ii*) *Any finite dimensional subspace U of* $\mathfrak{H}(G)$ *is contained in a finite dimensional Hopf subalgebra of* $\mathfrak{H}(G)$.

PROOF. Let $\mathcal{O}$ be the stalk of G at the neutral point e and let A be the formalization of G. If $\mathfrak{a}$ is the null space $D^\perp$ of D in A, $A/\mathfrak{a}$ is the dual space of D and hence is of a finite dimension. Therefore $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, where $\mathfrak{m}$ is the maximal ideal of A. It is well known that any $\mathfrak{m}$-primary ideal of A is the form $\mathfrak{q}A$ for some $\mathfrak{m}\cap\mathcal{O}$-primary ideal $\mathfrak{q}$ in $\mathcal{O}$, because A is the $(\mathfrak{m}\cap\mathcal{O})$-adic completion of $\mathcal{O}$. This means $\mathfrak{a}=(\mathfrak{a}\cap\mathcal{O})A$, and hence D is algebraic from the proof of Prop. 5. Next let $\{x_1,\dots,x_n\}$ be a basis for U over k. Then there is a positive integer N such that $<x_i, \mathfrak{m}^{p^N}>=0$ for all $i=1, 2,\ldots, n$. If we denote by $\mathfrak{m}^{(N)}$ the ideal of A generated by the elements a^{p^N} with a in $\mathfrak{m}$, we see $<U, \mathfrak{m}^{(N)}>=0$. If D_N is the null space of $\mathfrak{m}^{(N)}$ in $\mathfrak{H}(G)$. D_N is the algebraic Hopf subalgebra of $\mathfrak{H}(G)$. In fact the ideal $\mathfrak{m}^{(N)}$ satisfies the conditions in Th. 1 as seen easily. Then D_N is of finite dimension and U is a subspace of D_N. q. e. d.

Now let us recall the definition of *the shift* V of $\mathfrak{H}(G)$. If A is the formalization of G, let F be the Frobenius endomorphism of A such that $F(x)=x^p$ for any element x in A. Then the map F^* of the dual space A^* of A into itself defined by $<F^*(h), x>=<h, F(x)>^{1/p}$ is $1/p$-linear and moreover we see that $F^*(h)$ is in $\mathfrak{H}(A)$ if h is so. We denote by V the restriction of F^* to $\mathfrak{H}(A)=\mathfrak{H}(G)$ and we call *V the shift of* $\mathfrak{H}(G)$. It is easy to see that V is a $1/p$-linear Hopf algebra homomorphism of $\mathfrak{H}(G)$ into itself, and we denote by V^n the composite $V\cdot V\cdots V$ (n *times*).

THEOREM 3. *Let k and G be as in Lemma* 13, *and let V be the shift of* $\mathfrak{H}(G)$. *Then a Hopf subalgebra D of $\mathfrak{H}(G)$ is algebraic if and only if* $V^\infty(D)=\bigcap_{n=1}^{\infty} V^n(D)$ *is algebraic.*

PROOF. First we assume that $V^\infty(D)$ is algebraic. By Th. 2 and 3 in [9] there exist n sequences of divided powers $\{l_i^{(t)}|1\leq i\leq n,\ 0\leq t<p^{e_i}$ *for* $i\leq s$ *and* $0\leq t$ *for* $i\geq s+1\}$ in D such that $\{l_1^{(f_1)}\cdots l_n^{(f_n)}\}$ is a basis for D over k. Then we see easily that $\{l_{s+1}^{(f_{s+1})}\cdots l_n^{(f_n)}|f_i\geq 0\}$ is a basis for $V^\infty(D)$ over k. On the other hand if we denote by U the vector subspace of $\mathfrak{H}(G)$ generated by $\{l_1^{(f_1)}\cdots l_s^{(f_s)}|0\leq f_i<p^{e_i}\}$, U is of finite dimension. Then there exists a finite dimensional Hopf subalgebra D_1 of $\mathfrak{H}(G)$ containing U by Lemma 13, (ii). If we put $D_2=D_1\cap D$, D_2 contains U and D is generated by D_2 and $V^\infty(D)$ as k-algebras. Since $V^\infty(D)$ and D_2 are algebraic from our assumption and Lemma 13, (i), so is D by Cor. to Prop. 11. Conversely we assume that D is algebraic. Let $\mathfrak{q}$ be the null space of D in the formalization A of G. Then we see $\mathfrak{q}=(\mathfrak{q}\cap\mathcal{O})A$ where $\mathcal{O}$ is the stalk of G at e. If we put $\mathfrak{p}=\mathrm{rad}\,\mathfrak{q}$, $\mathfrak{p}$ is a prime ideal of A and the null space of $V^\infty(D)$ in A as seen easily. Then we can see easily that $\mathfrak{p}\cap\mathcal{O}$ is the radical of $\mathfrak{q}\cap\mathcal{O}$ and $\mathfrak{p}=(\mathfrak{p}\cap\mathcal{O})A$, because $(\mathfrak{p}\cap\mathcal{O})A$ is also a prime ideal of A. This means that $V^\infty(D)$ is algebraic. q. e. d.

REMARK. Let k be an algebraically closed field of a positive characteristic p, and let G be a group variety over k, i. e., a reduced and connected group scheme over k. Then the above theorem 3 shows that the condition $[k(G)^{\mathfrak{H}_0}: k(G)^{\mathfrak{H}'}]=\dim_k \mathfrak{H}'/\mathfrak{H}'\mathfrak{H}_0^+$ of Theorem 2 in [14] can be dropped. In fact, using the notations in [14], the equality $\dim_k L(\mathfrak{H}_0)+\dim V_{\mathfrak{H}'}=\dim G$ means that $\mathfrak{H}_0$ is algebraic by Theorem 1 in [14] and $\dim V_{\mathfrak{H}_0}=\dim V_{\mathfrak{H}'}$, and hence $\mathfrak{H}'$ is algebraic by Th. 3. In other words $\mathfrak{H}'$ is algebraic in wider sense in terms of [14].

§5. Rational representations of group schemes

Let $(G, \mu, \varepsilon, \gamma)$ be a reduced group scheme over k, and let $\mathcal{O}$ be the stalk of G at the neutral point e of G. Then the formalization A of G is isomorphic to a formal power series ring over k. In particular $\mathcal{O}$ is a regular local ring. If $\{a_1,\dots, a_n\}$ is a regular system of parameters of $\mathcal{O}$, A can be identified with the formal power series ring $k\ll a_1,\dots, a_n\gg$. Then there is a unique element $l_{e_1\cdots e_n}$ in $\mathfrak{H}(G)$ such that $<l_{e_1\cdots e_n}, a_1^{e_1},\dots, a_n^{e_n}>=1$ and $<l_{e_1\cdots e_n}, a_1^{e'_1},\dots, a_n^{e'_n}>=0$ if $(e_1,\dots, e_n)\neq(e'_1,\dots, e'_n)$ for any $(e_1,\dots, e_n)$. Then we can see easily that $\{l_{e_1\cdots e_n}| e_i\geq 0\}$ is a basis for $\mathfrak{H}(G)$ over k, which we call *the canonical basis for* $\mathfrak{H}(G)$ *with respect to* $\{a_1,\dots, a_n\}$.

PROPOSITION 12. *Let G, $\mathcal{O}$, A, $\{a_1,\dots, a_n\}$ and $\{l_{e_1\cdots e_n}\}$ be as above. Let C and D be Hopf subalgebras of $\mathfrak{H}(G)$. If C corresponds to the formal subgroup $A/(a_1,\dots, a_r)A$ of A, the followings are equivalent*:

(i) D is a Hopf subalgebra of C.

(ii) If $x=\sum_{(e)}\alpha_{e_1\cdots e_n}l_{e_1\cdots e_n}$ $(\alpha_{e_1\cdots e_n}\in k)$ is an element of D, we have $\alpha_{0\cdots 0\overset{i}{1}0\cdots 0}=0$ for $i\geq r$.

(iii) If $x=\sum_{(e)}\alpha_{e_1\cdots e_n}l_{e_1\cdots e_n}$ $(\alpha_{e_1\cdots e_n}\in k)$ is an element of D, we have $\alpha_{e_1\cdots e_n}=0$ for $e_i\neq 0$ with $i\leq r$.

The proof of this proposition is exactly similar to that of Corollary to Proposition 4 in [13] and hence we omit it.

COROLLARY. *Let G, $\mathfrak{H}(G)$ and $\mathcal{O}$ be as above. Then if H is a reduced group subscheme of G, there exists a regular system of parameters $\{a_1,\dots, a_n\}$ of $\mathcal{O}$ such that the defining ideal $\mathfrak{a}$ of H in $\mathcal{O}$ is $(a_1,\dots, a_r)\mathcal{O}$. Moreover if $\{l_{e_1\cdots e_n}| e_i\geq 0\}$ is the canonical basis for $\mathfrak{H}(G)$ with respect to $\{a_1,\dots, a_n\}$, the following conditions on a connected group subscheme K of G are equivalent*:

(i) K is a group subscheme of H.

(ii) If $x=\sum_{(e)}\alpha_{e_1\cdots e_n}l_{e_1\cdots e_n}$ $(\alpha_{e_1\cdots e_n}\in k)$ is an element of $\mathfrak{H}(K)$ identified with a Hopf subalgebra of $\mathfrak{H}(G)$, we have $\alpha_{0\cdots 0\overset{i}{1}0\cdots 0}=0$ for $i\leq r$.

(iii) If $x=\sum_{(e)}\alpha_{e_1\cdots e_n}l_{e_1\cdots e_n}$ $(\alpha_{e_1\cdots e_n}\in k)$ is an element of $\mathfrak{H}(K)$ identified with a Hopf subalgebra of $\mathfrak{H}(G)$, we have $\alpha_{e_1\cdots e_n}=0$ for $e_i\neq 0$ with $i\leq r$.

PROOF. Since $\mathfrak{a}$ is a prime ideal of $\mathcal{O}$, the existence of a regular system of parameters $\{a_1,\dots, a_n\}$ of $\mathcal{O}$ satisfying the property in our corollary follows from a well known result on regular system of parameters (cf. Serre [8], p. IV–41, Cor. to Prop. 22). The last assertion can be seen easily from Prop. 12, because K is a group subscheme of H if and only if $\mathfrak{H}(K)$ is contained in $\mathfrak{H}(H)$. q. e. d.

Let us consider the polynomial ring $k[t_{11},\dots, t_{nn}]$ of n^2 variables $t_{11},\dots, t_{1n},\dots, t_{n1},\dots, t_{nn}$ over k and denote by D the determinant of the matrix (t_{ij}). Then the affine scheme $\mathrm{Spec}(k[t_{11},\dots, t_{nn}, D^{-1}])$ is a group scheme over k. To see this it is enough to show that $k[t_{11},\dots, t_{nn}, D^{-1}]$ has a structure of a Hopf algebra over k whose antipode is an algebra homomorphism. Now we define k-linear maps Δ, η and c as follows:

$$\Delta(t_{ij}) = \sum_h t_{ih} \otimes t_{hj}, \quad \eta(t_{ij}) = \delta_{ij} \qquad \text{(Kronecker's delta)},$$

$$c(t_{ij}) = (-1)^{i+j} \det(t_{rs})_{r\neq i, s\neq j} D^{-1}.$$

We can easily see that these maps give a structure of a Hopf algebra over k to $k[t_{11},\dots, t_{nn}, D^{-1}]$ with the natural algebra structure over k. In other words if μ, ε and γ are the morphisms of affine schemes whose comorphisms are Δ, η and c respectively, $(\mathrm{Spec}(k[t_{11},\dots, t_{nn}, D^{-1}]), \mu, \varepsilon, \gamma)$ is an affine group scheme over k. We call this *the general linear group of order n* and denote it by GL_n. The neutral point e of GL_n corresponds to the maximal ideal of $k[t_{11},\dots, t_{nn}, D^{-1}]$ generated by $\{t_{ij}-\delta_{ij} | 1\leq i, j\leq n\}$. Therefore if we put $s_{ij}=t_{ij}-\delta_{ij}$ for $i, j= 1, 2,\dots, n$, $\{s_{ij} | 1\leq i, j\leq n\}$ is a regular system of parameters of the stalk $\mathcal{O}$ of GL_n at e.

Denote by $M_n(k)$ the ring of all the square matrices (α_{ij}) of size n with α_{ij} in k and by $\rho(x)$ the element $(\langle x, t_{ij}\rangle)$ of $M_n(k)$ for any element x in $\mathfrak{H}(GL_n)$. Then ρ is a k-linear map of $\mathfrak{H}(GL_n)$ to $M_n(k)$, which we call *the canonical representation of $\mathfrak{H}(GL_n)$ to $M_n(k)$ with respect to $\{t_{ij}\}$*.

PROPOSITION 13. *The canonical representation ρ of $\mathfrak{H}(GL_n)$ to $M_n(k)$ with respect to $\{t_{ij}\}$ is a ring homomorphism.*

PROOF. If Δ is the comultiplication of $k[t_{11},\dots, t_{nn}, D^{-1}]$, we have $\Delta(t_{ij}) = \sum_h t_{ih}\otimes t_{hj}$ and hence

$$\langle xy, t_{ij}\rangle = \langle x\otimes y, \Delta(t_{ij})\rangle = \sum_h \langle x, t_{ih}\rangle \langle y, t_{hj}\rangle$$

for any x and y in $\mathfrak{H}(GL_n)$. Therefore we see $\rho(xy)=\rho(x)\rho(y)$, and hence ρ is a ring homomorphism. q.e.d.

Now let V be a vector space of dimension n over k and let $GL(V)$ be the group of linear automorphisms of V. Let us fix a basis $\{v_1,\dots, v_n\}$ for V over k and a coordinate system $\{t_{ij}\}$ of $GL_n=\mathrm{Spec}(k[t_{11},\dots, t_{nn}, D^{-1}])$. If l is an element of $GL(V)$ such that $l(v)_i= \sum_{j=1}^{n} \lambda_{ij}v_j$ for $1\leq i\leq n$, we may identify l with the closed point of GL_n corresponding to the maximal ideal of $k[t_{11},\dots, t_{nn}, D^{-1}]$ generated by $\{t_{ij}-\lambda_{ij} | 1\leq i, j\leq n\}$. We denote this identification between $GL(V)$ and $GL_n(k)=\mathrm{Mor}(\mathrm{Spec}(k), GL_n)$ by $id(t_{ij}, v_i)$. If $\{w_1,\dots, w_n\}$ is another basis for

V over k, we denote by A the matrix (α_{ij}) in $M_n(k)$ such that $w_i = \sum_{j=1}^{n} \alpha_{ij} v_j$ $(1 \leq i \leq n)$. If we put $(t'_{ij}) = A(t_{ij})A^{-1}$, we see $\{t'_{ij}\}$ is another coordinate system of GL_n, i.e., we have $k[t_{11}, \ldots, t_{nn}, D^{-1}] = k[t'_{11}, \ldots, t'_{nn}, D^{-1}]$. Furthermore it is easy to see $id(t_{ij}, v_i) = id(t'_{ij}, w_i)$. We understand by *the group scheme GL_V of linear automorphisms of V* the group scheme GL_n with an identification $id(t_{ij}, v_i)$.

Let G be a group scheme over k. Then we say a homomorphism ϕ of G to $GL_V = (GL_n, id(t_{ij}, v_i))$ as group schemes to be *a rational representation of G in V*. We say also that *G acts rationally on V by ϕ*. Thus for any closed point x in G $\phi(x)$ is a linear automorphism of V. Now we fix a basis $\{v_1, \ldots, v_n\}$ for V over k and identify $M_n(k)$ with the $\mathrm{End}_k(V)$ of linear endomorphisms of V using $\{v_i\}$, i.e., we identify $A = (\alpha_{ij})$ of $M_n(k)$ with l of $\mathrm{End}_k(V)$ such that $l(v_i) = \sum_{j=1}^{n} \alpha_{ij} v_j$ $(1 \leq i \leq n)$. If ρ is the canonical representation of $\mathfrak{H}(GL_n)$ to $M_n(k)$ with respect to $\{t_{ij}\}$, we put $x_\phi(v) = \rho(\phi_*(x))(v)$ for any x in $\mathfrak{H}(G)$ and any v in V, where ϕ_* is the tangential homomorphism attached to ϕ. Then we have the following

LEMMA 14. *The notations being as above, we have*
(i) $(\alpha x + \alpha' x')_\phi(v) = \alpha x_\phi(v) + \alpha' x'_\phi(v)$,
(ii) $(xx')_\phi(v) = x_\phi(x'_\phi(v))$,
(iii) $x_\phi(\alpha v + \alpha' v') = \alpha x_\phi(v) + \alpha' x_\phi(v')$ *and*
(iv) $1_\phi(v) = v$,

where $\alpha, \alpha' \in k$, $x, x' \in \mathfrak{H}(G)$ and $v, v' \in V$. Moreover $x_\phi(v)$ depends only on ϕ and is independent of the choice of a basis $\{v_1, \ldots, v_n\}$ for V over k.

The proof of this lemma is easy and we omit it.

PROPOSITION 14. *Let V be a vector space of dimension n over k, and let U and W be vector subspaces of V such that $U \supset W$. Fixing a basis $\{v_1, \ldots, v_n\}$ for V over k, identify $M_n(k)$ with the ring $\mathrm{End}_k(V)$ of all linear endomorphisms of V using $\{v_i\}$. Then there exists a unique connected and reduced group subscheme H of $GL_n = \mathrm{Spec}(k[t_{11}, \ldots, t_{nn}, D^{-1}])$ satisfying the following conditions:*
(i) If ρ is the canonical representation of $\mathfrak{H}(GL_n)$ to $M_n(k)$ with respect to $\{t_{ij}\}$, we have $\rho(\mathfrak{H}(H)^\circ) = \{A \in M_n(k) | A(U) \subset W\}$ where $\mathfrak{H}(H)^\circ$ is the kernel of the coidentity of $\mathfrak{H}(H)$.
(ii) Let D be any Hopf subalgebra of $\mathfrak{H}(GL_n)$ such that $\rho(D^\circ) \subset \rho(\mathfrak{H}(H)^\circ)$ where D° is the kernel of the coidentity of D. Then D is contained in $\mathfrak{H}(H)$. In particular if H' is any connected group subscheme of GL_n such that $\rho(\mathfrak{H}(H')^\circ) \subset \{A \in M_n(k) | A(U) \subset W\}$, H' is group subscheme of H.
(iii) The subset of $GL(V)$ corresponding to the subset $H(k)$ of $GL_n(k)$ under the identification $id(t_{ij}, v_i)$ consists of the elements l of $GL(V)$ such that $l(W) = W$,

$l(U)=U$ and the induced linear endomorphism $\bar{l}$ of U/W by l is the identity map of U/W.

PROOF. By Lemma 14, we may assume that the subsets $\{v_1,\dots, v_r\}$ and $\{v_1, \dots, v_s\}$ are bases for W and U over k respectively, replacing $\{v_i\}$ by another one if necessary. Let l be an element of $GL(V)$ and let A_l be the matrix in $M_n(k)$ corresponding to l with respect to $\{v_i\}$. Denote by S the subset of $GL(V)$ consisting of the elements l such that A_l is the form $\begin{pmatrix} A & 0 & 0 \\ B & E & 0 \\ C & D & F \end{pmatrix}$, where $A \in M_r(k)$, $F \in M_{n-s}(k)$ and $E=$ *the unit matrix in* $M_{s-r}(k)$. Then it is easy to see that S is the set of the elements l in $GL(V)$ such that $l(U)=U$, $l(W)=W$ and $\{$*the induced linear endomorphism* $\bar{l}$ *of* U/W *by* $l\}=1_{U/W}$. Moreover S is a subgroup of $GL(V)$ and the corresponding subset T of $GL_n(k)$ to S under the identification $id(t_{ij}, v_i)$ is a closed subset of $GL_n(k)$. Therefore there is a reduced group subscheme H of GL_n such that $H(k)=T$. We shall show that H has also the properties (i) and (ii) of our proposition. It is easy to see that H is connected. Let $\mathcal{O}$ be the stalk of GL_n at the neutral point e of GL_n and put $s_{ij}=t_{ij}-\delta_{ij}$ $(1 \leq i, j \leq n)$ as before. Let $\mathfrak{a}$ be the ideal of $\mathcal{O}$ generated by $\{s_{ij} | i \leq s \text{ and } j \leq r+1\}$. Then it is easy to see that $\mathfrak{a}$ is the defining ideal of H in $\mathcal{O}$. The image $\{\bar{s}_{ij} | i \geq s+1 \text{ or } j \leq r\}$ of $\{s_{ij} | i \geq s+1 \text{ or } j \leq r\}$ by the canonical homomorphism of $\mathcal{O}$ to $\mathcal{O}/\mathfrak{a}$ is regular system of parameters of $\mathcal{O}/\mathfrak{a}$. Let $\{l_{a_{11}\cdots a_{ij}\cdots a_{nn}} | a_{ij} \geq 0\}$ be the canonical basis for $\mathfrak{H}(GL_n)$ over k with respect to $\{s_{11},\dots, s_{nn}\}$. Then we see easily that $\{l_{a_{11}\cdots a_{nn}} | a_{ij} \geq 0;\ a_{uv}=0$ *for* $u \leq s$ *and* $v \geq r+1\}$ is a basis for the subspace $\mathfrak{H}(H)$ of $\mathfrak{H}(GL_n)$ over k from the above and the definition of $\{l_{a_{11}\cdots a_{nn}}\}$. Now an element A of $M_n(k)$ satisfies $A(U) \subset W$ if and only if A has the form (α_{ij}), where $\alpha_{ij}=0$ for $i \leq s$ and $j \geq r+1$. Therefore we see $\rho(\mathfrak{H}(H)^{\circ})=\{A \in M_n(k) | A(U) \subset W\}$, because $\rho(l_{0\cdots 0 \overset{ij}{\check{1}} 0\cdots 0})$ is the matrix (α_{uv}) such that $\alpha_{ij}=1$ and $\alpha_{uv}=0$ for $(u, v) \neq (i, j)$. Lastly let D be any Hopf subalgebra of $\mathfrak{H}(GL_n)$ such that $\rho(D^{\circ}) \subset \rho(\mathfrak{H}(H))$. If $x=\sum_{(a)} \alpha_{a_{11}\cdots a_{nn}} \cdot l_{a_{11}\cdots a_{nn}}$ is in D°, we have $0=\langle x, s_{ij}\rangle=\alpha_{0\cdots 0 \overset{ij}{\check{1}} 0\cdots 0}$ for $i \leq s$ and $j \geq r+1$. This means that D is a Hopf subalgebra of $\mathfrak{H}(H)$ by Prop. 12. In particular if H' is a connected group subscheme of GL_n such that $\rho(\mathfrak{H}(H')^{\circ}) \subset \rho(\mathfrak{H}(H)^{\circ})$, we see that H' is a group subscheme of H by Cor. to Prop. 12. In fact the defining ideal of H in $\mathcal{O}$ is generated by $\{s_{ij} | i \leq s \text{ and } j \geq r+1\}$. q.e.d.

Let G be a group scheme over k and V a vector space of dimension n over k. Fixing a basis $\{v_i\}$ for V over k and a coordinate system $\{t_{ij}\}$ of GL_n, let ϕ be a rational representation of G to $GL_V=(GL_n, id(t_{ij}, v_i))$ and let ρ be the canonical representation of $\mathfrak{H}(GL_n)$ to $M_n(k)$ with respect to $\{t_{ij}\}$. If U and W are subspaces of V such that $U \supset W$, we denote by $\mathrm{Tr}_V(U, W)$ the subspace of $\mathrm{End}_k(V)$ consisting of l such that $l(U) \subset W$. We call $\mathrm{Tr}_V(U, W)$ *the transporter of* U *to* W *in* $\mathrm{End}_k(V)$. Then we have the following

THEOREM 4. *The notations being as above, let us identify* $M_n(k)$ *with* $\mathrm{End}_k(V)$ *using* $\{v_i\}$. *Then there exists a unique connected group subscheme* H *of* G *satisfying the following*:
(i) $\rho(\phi_*(\mathfrak{H}(H)^\circ)$ *is contained in* $\mathrm{Tr}_V(U, W)$, *where* ϕ_* *is the tangential homomorphism attached to* ϕ.
(ii) *If* H' *is any connected group subscheme of* G *such that* $\rho(\phi_*(\mathfrak{H}(H')^\circ))$ $\subset \mathrm{Tr}_V(U, W)$, *then* H' *is a group subscheme of* H.
(iii) *If* D *is any Hopf subalgebra of* $\mathfrak{H}(G)$ *such that* $\rho(\phi^*(D^\circ)) \subset \mathrm{Tr}_V(U, W)$, *then* D *is a Hopf subalgebra of* $\mathfrak{H}(H)$.

PROOF. Denoting by $\mathrm{Tr}_{GL_n}(U, W)$ the group subscheme of GL_n satisfying the conditions of Prop. 14, we have $\rho(\mathfrak{H}(\mathrm{Tr}_{GL_n}(U, W))^\circ) = \mathrm{Tr}_V(U, W)$. Then let H be the inverse image of $\mathrm{Tr}_{GL_n}(U, W)$ in G by ϕ. By Prop. 9, (ii) we see $\phi_*(\mathfrak{H}(H)^\circ) \subset \mathfrak{H}(\mathrm{Tr}_{GL_n}(U, W))^\circ$ and hence $\rho(\phi_*(\mathfrak{H}(H)^\circ) \subset \mathrm{Tr}_V(U, W)$. If H' is any connected group subscheme of G such that $\rho(\phi_*(\mathfrak{H}(H')^\circ)) \subset \mathrm{Tr}_V(U, W)$, we see $\phi_*(\mathfrak{H}(H')) \subset \mathfrak{H}(\mathrm{Tr}_{GL_n}(U, W))$. In fact $\phi_*(\mathfrak{H}(H'))$ is an algebraic Hopf subalgebra of $\mathfrak{H}(GL_n)$ corresponding to the direct image H_1 of H' by ϕ as seen from Prop. 9, (i). This means that $\phi_*(\mathfrak{H}(H')) = \mathfrak{H}(H_1)$ is contained in $\mathfrak{H}(\mathrm{Tr}_{GL_n}(U, W))$, because H_1 is a group subscheme of $\mathrm{Tr}_{GL_n}(U, W)$ by Prop. 14, (ii). Therefore H' is a group subscheme of H by Prop. 9, (ii). Similarly if D is any Hopf subalgebra of $\mathfrak{H}(G)$ such that $\rho(\phi_*(D^\circ)) \subset \mathrm{Tr}_V(U, W)$, $\phi_*(D)$ is a Hopf subalgebra of $\mathfrak{H}(\mathrm{Tr}_{GL_n}(U, W))$ by Prop. 14, (ii). Since we have $\mathfrak{H}(H) = h\text{-}\phi_*^{-1}(\mathfrak{H}(\mathrm{Tr}_{GL_n}(U, W))$ by Prop. 9, (ii), D is a Hopf subalgebra of $\mathfrak{H}(H)$ by Prop. 2. q.e.d.

We call the group subscheme H of G in Th. 4 *the transporter of* U *to* W *in* G *defined by* ϕ and denote it by $\mathrm{Tr}_{G,\phi}(U, W)$. In particular if $U = W$, we call $\mathrm{Tr}_{G,\phi}(U, U)$ *the normalizer of* U *in* G *defined by* ϕ and denote it by $N_{G,\phi}(U)$.

Since the image of the group $G(k)$ of the closed points of G by ϕ is contained in $GL_n(k) = GL(V)$, V may be considered as a $G(k)$-module. On the other hand we see $\rho(\phi_*(\mathfrak{H}(G))) \subset \mathrm{Tr}_V(V, V) = M_n(k)$ by Th. 4, (i) and hence V has the structure of an $\mathfrak{H}(G)$-module by Lemma 14 considering $\mathfrak{H}(G)$ as a k-algebra. We say that this structure of V as an $\mathfrak{H}(G)$-module is *the* $\mathfrak{H}(G)$*-structure of* V *attached to* ϕ. It is clear that a subspace W of V is an $\mathfrak{H}(G)$-submodule of V if and only if G coincides with $N_{G,\phi}(W)$.

PROPOSITION 15. *Let the notations be as above, and let* W *be a subspace of* V. *Assume that* G *is connected. Then if* W *is an* $\mathfrak{H}(G)$*-submodule,* W *is a* $G(k)$*-submodule. Conversely if* W *is a* $G(k)$*-submodule,* W *is an* $\mathfrak{H}(G_{\mathrm{red}})$*-submodule, where the* $\mathfrak{H}(G_{\mathrm{red}})$*-module structure of* V *is attached to the composite morphism of the natural immersion of* G_{red} *to* G *and* ϕ.

PROOF. If W is an $\mathfrak{H}(G)$-submodule, we have $G = N_{G,\phi}(W)$. Then $\phi(G(k))$

consists of elements l of $GL(V)$ satisfying $l(W)=W$ by Prop. 14, (iii). This means that W is a $G(k)$-submodule of V. Conversely assume that W is a $G(k)$-submodule of V. Then $\phi(G(k))$ is contained in $N_{GL_n,1_{GL_n}}(W)(k)$. Since $N_{G,\phi}(W)$ is the inverse image of $N_{GL_n,1_{GL_n}}(W)$ in G by ϕ as seen in the proof of Th. 4, we have $G(k)\subset N_{G,\phi}(W)(k)$ by Prop. 8 and hence $G(k)=N_{G,\phi}(W)(k)$. Therefore G_{red} is a group subscheme of $N_{G,\phi}(W)$, because they have the same underlying space. This means that W is an $\mathfrak{H}(G_{\text{red}})$-submodule of V. q.e.d.

COROLLARY. *In Prop.* 15 *we assume that* G *is reduced. Then* W *is a* $G(k)$*-submodule of* V *if and only if it is an* $\mathfrak{H}(G)$*-submodule of* V.

Later we need the following

LEMMA 15. *Let* G *be a group scheme over* k *and let* V *be a vector space of dimension* n *over* k. *Let* U *and* W *be subspaces of* V *such that* $U\supset W$. *Let* $\{v_1,\dots, v_n\}$ *be a basis for* V *over* k *such that* $\{v_1,\dots, v_r\}$ *and* $\{v_1,\dots, v_s\}$ *are bases for* W *and* U *over* k *respectively. Let* ϕ *be a rational representation of* G *to* GL_V $=(GL_n, id(t_{ij}, v_i))$ *and denote by* ψ *the linear map of* V *to* $k[t_{11},\dots, t_{nn}, D^{-1}]$ $\otimes_k V$ *given by* $\psi(v_i)=\sum_j t_{ij}\otimes v_j$ *and by* ϕ^* *the comorphism of* $k[t_{11},\dots, t_{nn}, D^{-1}]$ *to the stalk* $\mathcal{O}$ *of* G *at the neutral point* e *defined by* ϕ. *Let* $\mathfrak{a}$ *be the defining ideal of* $\mathrm{Tr}_{G,\phi}(U, W)$ *in* $\mathcal{O}$ *and let* π *be the natural homomorphism of* $\mathcal{O}$ *to* $\mathcal{O}/\mathfrak{a}$. *If* g *is the homomorphism of* V *to* $\mathcal{O}/\mathfrak{a}\otimes_k V$ *given by* $(\pi\phi^*\otimes 1_V)\psi$, *we have* $g(U)$ $\subset\mathcal{O}/\mathfrak{a}\otimes_k U$ *and* $g(W)\subset\mathcal{O}/\mathfrak{a}\otimes_k W$. *Further the induced map* $\bar{g}$ *of* U/W *to* $\mathcal{O}/\mathfrak{a}\otimes_k U/W$ *given by* g *satisfies* $\bar{g}(\bar{v})=1\otimes\bar{v}$ *for any element* $\bar{v}$ *of* U/W.

PROOF. We use the same notations as Prop. 14, Th. 4 and their proofs. Let $\mathfrak{a}_0$ be the ideal of $k[t_{11},\dots, t_{nn}, D^{-1}]$ generated by $\{s_{ij}|i\leq s \text{ and } j\geq r+1\}$. Then we see $\psi(U)\subset k[t_{11},\dots, t_{nn}, D^{-1}]\otimes U+\mathfrak{a}_0\otimes V$ and $\psi(W)\subset k[t_{11},\dots, t_{nn}, D^{-1}]\otimes W+\mathfrak{a}_0\otimes V$. Furthermore we have $\psi(v_i)\equiv 1\otimes v_i$ (mod $\mathfrak{a}_0\otimes V+k[t_{11},\dots, t_{nn}, D^{-1}]\otimes W$) for $r+1\leq i\leq s$, since $t_{ii}=s_{ii}+1$. On the other hand we see from the proof of Th. 4 that $\mathfrak{a}_0$ is mapped into $\mathfrak{a}$ by ϕ^*, because $\mathfrak{a}_0$ is the defining ideal of $\mathrm{Tr}_{GL_n}(U, W)$ in $k[t_{11},\dots, t_{nn}, D^{-1}]$. Therefore we see easily that our assertions are true. q.e.d.

§6. Adjoint representations of group schemes

In the following let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k, and let ϕ_G be the morphism given in §1. If x is any point of G, we denote by ϕ_x^* the comorphism of the stalk $\mathcal{O}$ of G at the neutral point e of G to the stalk $\mathcal{O}_{x\times e}$ of $G\times G$ at $x\times e$ obtained from ϕ_G. First we need some lemmas.

LEMMA 16. *If* $\mathfrak{m}$ *is the maximal ideal of* $\mathcal{O}$, *the image* $\phi_x^*(\mathfrak{m}^s)$ *of* $\mathfrak{m}^s$ *by* ϕ_x^* *is contained in* $(\mathcal{O}_x\otimes\mathfrak{m}^s)\mathcal{O}_{x\times e}$ *for any positive integer* s.

PROOF. Since (e, k) is a normal subscheme of G, we see $\phi_G(x\times e)=e$ for any point x in G, and we have $\phi_G(1_G\times\varepsilon)=\varepsilon\pi'$, where π' is the structure morphism of $G\times\mathrm{Spec}(k)$ to $\mathrm{Spec}(k)$. Let ε^* (resp. π'^* and $(1_G\times\varepsilon)^*$) be the comorphism of $\mathcal{O}$ to k (resp. k to $\mathcal{O}_x\otimes k$ and $\mathcal{O}_{x\times e}$ to $O_x\otimes k$) defined by ε (resp. π' and $1_G\times\varepsilon$). Then we have $\pi'^*\varepsilon^*=(1_G\times\varepsilon)^*\phi_x^*$. Since we see $\varepsilon^*(\mathfrak{m})=0$ and $\ker(1_G\times\varepsilon)^*=(\mathcal{O}_x\otimes\mathfrak{m})\mathcal{O}_{x\times e}$, $\phi_x^*(\mathfrak{m})$ is contained in $(\mathcal{O}_x\otimes\mathfrak{m})\mathcal{O}_{x\times e}$. Since ϕ_x^* is a ring homomorphism, we see easily from this that $\phi_x^*(\mathfrak{m}^s)$ is contained in $(\mathcal{O}_x\otimes\mathfrak{m}^s)O_{x\times e}$ for any positive integer s. q.e.d.

LEMMA 17. *Let G, $\mathcal{O}$, $\mathcal{O}_{x\times e}$ and $\mathfrak{m}$ be as above. Then $\mathcal{O}_{x\times e}/(\mathcal{O}_x\otimes\mathfrak{m}^s)\mathcal{O}_{x\times e}$ is canonically isomorphic to $\mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ for any positive integer s.*

PROOF. Since $\mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ is an integral extension of $\mathcal{O}_x\otimes_k k\cong\mathcal{O}_x$, any maximal ideal $\mathfrak{n}$ of $\mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ contains $\mathfrak{m}_x\otimes_k k\cong\mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$. On the other hand $\mathfrak{m}/\mathfrak{m}^s=k\otimes_k(\mathfrak{m}/\mathfrak{m}^s)$ is the unique prime ideal of $\mathcal{O}/\mathfrak{m}^s=k\otimes_k(\mathcal{O}/\mathfrak{m}^s)$. Therefore $\mathfrak{n}$ contains $k\otimes_k(\mathfrak{m}/\mathfrak{m}^s)$. This means that $\mathcal{O}\otimes(\mathfrak{m}/\mathfrak{m}^s)+\mathfrak{m}\otimes(\mathcal{O}/\mathfrak{m}^s)$ is the unique maximal ideal of $\mathcal{O}\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ and hence $\mathcal{O}\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ is a local ring. If we put $T=\mathcal{O}_x\otimes_k\mathcal{O}-(\mathcal{O}_x\otimes\mathfrak{m}+\mathfrak{m}_x\otimes\mathcal{O})$, $\mathcal{O}_{x\times e}/(\mathcal{O}x\otimes\mathfrak{m}^s)\mathcal{O}_{x\times e}$ is isomorphic to $(\mathcal{O}_x\otimes_k\mathcal{O}/\mathcal{O}_x\otimes\mathfrak{m}^s)_T$. However since $\mathcal{O}_x\otimes\mathcal{O}/\mathcal{O}_x\otimes\mathfrak{m}^s\cong\mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ is local, we have $(\mathcal{O}_x\otimes\mathcal{O}/\mathcal{O}_x\otimes\mathfrak{m}^s)_T=\mathcal{O}_x\otimes\mathcal{O}/\mathcal{O}_x\otimes\mathfrak{m}^s\cong\mathcal{O}\otimes_k(\mathcal{O}/\mathfrak{m}^s)$. q.e.d.

LEMMA 18. *Let ϕ_G and μ be as above and let L_x (resp. R_x) be the left (resp. right) translation of G for any closed point x in G. Then we have $\phi_G(\mu\times 1_G)=\phi_G(1_G\times\phi_G)$ and $L_xR_{x^{-1}}=\phi_G(x\pi_G\times 1_G)\Delta_G$.*

PROOF. Let p_i be the projection of $G\times G\times G$ to the i-th factor for $i=1, 2, 3$. Then we can see easily

$$\begin{aligned}1_{G\times G\times G} &= (p_1\times p_2\times p_3)(\Delta_{G\times G\times G}\times 1_{G\times G\times G})\Delta_{G\times G\times G}\\ &= (p_1\times p_2\times p_3)(1_{G\times G\times G}\times\Delta_{G\times G\times G})\Delta_{G\times G\times G}, \quad\text{and}\\ \phi_G(\mu\times 1_G) &= \phi_G(\mu\times 1_G)(p_1\times p_2\times p_3)(\Delta_{G\times G\times G}\times 1_{G\times G\times G})\Delta_{G\times G\times G}\\ &= \phi_G((p_1*p_2)\times p_3)\Delta_{G\times G\times G} = (p_1*p_2)*p_3*(p_1*p_2)^{-1}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\phi_G(1_G\times\phi_G) &= \phi_G(1_G\times\phi_G)(p_1\times p_2\times p_3)(1_{G\times G\times G}\times\Delta_{G\times G\times G})\Delta_{G\times G\times G}\\ &= \phi_G(p_1\times(p_2*p_3*p_2^{-1}))\Delta_{G\times G\times G}\\ &= p_1*(p_2*p_3*p_2^{-1})*p_1^{-1} = (p_1*p_2)*p_3*(p_1*p_2)^{-1}.\end{aligned}$$

Therefore we have the first equality. Next let x be any closed point of G which we identify with an element in $\mathrm{Mor}(\mathrm{Spec}(k), G)$. Then we see easily $L_xR_{x^{-1}}$

$=(x\pi_G)*1_G*(x\pi_G)^{-1}$. On the other hand we have in the same way as above

$$(x\pi_G)*1_G*(x\pi_G)^{-1} = \phi_G(x\pi_G \times 1_G)\Delta_G.$$

This means $L_x R_{x^{-1}} = \phi_G(x\pi_G \times 1_G)\Delta_G$. q.e.d.

LEMMA 19. *Let G, μ, ϕ_G, $\mathcal{O}$ and $\mathfrak{m}$ be as above. Let μ^*_{xy} be the comorphism of the stalk $\mathcal{O}_{\mu(x\times y)}$ of G to $\mathcal{O}_{x\times y}$ of $G\times G$ defined by μ for any closed points x and y in G. Then there exists a k-homomorphism $f_z^{(s)}$ of $\mathcal{O}/\mathfrak{m}^s$ to $\mathcal{O}_z\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ for any positive integer s and any closed point z in G such that the diagram*

$$\begin{array}{ccc} \mathcal{O}/\mathfrak{m}^s & \xrightarrow{f_x^{(s)}} & \mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s) \\ {\scriptstyle f^{(s)}_{\mu(x\times y)}}\big\downarrow & & \big\downarrow{\scriptstyle (h\otimes 1_{\mathcal{O}/\mathfrak{m}^s})(1_{\mathcal{O}_x}\otimes f_y^{(s)})} \\ \mathcal{O}_{\mu(x\times y)}\otimes_k(\mathcal{O}/\mathfrak{m}^s) & \xrightarrow[\mu^*_{xy}\otimes 1_{\mathcal{O}/\mathfrak{m}^s}]{} & \mathcal{O}_{x\times y}\otimes_k(\mathcal{O}/\mathfrak{m}^s) \end{array}$$

is commutative, where h is the natural homomorphism of $\mathcal{O}_x\otimes_k\mathcal{O}_y$ to $\mathcal{O}_{x\times y}$.

PROOF. By Lemma 16 ϕ_x^* gives a homomorphism $g_x^{(s)}$ of $\mathcal{O}/\mathfrak{m}^s$ to $\mathcal{O}_{x\times e}/(\mathcal{O}_x\otimes\mathfrak{m}^s)\mathcal{O}_{x\times e}$, which is isomorphic to $\mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ by Lemma 17. Therefore we obtain a homomorphism $f_x^{(s)}$ of $\mathcal{O}/\mathfrak{m}^s$ to $\mathcal{O}_x\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ from $g_x^{(s)}$. On the other hand we have the following commutative diagram from Lemma 18:

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\phi_x^*} & \mathcal{O}_{x\times e} \\ {\scriptstyle \phi^*_{\mu(x\times y)}}\big\downarrow & & \big\downarrow{\scriptstyle (1_G\times\phi_G)^*} \\ \mathcal{O}_{\mu(x\times y)\times e} & \xrightarrow[(\mu\times 1_G)^*]{} & \mathcal{O}_{x\times y\times e}\,. \end{array}$$

Since we see $(1_G\times\phi_G)^*((\mathcal{O}_x\otimes\mathfrak{m}^s)\mathcal{O}_{x\times e})\subset(\mathcal{O}_x\otimes\mathcal{O}_y\otimes\mathfrak{m}^s)\mathcal{O}_{x\times y\times e}$ and $(\mu\times 1_G)^*\cdot((\mathcal{O}_{\mu(x\times y)}\otimes\mathfrak{m}^s)\mathcal{O}_{\mu(x\times y)\times e})\subset((\mathcal{O}_{x\times y}\otimes\mathfrak{m}^s)\mathcal{O}_{x\times y\times e}$, this commutative diagram gives the one in our lemma. q.e.d.

LEMMA 20. *Let G, ϕ_G and $f_x^{(s)}$ be as above. Let $\{e_1,\dots, e_n\}$ be a basis for $\mathcal{O}/\mathfrak{m}^s$ over k. Then there exist n^2 global sections a_{ij} $(1\leq i, j\leq n)$ of the structure sheaf $\mathcal{O}_G$ satisfying the followings:*

(i) If $a_{ij,x}$ is the image of a_{ij} in the stalk $\mathcal{O}_x$ of G at x, we have $f_x^{(s)}(e_i)=\sum_{j=1}^{n} a_{ij,x}\otimes e_j$ for each $i=1, 2,\dots, n$.

(ii) Let $\Gamma(G)$ and $\Gamma(G\times G)$ be the rings of the global sections of $\mathcal{O}_G$ and $\mathcal{O}_{G\times G}$ respectively, and let ψ be the canonical homomorphism of $\Gamma(G)\otimes_k\Gamma(G)$ to $\Gamma(G\times G)$. Then if μ^* is the comorphism of $\Gamma(G)$ to $\Gamma(G\times G)$ defined by μ, we have $\mu^*(a_{ij})=\psi(\sum_{h=1}^{n} a_{ih}\otimes a_{hj})$ for $1\leq i, j\leq n$.

PROOF. Let x be any closed point of G and let $U=\mathrm{Spec}(A)$ be an affine open neighborhood of $x\times e$ in $G\times G$. Let $V=\mathrm{Spec}(B)$ be an affine open neighborhood of e in G such that $\phi_G(U)\subset V$, and denote by ϕ^* the comorphism of B to A defined by ϕ_G. If $\mathfrak{a}$ and $\mathfrak{b}$ are the defining ideals of closed subschemes $(G\times\mathrm{Spec}(\mathcal{O}/\mathfrak{m}^s))\cap U$ of U and $\mathrm{Spec}(\mathcal{O}/\mathfrak{m}^s)$ of V in A and B respectively, we see $\phi^*(\mathfrak{b})\subset\mathfrak{a}$ in the same way as the proof of Lemma 16. On the other hand we see easily that $B/\mathfrak{b}$ is isomorphic to $\mathcal{O}/\mathfrak{m}^s$ and that $A/\mathfrak{a}$ is isomorphic to $A'\otimes_k(\mathcal{O}/\mathfrak{m}^s)$, where $U'=\mathrm{Spec}(A')$ is an affine open neighborhood of x in G. Let $g_{U'}^{(s)}$ be the homomorphism of $B/\mathfrak{b}$ to $A/\mathfrak{a}$ obtained from ϕ^* and let $f_{U'}^{(s)}$ be the one of $\mathcal{O}/\mathfrak{m}^s$ to $A'\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ given by $g_{U'}^{(s)}$ identifying $B/\mathfrak{b}$ and $A/\mathfrak{a}$ with $\mathcal{O}/\mathfrak{m}^s$ and $A'\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ respectively. Then if we denote by h_y the natural homomorphism of A' to $\mathcal{O}_y$ for any closed point y in U', it is easy to see $f_y^{(s)}=(h_y\otimes 1_{\mathcal{O}/\mathfrak{m}^s})f_{U'}^{(s)}$ from the definitions of $f_y^{(s)}$ and $f_{U'}^{(s)}$. Therefore if we put $f_{U'}^{(s)}(e_i)=\sum_j a_{ij,U'}\otimes e_j$, we see $f_y^{(s)}(e_i)=\sum_j h_y(a_{ij,U'})\otimes e_j$ for any closed point y in U'. Since $h_y(a_{ij,U'})$ is independent of the choice of V and U, there exists a global section a_{ij} of $\mathcal{O}_G$ such that the restriction of a_{ij} to U' coincides with $a_{ij,U'}$. This means that the assertion (i) holds true. To see (ii), it is sufficient to show $\mu_{xy}^*(a_{ij,\mu(x\times y)})=\sum_h a_{ih,x}\otimes a_{hj,y}$, where μ_{xy}^* is the comorphism of $\mathcal{O}_{\mu(x\times y)}$ to $\mathcal{O}_{x\times y}$ defined by μ. By Lemma 19, we have for $1\le i\le n$

$$(\mu_{xy}^*\otimes 1_{\mathcal{O}/\mathfrak{m}^s})f_{\mu(y\times y)}^{(s)}(e_i)=(h\otimes 1_{\mathcal{O}/\mathfrak{m}^s})(1_{\mathcal{O}_x}\otimes f_y^{(s)})f_x^{(s)}(e_i)$$

and hence

$$\sum_j \mu_{xy}^*(a_{ij,\mu(x\times y)})\otimes e_j=\sum_{u,v} a_{iu,x}\otimes a_{uv,y}\otimes e_v.$$

Comparing the coefficients of e_j in both sides, we have

$$\mu_{xy}^*(a_{ij,\mu(x\times y)})=\sum_h a_{ih,x}\otimes a_{hj,y}. \qquad \text{q.e.d.}$$

LEMMA 21. *Let X be an algebraic scheme over k and let Y be an affine algebraic scheme* $\mathrm{Spec}(A)$ *over k. If Γ is the ring of global sections of the structure sheaf $\mathcal{O}_X$ of X, there is a natural bijection between* $\mathrm{Mor}(X, Y)$ *and* $\mathrm{Hom}_{k\text{-}alg}(A, \Gamma)$.

For the proof of this see Mumford [7], Chap. II, §2, Th. 1.

THEOREM 5. *Let G, $\mathcal{O}$, $\mathfrak{m}$ and $f_x^{(s)}$ be as above. Then G acts on $\mathcal{O}/\mathfrak{m}^s$ rationally by a representation ρ_s such that $\rho_s(x)=\overline{f_x^{(s)}}$ for any closed point x in G considering them as linear transformations of $\mathcal{O}/\mathfrak{m}^s$, where $\overline{f_x^{(s)}}$ is the linear transformation of $\mathcal{O}/\mathfrak{m}^s$ obtained from $f_x^{(s)}$ naturally.*

PROOF. Let $\{e_1,\dots, e_n\}$ be a basis for $\mathcal{O}/\mathfrak{m}^s$ over k, and let $a_{ij}(1\le i, j\le n)$ be global sections of $\mathcal{O}_G$ satisfying the conditions of Lemma 20. First we show that $\det(a_{ij})$ is an invertible element in the ring $\Gamma(G)$ of the global sections of $\mathcal{O}_G$. To see this it is sufficient to show that the image $\det(a_{ij,x})$ of $\det(a_{ij})$ in $\mathcal{O}_x$ is invertible for any closed point x in G. As seen in the proof of Lemma 20 we have $\mu^*_{xx^{-1}}(a_{ij,e})=\sum_h a_{ih,x}\otimes a_{hj,x^{-1}}$. This means $a_{ij,e}(e)=\sum_h a_{ih,x}(x)a_{hj,x^{-1}}(x^{-1})$ and hence $\det(a_{ij,e}(e))=\det(a_{ij,x}(x))\times\det(a_{ij,x^{-1}}(x^{-1}))$, where $a_{ij,x}(x)$ is the residue class of $a_{ij,x}$ in $\mathcal{O}_x$ modulo the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$. Therefore it is sufficient to show that $\det(a_{ij,e}(e))$ is the unit element of k. Thus let g be the canonical homomorphism of $\mathcal{O}_{e\times e}$ to $\mathcal{O}$ which is isomorphic to $k\otimes_k\mathcal{O}\cong\mathcal{O}_{e\times e}/(\mathfrak{m}\otimes\mathcal{O})\mathcal{O}_{e\times e}$. By Lemma 18 we see $1_G=\phi_G(\varepsilon\pi_G\times 1_G)\Delta_G$. It is easy to see that g is the comorphism of $\mathcal{O}_{e\times e}$ to $\mathcal{O}$ defined by $(\varepsilon\pi_G\times 1_G)\Delta_G$. Therefore the comorphism $g\phi^*_e$ of $\mathcal{O}$ to itself defined by $\phi_G(\varepsilon\pi_G\times 1_G)\Delta_G$ is identity $1_{\mathcal{O}}$. This means that $(a_{ij,e}(e))$ is the unit matrix E_n of $M_n(k)$ as seen easily from the definitions of $f^{(s)}_e$ and $a_{ij,}e$.

Now we identify $GL(\mathcal{O}/\mathfrak{m}^s)$ with $GL_n(k)$ by a coordinate system $\{t_{ij}\}$ of GL_n and a basis $\{e_1,\dots,e_n\}$ for $\mathcal{O}/\mathfrak{m}^s$ over k as in § 5. Let ρ^*_s be the k-algebra homomorphism of $k[t_{11},\dots,t_{nn},D^{-1}]$ to $\Gamma(G)$ such that $\rho^*_s(t_{ij})=a_{ij}$ and $\rho^*_s(D)=\det(a_{ij})$, where $D=\det(t_{ij})$. Then, by Lemma 21, there exists a unique morphism ρ_s of G to $GL_n=\operatorname{Spec}(k[t_{ij},D^{-1}])$ such that ρ^*_s is the comorphism of $k[t_{ij},D^{-1}]$ to $\Gamma(G)$ defined by ρ_s. Let μ^*, ψ and $\Gamma(G\times G)$ be as in Lemma 20 and let μ^*_n be the comorphism of $k[t_{ij},D^{-1}]$ to $k[t_{ij},D^{-1}]\otimes_k k[t_{ij},D^{-1}]$ defined by the multiplication μ_n of GL_n. Since $\mu^*_n(t_{ij})=\sum_h t_{ih}\otimes t_{hj}$ for $1\le i,j\le n$, we see $\mu^*\rho^*_s=\psi(\rho^*_s\otimes\rho^*_s)\mu^*_n$ by Lemma 20. On the other hand we see $\psi(\rho^*_s\otimes\rho^*_s)$ is the comorphism of $k[t_{ij},D^{-1}]\otimes k[t_{ij},D^{-1}]$ to $\Gamma(G\times G)$ defined by $\rho_s\times\rho_s$. Therefore we have $\rho_s\mu=\mu_n(\rho_s\times\rho_s)$ again by Lemma 21, and hence ρ_s is a homomorphism of G to GL_n as group schemes. This means that $(\rho_s, id(t_{ij},e_i))$ is a rational representation of G to $GL_{\mathcal{O}/\mathfrak{m}^s}$. The equality $\rho_s(x)=\overline{f^{(s)}_x}$ follows easily from the definition of ρ_s. q.e.d.

Let $(G,\mu,\varepsilon,\gamma)$, $\mathcal{O}$, $\mathfrak{m}$ and ϕ_G be as above. Then it is easy to see that the representation ρ_s of G to $GL_{\mathcal{O}/\mathfrak{m}^s}$ given in the above theorem is determined independently of the choice of the basis $\{e_1,\dots,e_n\}$ for $\mathcal{O}/\mathfrak{m}^s$. We call ρ_s *the adjoint representation of G of degree s* and denote it by Ad_s. If we denote by $\pi_{ss'}$ the canonical homomorphism of $\mathcal{O}/\mathfrak{m}^s$ to $\mathcal{O}/\mathfrak{m}^{s'}$, we see easily

$$\mathrm{Ad}_{s'}(x)\pi_{ss'}(v)=\pi_{ss'}\mathrm{Ad}_x(x)(v)$$

for any closed point x in G and any element v in $\mathcal{O}/\mathfrak{m}^s$.

PROPOSITION 16. *Let G, $\mathcal{O}$, $\mathfrak{m}$ and Ad_s be as above. Let $\{e_1,\dots,e_n\}$ be a basis for $\mathcal{O}/\mathfrak{m}^s$ over k and identify $GL_n=\operatorname{Spec}(k[t_{11},\dots,t_{nn},D^{-1}])$ with $GL_{\mathcal{O}/\mathfrak{m}^s}$ by $id(t_{ij},e_i)$. Then if $f^{(s)}_e$ and $a_{ij,e}$ $(1\le i,j\le n)$ are as in Lemma 20, the $\mathfrak{H}(G)$-structure of $\mathcal{O}/\mathfrak{m}^s$ attached to Ad_s is given by the matrix $(\langle x,a_{ij,e}\rangle)$ for any element x in $\mathfrak{H}(G)$, identifying $M_n(k)$ with $\mathrm{End}_k(\mathcal{O}/\mathfrak{m}^s)$ with respect to $\{e_i\}$.*

PROOF. By the definition in § 5 the $\mathfrak{H}(G)$-structure of $\mathcal{O}/\mathfrak{m}^s$ attached to Ad_s is given by the matrix $(<\mathrm{Ad}_{s*}(x), t_{ij}>)$ for any element x in $\mathfrak{H}(G)$, where Ad_{s*} is the tangential homomorphism of $\mathfrak{H}(G)$ to $\mathfrak{H}(GL_n)$ defined by Ad_s. Since Ad_{s*} is the transpose of the comorphism Ad_s^* of $\mathcal{O}_{GL_n,e}$ to $\mathcal{O}_{G,e}$ defined by Ad_s, we see

$$<\mathrm{Ad}_{s*}(x), t_{ij}> = <x, \mathrm{Ad}_s^*(t_{ij})> = <x, a_{ij,e}> .$$

This means that our assertion is true. q.e.d.

PROPOSITION 17. *Let the notations be as above. Let x be an element of $\mathfrak{H}(G)$ and put $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$, where Δ is the comultiplication of $\mathfrak{H}(G)$. Then if x' is an element of the dual space $(\mathcal{O}/\mathfrak{m}^s)^*$ of $\mathcal{O}/\mathfrak{m}^s$, we have the followings:*

(*i*) *If c is the antipode of $\mathfrak{H}(G)$, $\sum_{(x)} x_{(1)}x'c(x_{(2)})$ is contained in $(\mathcal{O}/\mathfrak{m}^s)^*$.*

(*ii*) $<\sum_{(x)} x_{(1)}x'c(x_{(2)}), e_i> = \sum_j <x, a_{ij,e}> <x', e_j> .$

PROOF. By the definition of ϕ_G, we see that the tangential homomorphism ϕ_{G*} is given by $\mu_*(\mu_*\otimes 1_{\mathfrak{H}(G)})(1_{\mathfrak{H}(G)}\otimes 1_{\mathfrak{H}(G)}\otimes c)(1_{\mathfrak{H}(G)}\otimes S_*)(\Delta\otimes 1_{\mathfrak{H}(G)})$. This means that $\phi_{G*}(x\times y)=\sum_{(x)} x_{(1)}yc(x_{(2)})$ for any x and y in $\mathfrak{H}(G)$. On the other hand since the homomorphism $f_e^{(s)}$ of $\mathcal{O}/\mathfrak{m}^s$ to $\mathcal{O}\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ is obtained naturally from the comorphism ϕ_G^* of $\mathcal{O}$ to $\mathcal{O}_{e\times e}$, the transpose ϕ_s of $f_e^{(s)}$ is the restriction of ϕ_{G*} to $(\mathcal{O}\otimes_k(\mathcal{O}/\mathfrak{m}^s))^c=\mathcal{O}^c\otimes_k(\mathcal{O}/\mathfrak{m}^s)^c=\mathfrak{H}(G)\otimes(\mathcal{O}/\mathfrak{m}^s)^*$, whose image is contained in $(\mathcal{O}/\mathfrak{m}^s)^c=(\mathcal{O}/\mathfrak{m}^s)^*$. Therefore we see the first assertion. As to the second we see

$$\begin{aligned} <\sum_{(x)} x_{(1)}x'c(x_{(2)}), e_i> &= <\phi_{G*}(x\otimes x'), e_i> \\ &= <\phi_s(x\otimes x'), e_i> \\ &= <x\otimes x', f_e^{(s)}(e_i)> \\ &= \sum_j <x, a_{ij,e}> <x', e_j>. \end{aligned}$$

q.e.d.

THEOREM 6. *Let G be a group scheme over k, and let U and W be subspaces of $\mathfrak{H}(G)$ such that $U\supset W$. Then there exists a connected group subscheme H of G satisfying the following conditions:*

(*i*) *If c and Δ are the antipode and the comultiplication of $\mathfrak{H}(G)$ respectively, $\sum_{(x)} x_{(1)}Uc(x_{(2)})$ is contained in W for any element x in the kernel $\mathfrak{H}(H)^\circ$ of the coidentity of $\mathfrak{H}(H)$, where $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$.*

(*ii*) *If D is any Hopf subalgebra of $\mathfrak{H}(G)$ such that $\sum_{(x)} x_{(1)}Uc(x_{(2)})\subset W$ for any element x in the kernel D° of the coidentity of D with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$, D is contained in $\mathfrak{H}(H)$.*

PROOF. Put $U_s=U\cap(\mathcal{O}/\mathfrak{m}^s)^*$ and $W_s=W\cap(\mathcal{O}/\mathfrak{m}^s)^*$ for any positive integer

s, where $(\mathcal{O}/\mathfrak{m}^s)^*$ is the dual space of $(\mathcal{O}/\mathfrak{m}^s)$ and is identified naturally with a subspace of $\mathfrak{H}(G)$. Let T_s and V_s be the null spaces $U_s^\perp$ and $W_s^\perp$ of U_s and W_s in $\mathcal{O}/\mathfrak{m}^s$ respectively. Then we see $U_s=T_s^\perp$ and $W_s=V_s^\perp$. Now fix an s and let $\{e_1,\dots, e_n\}$ be a basis for $\mathcal{O}/\mathfrak{m}^s$ over k such that the subsets $\{e_1,\dots, e_l\}$ and $\{e_1,\dots, e_m\}$ are bases for T_s and V_s respectively. If x is an element of $\mathfrak{H}(G)$ with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$, $\sum_{(x)} x_{(1)}x'c(x_{(2)})$ is contained in $(\mathcal{O}/\mathfrak{m}^s)^*$ for any element x' in $(\mathcal{O}/\mathfrak{m}^s)^*$ by Prop. 17. Therefore $\sum_{(x)} x_{(1)}x'c(x_{(2)})$ is in W if and only if we have $<\sum_{(x)} x_{(1)}x'c(x_{(2)}),\ e_i> =0$ *for* $0\leq i\leq m$. From Prop. 17, (ii) and the equality $U_s^\perp=T_s=ke_1+\cdots+ke_l$, we see that $\sum_{(x)} x_{(1)}U_sc(x_{(2)})\subset W_s$ if and only if $\sum_{j=l+1}^{n} <x,\ a_{ij,e}>\ <x',\ e_j> =0$ for any x' in U_s and $1\leq i\leq m$, using the notations in Prop. 17. Since the set of the vectors $\{(<x', e_{l+1}>,\dots, <x', e_n>)|x'\in U_s\}$ coincides with the full space k^{n-l} as seen easily, the last condition is equivalent to $<x, a_{ij,e}> =0$ for $1\leq i\leq m$ and $l+1\leq j\leq n$, i.e., $(<x, a_{ij,e}>)\in \mathrm{Tr}_{\mathcal{O}/\mathfrak{m}^s}(V_s, T_s)$, identifying $M_n(k)$ with $\mathrm{End}_k(\mathcal{O}/\mathfrak{m}^s)$ with respect to $\{e_i\}$. This means by Prop. 16 that x maps V_s into T_s considering $\mathcal{O}/\mathfrak{m}^s$ as an $\mathfrak{H}(G)$-module by Ad_s if and only if we have $\sum_{(x)} x_{(1)}U_sc(x_{(2)})\subset W_s$ with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. Therefore if H_s is the transporter $\mathrm{Tr}_G(V_s, T_s)$ of V_s to T_s in G given by Ad_s, we see by Th. 4 that $\mathfrak{H}(H_s)$ is the largest Hopf subalgebra D of $\mathfrak{H}(G)$ such that x maps U_s to W_s for any element x in D°. In other words $\mathfrak{H}(H_s)$ is the largest Hopf subalgebra D of $H(G)$ satisfying $\sum_{(x)} x_{(1)}U_sc(x_{(2)})\subset W_s$ for any element x in D°. Now if we put $H=\underset{s\geq 0}{I}(H_s)$, we have $\mathfrak{H}(H)=\bigcap_{s\geq 0}\mathfrak{H}(H_s)$ as seen in §4. Since we have $U=\bigcup_s U_s$ and $W=\bigcup_s W_s$, we see easily from the above that $\mathfrak{H}(H)$ is the largest Hopf subalgebra D of $\mathfrak{H}(G)$ satisfying $\sum_{(x)} x_{(1)}Uc(x_{(2)})\subset W$ for any x in D°. q.e.d.

We call the connected group subscheme H of G given in Th. 6 *the transporter of U to W in G by the adjoint representations* and denote it by $\mathrm{Tr}_{\mathrm{Ad}}(U, W)$. In particular if $U=W$, we put $\mathrm{N}_{\mathrm{Ad}}(U)=\mathrm{Tr}_{\mathrm{Ad}}(U, U)$ and call it *the normalizer of U in G by the adjoint representations.*

§7. Normalizers of Hopf subalgebras and group subschemes

Let $(B, m, i, \Delta, \varepsilon, c)$ be a cocommutative Hopf algebra over k, and put

$$\phi_B = m(m\otimes 1_B)(1_B\otimes 1_B\otimes c)(1_B\otimes S)(\Delta\otimes 1_B),$$

where S is the exchange of the factors of $B\otimes_k B$. Moreover if $(A, \lambda, \eta, \sigma)$ is a formal group over k, we denote by ϕ_A the transpose of $\phi_{\mathfrak{H}(A)}$. If $\mathfrak{m}$ is the maximal ideal of A, let $\bar{A}'$ be the $(A\otimes\mathfrak{m}+\mathfrak{m}\otimes A)$-adic completion of $A\otimes_k A$. Then ϕ_A is a local homomorphism of $A=\mathfrak{H}(A)^*$ to $\bar{A}'=(\mathfrak{H}(A)\otimes_k\mathfrak{H}(A))^*$ as seen easily.

Denoting by $\mathrm{Hom}_{coal}(C_1, C_2)$ the set of coalgebra homomorphisms of a cocommutative coalgebra C_1 to another C_2, $\mathrm{Hom}_{coal}(C_1, B)$ has a structure of a group with the composition $f*g=m(f\otimes g)\Delta_{C_1}$ for f and g in $\mathrm{Hom}_{coal}(C_1, B)$ where Δ_{C_1} is the comultiplication of C_1. Similarly if we denote by $\mathrm{Hom}_{loc}(R_1, R_2)$ the set of local k-homomorphisms of a local ring R_1 containing k to another R_2, $\mathrm{Hom}_{loc}(A, R_1)$ has a group structure using λ instead of m as seen easily.

PROPOSITION 18. *Let D and E be Hopf subalgebras of a cocommutative Hopf algebra B over k. Then the followings are equivalent:*
(i) $\phi_B(D\times E)$ is contained in E.
(ii) Let C be any cocommutative coalgebra over k. Then $f*g*f^{-1}$ is contained in the subgroup $\mathrm{Hom}_{coal}(C, E)$ of the group $\mathrm{Hom}_{coal}(C, B)$ for any elements f in $\mathrm{Hom}_{coal}(C, D)$ and g in $\mathrm{Hom}_{coal}(C, E)$.

PROOF. (*i*)⇒(*ii*). If Δ_C is the comultiplication of C, we see $f*g*f^{-1}=\phi_B(f\otimes g)\Delta_C$ and hence $(f*g*f^{-1})(x)=\phi_B(f\otimes g)\Delta_C(x)=\sum_{(x)}\phi_B(f(x_{(1)})\otimes g(x_{(2)}))$ for any x in C with $\Delta_C(x)=\sum_{(x)}x_{(1)}\otimes x_{(2)}$. Since $f(x_{(1)})\in D$ and $g(x_{(2)})\in E$, we see $(f*g*f^{-1})(x)\in E$ by the assumption. This means that $f*g*f^{-1}$ is an element of $\mathrm{Hom}_{coal}(C, E)$.
(*ii*)⇒(*i*). Put $C=D\otimes_k E$, and let p_1 and p_2 be the projection of C to D and E as coalgebras respectively. Then p_1 and p_2 are in $\mathrm{Hom}_{coal}(C, D)$ and $\mathrm{Hom}_{coal}(C, E)$ respectively, and hence $p_1*p_2*p_1^{-1}$ is in $\mathrm{Hom}_{coal}(C, E)$ by our assumption. On the other hand we have $(p_1\otimes p_2)\Delta_C=1_C$. Therefore if x and y are elements in D and E respectively, we see $\phi_B(x\otimes y)=\phi_B(p_1\otimes p_2)\Delta_C(x\otimes y)=(p_1*p_2*p_1^{-1})(x\otimes y)$. This means $\phi_B(x\otimes y)\in E$. q.e.d.

PROPOSITION 19. *Let $A/\mathfrak{a}$ and $A/\mathfrak{b}$ be formal subgroups of a formal group A over k. Then the followings are equivalent:*
(i) If ρ is the canonical homomorphism of $\bar{A}'$ to $\bar{A}'/(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$, the kernel of $\rho\phi_A$ contains $\mathfrak{b}$.
(ii) Let R be any complete local ring containing k. Then $f*g*f^{-1}$ is in $\mathrm{Hom}_{loc}(A/\mathfrak{b}, R)$ for any elements f in $\mathrm{Hom}_{loc}(A/\mathfrak{a}, R)$ and g in $\mathrm{Hom}_{loc}(A/\mathfrak{b}, R)$.

This is the dual of Prop. 18 and the proof is the same as above. Therefore we omit the proof.

PROPOSITION 20. *Let H and K be group subschemes of a group scheme G over k. Let i_H and i_K be the canonical immersions of H and K into G respectively. Then the followings are equivalent:*
(i) $\phi_G(i_H\times i_K)$ decomposes through K.
(ii) Let X be any algebraic scheme over k. Then $f*g*f^{-1}$ is contained in $\mathrm{Mor}(X, K)$ for any elements f in $\mathrm{Mor}(X, H)$ and g in $\mathrm{Mor}(X, K)$.

PROOF. $(i)\Rightarrow(ii)$. Since we have $f*g*f^{-1}=\phi_G(i_H\times i_K)(f\times g)\Delta_X$ as seen easily, we see $f*g*f^{-1}\in \mathrm{Mor}(X, K)$ from the assumption (i).
$(ii)\Rightarrow(i)$. Put $X=H\times K$ and let p_1 and p_2 be the projections of X to H and K respectively. Then, using $(p_1\times p_2)\Delta_X=1_X$, we see $\phi_G(i_H\times i_K)=p_1*p_2*p_1^{-1}$. This means that $\phi_G(i_H\times i_K)$ decomposes through K by our assertion (ii). q.e.d.

If H and K are group subschemes of a group scheme G over k satisfying the equivalent conditions in Prop. 20, we say that H *normalizes* K. Similarly we say that *a Hopf subalgebra D of a cocommutative Hopf algebra B over k (resp. a formal subgroup $A/\mathfrak{a}$ of a formal group A over k) normalizes another E (resp.* $A/\mathfrak{b}$) if they satisfy the equivalent conditions in Prop. 18 (resp. Prop. 19). If there exists the largest group subscheme H of G such that H normalizes a group subscheme K of G, we call H *the normalizer* of K *in* G and denote it by $\mathrm{N}_G(K)$. Similarly we define the *normalizers* $\mathrm{N}_B(E)$ and $\mathrm{N}_A(A/\mathfrak{b})$ of a Hopf subalgebra E in B and a formal subgroup $A/\mathfrak{b}$ in A respectively. We see easily that a group subscheme H of G is normal in G if and only if the normalizer of H in G is G itself. Similarly we call a Hopf subalgebra D of B and a formal subgroup $A/\mathfrak{a}$ of A *normal* if $\mathrm{N}_B(D)=B$ and $\mathrm{N}_A(A/\mathfrak{a})=A$ respectively.

PROPOSITION 21. *If E is any Hopf subalgebra of a cocommutative Hopf algebra B over k, there exists the normalizer* $\mathrm{N}_B(E)$ *of E in B.*

PROOF. Let $\mathscr{F}$ be the family of Hopf subalgebras D_λ of B which normalize E. Since $\mathscr{F}$ contains E, $\mathscr{F}$ is not empty. Now let D_1 and D_2 be elements in F and put $D=J(D_1, D_2)$. Then D is also an element of F. In fact if m is the multiplication of B, we see easily $\phi_B(m\otimes 1_B)=\phi_B(1_B\otimes\phi_B)$ in the same way as the proof of Lemma 18. Therefore we have $\phi_B(xx'\otimes y)=\phi_B(x\otimes\phi_B(x'\otimes y))\in E$ for x and x' in D_1+D_2 and for y in E and hence we see easily that $J(D_1, D_2)$ normalizes E repeating similar calculations. Moreover if $\mathscr{F}_0$ is a totally ordered subset of $\mathscr{F}$ with respect to inclusion, $J(D_\lambda)=\bigcup_{D_\lambda\in\mathscr{F}_0} D_\lambda$ belongs to $\mathscr{F}$ as seen easily. Therefore, by Zorn's lemma, there exists a maximal element D in $\mathscr{F}$ which is the largest one in $\mathscr{F}$ from the above. q.e.d.

PROPOSITION 22. *If $A/\mathfrak{b}$ is a formal subgroup of a formal group A over k, there exists the normalizer* $\mathrm{N}_A(A/\mathfrak{b})$ *of $A/\mathfrak{b}$ in A.*

PROOF. Let E be the null space of $\mathfrak{b}$ in $B=\mathfrak{H}(A)$, and put $D=\mathrm{N}_B(E)$. If $\mathfrak{a}$ is the null space of D in $A=B^*=\mathfrak{H}(A)^*$, $A/\mathfrak{a}$ is a formal subgroup of A as seen easily. Then it is easy to see $A/\mathfrak{a}=\mathrm{N}_A(A/\mathfrak{b})$. q.e.d.

Now let G be a group scheme over k, and let $\mathcal{O}$ and $\mathcal{O}'$ be the stalks of G and $G\times G$ at the neutral points e and $e\times e$ respectively. Denote by ϕ_G^* the comorphism of $\mathcal{O}$ to $\mathcal{O}'$ defined by ϕ_G. Then the following proposition gives a similar criterion to Th. 1 that a connected group subscheme H of G normalizes another connected one K.

PROPOSITION 23. *Let G, $\mathcal{O}$, $\mathcal{O}'$, H and K be as above. Denote by $\mathfrak{a}$ and $\mathfrak{b}$ the defining ideals of H and K in $\mathcal{O}$. Then H normalizes K if and only if $\phi_G^*(\mathfrak{b})$ is contained in $(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$.*

PROOF. If H normalizes K, $\phi_G(i_H\times i_K)$ decomposes through K. Therefore it is easy to see $\phi_G^*(\mathfrak{b})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$. Conversely if we have $\phi_G^*(\mathfrak{b})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$, there exist an open subset U of K and an open subset V of $H\times K$ such that $\phi_G\cdot i_V$ is a morphism of V to U. Since $H\times K$ is irreducible, V is dense in $H\times K$. This means that $\phi_G(H\times K)$ is contained in the closure of $\phi_G(V)$, and hence we see $\phi_G(H\times K)\subset K$ as sets. In particular the subgroup $H(k)$ of $G(k)$ normalizes the subgroup $K(k)$. On the other hand let x be the generic point of $H\times K$ and put $y=\phi_G(x)$. Denote by ϕ_x^* the comorphism of the stalk $\mathcal{O}_y$ of G at y to the stalk $\mathcal{O}_x$ of $G\times G$ at x defined by ϕ_G. Then we see $\phi_x^*(\mathfrak{b}\mathcal{O}_y)\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}_x$, because $\mathcal{O}_x$ and $\mathcal{O}_y$ are localizations of $\mathcal{O}'$ and $\mathcal{O}$ respectively. Let a and b be closed points of H and K respectively, and let $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ be affine neighborhoods of $aba^{-1}=\phi_G(a\times b)$ and $a\times b$ in G and $G\times G$ respectively such that $\phi_G(\mathrm{Spec}(S))\subset\mathrm{Spec}(R)$. If $\mathfrak{q}$ and $\mathfrak{q}'$ are the defining ideals of K and $H\times K$ in R and S respectively, they are primary. If $\mathfrak{p}$ and $\mathfrak{p}'$ are the prime ideals of R and S corresponding to y and x respectively, we see $\mathcal{O}_x=S_{\mathfrak{p}'}$, $\mathcal{O}_y=R_{\mathfrak{p}}$, $\mathfrak{b}\mathcal{O}_y=\mathfrak{q}R_{\mathfrak{p}}$ and $(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}_x=\mathfrak{q}'S_{\mathfrak{p}'}$. If we denote by ϕ_S^* the comorphism of R to S defined by ϕ_G, we have $\phi_S^*(\mathfrak{q})\subset S\cap\mathfrak{q}'S_{\mathfrak{p}'}=\mathfrak{q}'$ because of $\phi_x^*(\mathfrak{q}R_{\mathfrak{p}})\subset\mathfrak{q}'S_{\mathfrak{p}'}$. This means that ϕ_G induces a morphism of $(H\times K)\cap\mathrm{Spec}(S)$ to $K\cap\mathrm{Spec}(R)$, and hence we see that $\phi_G(i_H\times i_K)$ decomposes through K. q.e.d.

COROLLARY. *Let G, H, $\mathfrak{a}$, $\mathcal{O}$, $\mathcal{O}'$ and ϕ_G be as above. Then H is a normal group subscheme of G if and only if we have $\phi_G^*(\mathfrak{a})\subset(\mathcal{O}\otimes\mathfrak{a})\mathcal{O}'$.*

PROPOSITION 24. *If K is a connected group subscheme of a group scheme G over k, $\mathrm{N}_{\mathfrak{H}(G)}(\mathfrak{H}(K))$ is an algebraic Hopf subalgebra of $\mathfrak{H}(G)$.*

PROOF. Put $B=\mathfrak{H}(G)$, and let Δ and c be the comultiplication and the antipode of B respectively. Then we have $\phi_B(x\otimes y)=\sum_{(x)}x_{(1)}yc(x_{(2)})$ for x and y in B with $\Delta(x)=\sum_{(x)}x_{(1)}\otimes x_{(2)}$. Therefore if we put $H=\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))$, we see $\phi_B(\mathfrak{H}(H)^\circ\otimes\mathfrak{H}(K))\subset\mathfrak{H}(K)$ by Th. 6 and the definition of $\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))$. If x is contained in the image of the identity of B, we see $\phi_B(x\otimes y)=xy$ for any y in B. Therefore we see $\mathfrak{H}(H)$ is contained in $\mathrm{N}_B(\mathfrak{H}(K))$. Conversely $\mathrm{N}_B(\mathfrak{H}(K))$ is contained in $\mathfrak{H}(H)$ by Th. 6, (ii). This means that $\mathrm{N}_B(\mathfrak{H}(K))$ coincides with $\mathfrak{H}(H)$. q.e.d.

PROPOSITION 25. *Let K be a connected group subscheme of a group* *scheme G over k. Then there exists the largest connected group subscheme H of G which normalizes K. Moreover we have $\mathfrak{H}(H)=\mathrm{N}_{\mathfrak{H}(G)}(\mathfrak{H}(K))$.*

PROOF. Let $\mathcal{O}$, $\mathcal{O}'$, B, Δ, c and ϕ_G^* be as above. If we put $H=\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))$, we have $\mathrm{N}_{\mathfrak{H}(G)}(\mathfrak{H}(K))=\mathfrak{H}(H)$ as seen in the proof of Prop. 24. Therefore we must show that H is the largest connected group subscheme of G which normalizes K. Now put $U_s=\mathfrak{H}(K)\cap(\mathcal{O}/\mathfrak{m}^s)^*$ and let V_s be the null space $U_s^{\perp}$ of U_s in $\mathcal{O}/\mathfrak{m}^s$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Then if $\mathfrak{b}$ is the defining ideal of K in $\mathcal{O}$, we have $V_s=(\mathfrak{b}+\mathfrak{m}^s)/\mathfrak{m}^s$. In fact if A is the $\mathfrak{m}$-adic completion of $\mathcal{O}$, A is the dual space of $B=\mathfrak{H}(G)$. Since $\mathfrak{b}A$ and $\mathfrak{m}^sA$ are the null spaces of $\mathfrak{H}(K)$ and $(\mathcal{O}/\mathfrak{m}^s)^*$ in A, we have $\mathfrak{b}A+\mathfrak{m}^sA=(\mathfrak{H}(K)\cap(\mathcal{O}/\mathfrak{m}^s)^*)^{\perp}$, and hence the null space of $\mathfrak{H}(K)\cap(\mathcal{O}/\mathfrak{m}^s)^*$ in $\mathcal{O}$ coincides with $\mathfrak{b}+\mathfrak{m}^s$. This means that the null space V_s of $U_s=\mathfrak{H}(K)\cap(\mathcal{O}/\mathfrak{m}^s)^*$ in $\mathcal{O}/\mathfrak{m}^s$ is $(\mathfrak{b}+\mathfrak{m}^s)/\mathfrak{m}^s$. Thus we see that $H=\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))$ is the intersection $\underset{s\geq 1}{I}(\mathrm{N}_{G,\mathrm{Ad}_s}(V_s))$ of the normalizers $H_s=\mathrm{N}_{G,\mathrm{Ad}_s}(V_s)$ for $s\geq 1$ as seen from the proof of Th. 6. The proof shows also that $\mathfrak{H}(H_s)$ is the largest Hopf subalgebra D of $\mathfrak{H}(G)$ such that $\sum_{(x)} x_{(1)}V_s c(x_{(2)})\subset V_s$ for any element x in D°. Now we show that H normalizes K. If $\mathfrak{a}_s$ and $\mathfrak{a}$ are the defining ideals of H_s and H in $\mathcal{O}$ respectively, we have $\mathfrak{a}=\bigcup_s \mathfrak{a}_s$ as seen in §4. Let π_s and π be the natural homomorphisms of $\mathcal{O}$ to $\mathcal{O}/\mathfrak{a}_s$ and $\mathcal{O}/\mathfrak{a}$ respectively. If $f^{(s)}=f_e^{(s)}$ is the homomorphism of $\mathcal{O}/\mathfrak{m}^s$ to $\mathcal{O}\otimes_k(\mathcal{O}/\mathfrak{m}^s)$ defined by ϕ_G^* as given in Lemma 19, we see that the homomorphism g given in Lemma 15 coincides with $(\pi_s\otimes 1_{\mathcal{O}/\mathfrak{m}^s})f^{(s)}$ for $V=\mathcal{O}/\mathfrak{m}^s$, $U=W=V_s$ and $\phi=\mathrm{Ad}_s$ by Lemma 20, (i) and the definition of Ad_s. Therefore we have $(\pi_s\otimes 1_{\mathcal{O}/\mathfrak{m}^s})f^s(V_s)\subset(\mathcal{O}/\mathfrak{a}_s)\otimes_k V_s$ by Lemma 15, and hence $(\pi\otimes 1_{\mathcal{O}/\mathfrak{m}^s})f^{(s)}(V_s)\subset(\mathcal{O}/\mathfrak{a})\otimes_k V_s$. Since we have $\phi_G^*(\mathfrak{m}^s)\subset(\mathcal{O}\otimes\mathfrak{m}^s)\mathcal{O}'$ by Lemma 16, this means $\phi_G^*(\mathfrak{b})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes(\mathfrak{b}+\mathfrak{m}^s))\mathcal{O}'$ for any $s>0$ from the definition of $f^{(s)}$. Therefore we see $\phi_G^*(\mathfrak{b})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$, and hence H normalizes K by Prop. 23. Next we show that H is the largest connected group subscheme of G which normalizes K. If N is a connected group subscheme of G normalizing K, we see that $\phi_G(i_N\times i_K)$ decomposes through K. Therefore we have $\phi_{\mathfrak{H}(G)}(\mathfrak{H}(N)\otimes\mathfrak{H}(K))\subset\mathfrak{H}(K)$, because the transpose of $\phi_{\mathfrak{H}(G)}$ coincides with the continuous extension of ϕ_G^* to the $\mathfrak{m}$-adic completion A of $\mathcal{O}$. This means by Th. 6, (ii) that $\mathfrak{H}(N)$ is contained in $\mathfrak{H}(\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))=\mathfrak{H}(H)$, and hence we see that N is a group subscheme of H. q.e.d.

COROLLARY 1. *Let G and K be as above. If H_1 and H_2 are connected group subschemes of G normalizing K, then the join $J(H_1, H_2)$ of H_1 and H_2 normalizes K.*

PROOF. If H is the largest connected group subscheme of G which normalizes K, H_1 and H_2 are group subschemes of H. Therefore $J(H_1, H_2)$ is also a group subscheme of H. But any connected group subscheme of H normalizes K as seen easily from Prop. 23. q.e.d.

COROLLARY 2. *Let H and K be connected group subschemes of a group scheme G over k. Then H normalizes K if and only if $\mathfrak{H}(H)$ normalizes $\mathfrak{H}(K)$.*

PROOF. If H normalizes K, $\mathfrak{H}(H)$ normalizes $\mathfrak{H}(K)$ as seen in the last part of the proof of Prop. 25. Conversely if $\mathfrak{H}(H)$ normalizes $\mathfrak{H}(K)$, $\mathfrak{H}(H)$ is a Hopf subalgebra of $\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))$. But we know by Prop. 25 that $\mathrm{N}_{\mathrm{Ad}}(\mathfrak{H}(K))$ corresponds to the largest connected group subscheme H_0 which normalizes K. Therefore H is a group subscheme of H_0 and hence H normalizes K by Prop. 23. q.e.d.

PROPOSITION 26. *Let G be a group scheme over k, and let D and E be Hopf subalgebras of $\mathfrak{H}(G)$ such that D normalizes E. Then the algebraic hull $\mathscr{A}(D)$ of D normalizes E. In particular $\mathrm{N}_{\mathfrak{H}(G)}(E)$ is algebraic.*[5]

PROOF. We see $\mathrm{N}_{\mathfrak{H}(G)}(E)=\mathfrak{H}(\mathrm{N}_{\mathrm{Ad}}(E))$ in the exactly same way as the proof of Prop. 24 by replacing $\mathfrak{H}(K)$ with E. Therefore D is a Hopf subalgebra of $\mathfrak{H}(\mathrm{N}_{\mathrm{Ad}}(E))$. Since $\mathscr{A}(D)$ is the smallest algebraic Hopf subalgebra of $\mathfrak{H}(G)$ containing D, $\mathscr{A}(D)$ is a Hopf subalgebra of $\mathrm{N}_{\mathfrak{H}(G)}(E)=\mathfrak{H}(\mathrm{N}_{\mathrm{Ad}}(E))$. This means that $\mathscr{A}(D)$ normalizes E. q.e.d.

PROPOSITION 27. *Let G, D and E be as above. Then if D normalizes E, the algebraic hull $\mathscr{A}(D)$ of D normalizes that of E.*[6]

PROOF. Let A be the formalization of G, and let $A/\bar{\mathfrak{a}}$ and $A/\bar{\mathfrak{b}}$ be the formal subgroups of A corresponding to D and E respectively. Let $\mathcal{O}$, $\mathcal{O}'$, ϕ_G and ϕ_G^* be as above, and put $\mathfrak{a}=\mathcal{O}\cap\bar{\mathfrak{a}}$ and $\mathfrak{b}=\mathcal{O}\cap\bar{\mathfrak{b}}$. Then $\mathscr{A}(D)$ and $\mathscr{A}(E)$ correspond to $A/\mathfrak{a}A$ and $A/\mathfrak{b}A$ respectively as seen from the proof of Prop. 5. Since D normalizes E, $A/\bar{\mathfrak{a}}$ normalizes $A/\bar{\mathfrak{b}}$ by duality. This means $\phi_A(\bar{\mathfrak{b}})\subset(\bar{\mathfrak{a}}\otimes A+A\otimes\bar{\mathfrak{b}})\bar{A}'$, where ϕ_A is the formal comorphism defined by ϕ_G from A to the completion $\bar{A}'$ of $\mathcal{O}'$ with respect to the maximal ideal. As in the proof of Prop. 4, we may consider $\mathcal{O}\otimes_k\mathcal{O}$, $A\otimes_k A$ and $\mathcal{O}'$ as subrings of $\bar{A}'$. Then since ϕ_G^* is the restriction of ϕ_A to $\mathcal{O}$, $\phi_G^*(\mathfrak{b})$ is contained in $(\bar{\mathfrak{a}}\otimes A+A\otimes\bar{\mathfrak{b}})\bar{A}'\cap\mathcal{O}'$. Now we assume that D contains E, i.e., $\bar{\mathfrak{b}}$ contains $\bar{\mathfrak{a}}$. In this case a similar argument to the proof of Prop. 4 shows $\phi_G^*(\mathfrak{b})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$ as seen easily. Therefore we see $\phi_A(\mathfrak{b}A)\subset(\mathfrak{a}A\otimes A+A\otimes\mathfrak{b}A)\bar{A}'$, and hence $\mathscr{A}(D)$ normalizes $\mathscr{A}(E)$. In general case we put $D_1=J(D, E)$. Since D and E normalize E, it is easy to see that D_1 normalizes E. From the above case, $\mathscr{A}(D_1)$ normalizes $\mathscr{A}(E)$. Since $\mathscr{A}(D_1)$ contains D, $\mathscr{A}(D)$ is contained in $\mathscr{A}(D_1)$. This means that $\mathscr{A}(D)$ normalizes $\mathscr{A}(E)$. q.e.d.

COROLLARY. *Let G and D be as above. Then if D is a normal Hopf sub-*

5) The fact that $\mathrm{N}_{\mathfrak{H}(G)}(E)$ is algebraic was shown in (3.6.2) of [11].

6) If D and E are reduced Hopf subalgebras of $\mathfrak{H}(G)$ attached to an affine algebraic group G, this result is given in Prop. 6, Chap. IV in [4].

algebra of $\mathfrak{H}(G)$, *so is* $\mathscr{A}(D)$.

§8. Centralizers of Hopf subalgebras and group subschemes

In this section we shall show similar results on centralizers of Hopf subalgebras and group subschemes to those on normalizers of them treated in the previous section. We use the same notations as before.

PROPOSITION 28. *Let D and E be Hopf subalgebras of a cocommutative Hopf algebra* $(B, m, i, \Delta, \varepsilon, c)$ *over k. Then the followings are equivalent*:
(*i*) $xy = yx$ *for any elements x in D and y in E.*
(*ii*) $\varepsilon(x)y = \phi_B(x \otimes y)$ *for any elements x in D and y in E.*
(*ii*)$'$ $\phi_B(x \otimes y) = 0$ *for any elements x in* $D^\circ = \ker \varepsilon \cap D$ *and y in E.*
(*iii*) $\varepsilon(y)x = \phi_B(y \otimes x)$ *for any elements x in D and y in E.*
(*iii*)$'$ $\phi_B(y \otimes x) = 0$ *for any elements x in D and y in* $E^\circ = \ker \varepsilon \cap E$.
(*iv*) *Let C be any cocommutative coalgebra over k. Then we have* $f*g = g*f$ *for any elements f in* $\mathrm{Hom}_{coal}(C, D)$ *and g in* $\mathrm{Hom}_{coal}(C, E)$.

PROOF. (*iv*)⇒(*i*). Put $C = D\otimes_k E$ and let Δ_C be the comultiplication of the coalgebra C. If p_1 and p_2 are the projections of $D\otimes_k E$ to D and E respectively as coalgebras over k, we have $(p_1\otimes p_2)\Delta_C = 1_C$ and $(p_2\otimes p_1)\Delta_C$ coincides with the exchange of the factors of $C = D\otimes_k E$. Since we have $p_1*p_2 = p_2*p_1$ by our assumption, we see for x in D and y in E

$$xy = m(x \otimes y) = m(p_1 \otimes p_2)\Delta_C(x \otimes y) = (p_1*p_2)(x \otimes y)$$
$$= (p_2*p_1)(x \otimes y) = m(p_2 \otimes p_1)\Delta_C(x \otimes y) = m(y \otimes x) = yx.$$

(*i*)⇒(*ii*). If x is an element of D, we may put $\Delta(x) = \sum_{(x)} x_{(1)}\otimes x_{(2)}$ with $x_{(1)}$ and $x_{(2)}$ in D. Since we have $\varepsilon(x) = \sum_{(x)} x_{(1)}c(x_{(2)})$, we see $\phi_B(x\otimes y) = \sum_{(x)} x_{(1)}yc(x_{(2)}) = \sum_{(x)} x_{(1)}c(x_{(2)})y = \varepsilon(x)y$ for any element y in E.
(*ii*)⇒(*iv*). Let z be an element of C with $\Delta_C(z) = \sum_{(z)} z_{(1)}\otimes z_{(2)}$. Then we have $z = \sum_{(z)} \varepsilon_C(z_{(1)})z_{(2)}$ and $\varepsilon(f(z_{(1)})) = \varepsilon_C(z_{(1)})$ for f in $\mathrm{Hom}_{coal}(C, D)$, where ε_C is the coidentity of C. Since we have $f*g*f^{-1} = \phi_B(f\otimes g)\Delta_C$ for f in $\mathrm{Hom}_{coal}(C, D)$ and g in $\mathrm{Hom}_{coal}(C, E)$ as seen in the proof of Prop. 18, we see by (*ii*)

$$f*g*f^{-1}(z) = \phi_B(f\otimes g)\Delta_C(z) = \sum_{(z)} \phi_B(f(z_{(1)}) \otimes g(z_{(2)}))$$
$$= \sum_{(z)} \varepsilon(f(z_{(1)}))g(z_{(2)}) = \sum_{(z)} g(\varepsilon_C(z_{(1)})z_{(2)}) = g(z).$$

This means that the assertion (*iv*) is true.

Similarly we see that (*i*) is equivalent to (*iii*). Since we have $D = i(k)\oplus D^\circ$

(resp. $E=i(k)\oplus E^{\circ}$), we see easily that (*ii*) (resp. (*iii*)) is equivalent to (*ii*)′ (resp. (*iii*)′). q.e.d.

PROPOSITION 29. *Let* $(A, \lambda, \eta, \sigma)$, $\bar{A}'$, ϕ_A, $A/\mathfrak{a}$, $A/\mathfrak{b}$ *and* $B=\mathfrak{H}(A)$ *be the same as in Prop.* 19. *Let* ρ_1 *and* ρ_2 *be the natural homomorphisms of* $\bar{A}'$ *to* $\bar{A}'/(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$ *and* $\bar{A}'/(\mathfrak{b}\otimes A+A\otimes\mathfrak{a})\bar{A}'$ *respectively. Then the followings are equivalent*:

(*i*) *If S is the isomorphism of* $\bar{A}'/(\mathfrak{b}\otimes A+A\otimes\mathfrak{a})\bar{A}'$ *to* $A'/(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})A'$ *given naturally from the exchange of the factors of* $A\otimes_k A$, *we have* $\rho_1\lambda=S\rho_2\lambda$.

(*ii*) *The kernel of* $\rho_1\phi_A$ *contains* $\mathfrak{b}$, *and the induced homomorphism of* $A/\mathfrak{b}$ *to* $\bar{A}'/(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$ *by* $\rho_1\phi_A$ *coincides with the one given naturally from the homomorphism of* $A/\mathfrak{b}$ *to* $A/\mathfrak{a}\otimes A/\mathfrak{b}$ *mapping any* α *in* $A/\mathfrak{b}$ *to* $1\otimes\alpha$.

(*iii*) *The kernel of* $\rho_2\phi_A$ *contains* $\mathfrak{a}$, *and the induced homomorphism of* $A/\mathfrak{a}$ *to* $\bar{A}'/(\mathfrak{b}\otimes A+A\otimes\mathfrak{a})\bar{A}'$ *by* $\rho_2\phi_A$ *coincides with the one given naturally from the homomorphism of* $A/\mathfrak{a}$ *to* $A/\mathfrak{b}\otimes A/\mathfrak{a}$ *mapping any* α *in* $A/\mathfrak{a}$ *to* $1\otimes\alpha$.

(*iv*) *Let R be any complete noetherian local ring containing k. Then we have* $f*g=g*f$ *for any elements f in* $\mathrm{Hom}_{loc}(A/\mathfrak{a}, R)$ *and g in* $\mathrm{Hom}_{loc}(A/\mathfrak{b}, R)$.

This is the dual of Prop. 28 and the proof is exactly similar to that of it. Therefore we omit the proof.

PROPOSITION 30. *Let* $(G, \mu, \varepsilon, \gamma)$, H, K, i_H *and* i_K *be the same as in Prop.* 20. *Then the followings are equivalent*:

(*i*) *If* σ *is the isomorphism of* $H\times K$ *to* $K\times H$ *given by the exchange of the factors, we have* $\mu(i_H\times i_K)=\mu(i_K\times i_H)\sigma$.

(*ii*) $\phi_G(i_H\times i_K)$ *decomposes through K and it coincides with the projection* p_K *of* $H\times K$ *to K*.

(*iii*) $\phi_G(i_K\times i_H)$ *decomposes through H and it coincides with the projection* p'_H *of* $K\times H$ *to H*.

(*iv*) *If X is any algebraic scheme over k, we have* $f*g=g*f$ *for any elements f in* $\mathrm{Mor}(X, H)$ *and g in* $\mathrm{Mor}(X, K)$.

PROOF. (*i*)⇒(*iv*). If σ_X is the exchange of the factors of $X\times X$, we see by (i)

$$\begin{aligned} f*g &= \mu(i_H f\times i_K g)\Delta_X=\mu(i_H\times i_K)(f\times g)\Delta_X \\ &= \mu(i_K\times i_H)\sigma(f\times g)\Delta_X=\mu(i_K\times i_H)(g\times f)\sigma_X\Delta_X \\ &= \mu(i_K\times i_H)(g\times f)\Delta_X=g*f \end{aligned}$$

for any f in $\mathrm{Mor}(X, H)$ and g in $\mathrm{Mor}(X, K)$.

(*iv*)⇒(*i*). A similar way to the verification of (*iv*)⇒(*i*) in the proof of Prop. 28 can be applicable, but we omit the detail.

(ii)⇒*(iv)*. If p_2 is the projection of $X \times X$ to the second factor, we have

$$f*g*f^{-1} = \phi_G(i_H \times i_K)(f \times g)\Delta_X = p_K(f \times g)\Delta_X = g p_2 \Delta_X = g$$

for any f in $\mathrm{Mor}(X, H)$ and any g in $\mathrm{Mor}(X, K)$ by *(ii)*. This means that the assertion (iv) is true.
(iv)⇒*(ii)*. Put $X = H \times K$ and let p_H be the projection of $X = H \times K$ to H. Then we see $(p_H \times p_K)\Delta_X = 1_X$ and hence by (iv)

$$\phi_G(i_H \times i_K) = \phi_G(i_H \times i_K)(p_H \times p_K)\Delta_X = (i_H p_H)*(i_K p_K)*(i_H p_H)^{-1}$$
$$= i_K p_K = p_K.$$

This means that *(ii)* is true.
Similarly we can see that *(iii)* is equivalent to (iv). q.e.d.

If H and K are group subschemes of a group scheme G over k satisfying the equivalent conditions in Prop. 30, we say that *H and K centralize each other or commute with each other.* Similarly we say that *Hopf subalgebras D and E of a cocommutative Hopf algebra B over k (resp. formal subgroups $A/\mathfrak{a}$ and $A/\mathfrak{b}$ of a formal group A over k) centralize each other or commute with each other*, if they satisfy the equivalent conditions in Prop. 28 (resp. Prop. 29). If there exists the largest group subscheme H of G commuting with K, we call H *the centralizer* of K in G and denote it by $\mathrm{C}_G(K)$. Similarly we define *the centralizers* $\mathrm{C}_B(E)$ and $\mathrm{C}_A(A/\mathfrak{b})$ of E in B and $A/\mathfrak{b}$ in A respectively. In particular we call $\mathrm{C}_G(G)$ (resp. $\mathrm{C}_B(B)$ and $\mathrm{C}_A(A)$) *the center* of G (resp. B and A).

PROPOSITION 31. *If B and E are as above, there exists the centralizer $\mathrm{C}_B(E)$ of E in B.*

PROPOSITION 32. *If A and $A/\mathfrak{b}$ are as above, there exists the centralizer $\mathrm{C}_A(A/\mathfrak{b})$ of $A/\mathfrak{b}$ in A.*

These propositions are proved in similar ways to the proofs of Prop. 21 and 22, but we omit the detail.

Now we give a corresponding result to Prop. 23. Let G, H, K and ϕ_G be as above and assume that H and K are connected. Let $\mathcal{O}$, $\mathcal{O}'$ and ϕ_G^* be the same as in § 12. If $\mathfrak{a}$ and $\mathfrak{b}$ are the defining ideals of H and K in $\mathcal{O}$ respectively, we denote by ρ the canonical homomorphism of $\mathcal{O}'$ to $\mathcal{O}'/(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$. Moreover let h be the comorphism of $\mathcal{O}$ to $\mathcal{O}'$ defined by the projection of $G \times G$ to the second factor. Then we have the following

PROPOSITION 33. *Let the notations be as above. Then the followings are equivalent*:

(*i*) *H commutes with K.*
(*ii*) *The kernel of* $\rho\phi_G^*$ *contains* $\mathfrak{b}$, *and the induced homomorphism of* $\mathcal{O}/\mathfrak{b}$ *to* $\mathcal{O}'/(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$ *given by* $\rho\phi_G^*$ *coincides with the one obtained naturally from h.*

PROOF. (*i*)⇒(*ii*). If i_H and i_K are the canonical immersions of H and K into G respectively, ρ is the comorphism of $i_H\times i_K$. This means that the kernel of $\rho\phi_G^*$ contains $\mathfrak{b}$, and that the induced homomorphism of $\mathcal{O}/\mathfrak{b}$ to $\mathcal{O}'/(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$ is given by h from Prop. 30, (ii).
(*ii*)⇒(*i*). If the assertion (ii) is true, there exists an open neighborhood U of $e\times e$ in $G\times G$ such that the restriction of $\phi_G(i_H\times i_K)$ to $U\cap(H\times K)$ coincides with the projection p_K of $H\times K$ to K. Then the induced morphism $(\phi_G(i_H\times i_K))_{\mathrm{red}}$ of $H_{\mathrm{red}}\times K_{\mathrm{red}}$ to K_{red} defined by $\phi_G(i_H\times i_K)$ is equal to $(p_K)_{\mathrm{red}}$ defined by p_K, because $H\times K$ is separated. In particular we have $\phi_G(i_H\times i_K)(x\times y)=\phi_G(x\times y)=y$ for any closed point $x\times y$ in $H\times K$. Let $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ be affine open neighborhoods of $x\times y$ and y in $G\times G$ and G respectively such that $\phi_G(\mathrm{Spec}(R))\subset\mathrm{Spec}(S)$ and $p_2(\mathrm{Spec}(R))\subset\mathrm{Spec}(S)$, where p_2 is the projection of $G\times G$ to the second factor. Let $\mathfrak{c}_R$ and $\mathfrak{b}_S$ be the defining ideals of $H\times K$ and K in R and S respectively. Since H normalizes K by Prop. 23, we see $\phi^*(\mathfrak{b}_S)\subset\mathfrak{c}_R$ denoting by ϕ^* the comorphism of B to A defined by ϕ_G. We see easily that $\mathfrak{c}_R$ and $\mathfrak{b}_S$ are primary and hence that $\mathrm{rad}(\mathfrak{c}_R)=\mathfrak{p}$ and $\mathrm{rad}(\mathfrak{b}_S)=\mathfrak{q}$ are prime ideals. If we put $\mathfrak{p}_1=\mathfrak{p}/\mathfrak{c}_R$ and $\mathfrak{q}_1=\mathfrak{q}/\mathfrak{b}_S$, $(R/\mathfrak{c}_R)_{\mathfrak{p}_1}$ and $(S/\mathfrak{b}_S)_{\mathfrak{q}_1}$ are the stalks of $H\times K$ and K at the generic points respectively. Then the homomorphisms $\bar{\phi}$ and $\bar{p}_K$ of $(S/\mathfrak{b}_S)_{\mathfrak{q}_1}$ to $(R/\mathfrak{c}_R)_{\mathfrak{p}_1}$ given naturally from ϕ^* and the comorphism of p_K respectively are equal to each other by our assumption. Therefore we see easily that the comorphisms of $S/\mathfrak{b}_S$ to $R/\mathfrak{c}_R$ defined by $\phi_G(i_H\times i_K)$ and p_K are equal to each other, because the set of the zero-divisors in $R/\mathfrak{c}_R$ is $\mathfrak{p}_1$. This means that $\phi_G(i_H\times i_K)$ and p_K are the same morphism on $(H\times K)\cap\mathrm{Spec}(R)$, and hence on $H\times K$. By Prop. 30, (ii), H centralizes K. q. e. d.

PROPOSITION 34. *If K is a connected group subscheme of a group scheme G over k,* $C_{\mathfrak{H}(G)}(\mathfrak{H}(K))$ *is an algebraic Hopf subalgebra of* $\mathfrak{H}(G)$.

PROOF. If we put $H=\mathrm{Tr}_{\mathrm{Ad}}(\mathfrak{H}(K),0)$, we see $\phi_{\mathfrak{H}(G)}(\mathfrak{H}(H)^{\circ}\otimes\mathfrak{H}(K))\subset 0$ by Th. 6 and the definition of $\mathrm{Tr}_{\mathrm{Ad}}(\mathfrak{H}(K),0)$ in the same way as the proof of Prop. 24. Therefore we see easily $\phi_{\mathfrak{H}(G)}(x\otimes y)=\varepsilon(x)y$ for any x in $\mathfrak{H}(H)$ and y in $\mathfrak{H}(K)$, and hence $\mathfrak{H}(H)$ is contained in $C_{\mathfrak{H}(G)}(\mathfrak{H}(K))$ by Prop. 28, (ii). Conversely $C_{\mathfrak{H}(G)}(\mathfrak{H}(K))$ is contained in $\mathfrak{H}(H)$ by Th. 6, (ii) and Prop. 28. This means that $C_{\mathfrak{H}(G)}(\mathfrak{H}(K))$ coincides with $\mathfrak{H}(H)$. q. e. d.

PROPOSITION 35. *If K is a connected group subscheme of a group scheme G over k, there exists the largest connected group subscheme H of G which* *centralizes* K. *Moreover we have* $\mathfrak{H}(H)=C_{\mathfrak{H}(G)}(\mathfrak{H}(K))$.

The proof of this proposition can be given in a exactly similar way to the proof of Prop. 25 using Prop. 34 and 33 instead of Prop. 24 and 23. We can see that $H=\mathrm{Tr}_{\mathrm{Ad}}(\mathfrak{H}(K), 0)$ satisfies our conditions, but we omit the detail.

COROLLARY 1. *Let* G *and* K *be as above. If* H_1 *and* H_2 *are connected group subschemes of* G *centralizing* K, *then the join* $J(H_1, H_2)$ *also centralizes* K.

COROLLARY 2. *Let* H *and* K *be connected group subschemes of a group scheme* G *over* k. *Then* H *commutes with* K *if and only if* $\mathfrak{H}(H)$ *commutes with* $\mathfrak{H}(K)$. *In particular a connected group scheme* G *is commutative if and only if* $\mathfrak{H}(G)$ *is commutative.*

The proofs of these corollaries are similar to those of Cor. 1 and 2 to Prop. 25 and hence we omit them.

PROPOSITION 36. *Let* G *be a group scheme over* k, *and let* D *and* E *be Hopf subalgebras of* $\mathfrak{H}(G)$. *Then if* D *commutes with* E, *the algebraic hull* $\mathscr{A}(D)$ *of* D *commutes with that of* E. *In particular* $C_{\mathfrak{H}(G)}(E)$ *is algebraic.*[7]

PROOF. Let the notations be the same as those in the proof of Prop. 27. Then since D commutes with E, $A/\bar{\mathfrak{a}}$ commutes with $A/\bar{\mathfrak{b}}$. In particular $A/\bar{\mathfrak{a}}$ normalizes $A/\bar{\mathfrak{b}}$. Therefore we have $\phi_G^*(\mathfrak{b})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$ as seen in the proof of Prop. 27. Moreover we see easily $(\bar{\mathfrak{a}}\otimes A+A\otimes\bar{\mathfrak{b}})\cap\mathcal{O}'=(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}'$, and hence we see from Prop. 29, (ii) that the condition (ii) of Prop. 33 is satisfied. Therefore we see that $A/\mathfrak{a}A$ commutes with $A/\mathfrak{b}A$. In other words $\mathscr{A}(D)$ commutes with $\mathscr{A}(E)$. The last assertion follows from the above easily by the definition of $C_{\mathfrak{H}(G)}(E)$. q.e.d.

§9. Commutators of Hopf subalgebras and group subschemes

Let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k. If we denote by μ_4 the morphism $\mu(\mu\times 1_G)(\mu\times 1_G\times 1_G)$ of $G\times G\times G\times G$ to G, we put

$$\psi_G = \mu_4(1_G\times S\times 1_G)(1_G\times\gamma\times 1_G\times\gamma)(\Delta_G\times\Delta_G),$$

where S is the exchange of the factors of $G\times G$. Similarly if $(B, m, i, \Delta, \varepsilon, c)$ is a cocommutative Hopf algebra over k, we put

$$\psi_B = m_4(1_B\otimes\sigma\otimes 1_B)(1_B\otimes c\otimes 1_B\otimes c)(\Delta\otimes\Delta),$$

where m_4 and σ are the multiplication of $B\otimes B\otimes B\otimes B$ to B and the exchange of

7) The fact that $C_{\mathfrak{H}(G)}(E)$ is algebraic was shown in (3.6.2) of [11].

the factors of $B\otimes_k B$ respectively. If $(A, \lambda, \eta, \sigma)$ is a formal group over k, we define ψ_A as the transpose of $\psi_{\mathfrak{D}(A)}$ which is a k-linear continuous homomorphism of A to the $(\mathfrak{m}\otimes A+A\otimes\mathfrak{m})$-adic completion $\bar{A}'$ of $A\otimes_k A$ where $\mathfrak{m}$ is the maximal ideal of A. Moreover let ϕ_G, ϕ_B and ϕ_A be the same as in previous sections.

PROPOSITION 37. *If D, E and F are Hopf subalgebras of a cocommutative Hopf algebra $(B, m, i, \Delta, \varepsilon, c)$ over k, the followings are equivalent:*
(i) $\psi_B(D\otimes E)\subset F$.
(ii) If C is any cocommutative coalgebra over k, $[f, g]=f*g*f^{-1}*g^{-1}$ is in $\mathrm{Hom}_{coal}(C, F)$ for any f in $\mathrm{Hom}_{coal}(C, D)$ and any g in $\mathrm{Hom}_{coal}(C, E)$.
(iii) Let i_D, i_E and i_D be the natural injections of D, E and F into B respectively and let p_E be the projection of $D\otimes_k E$ to E as coalgebras. Then there exists an element h in $\mathrm{Hom}_{coal}(D\otimes E, F)$ satisfying $\phi_B(i_D\otimes i_E)=(i_F h)*(i_E p_E)$.

PROOF. *(i)⇒(ii)*. If Δ_C is the comultiplication of C, we have $\Delta g=(g\otimes g)\Delta_C$ for any g in $\mathrm{Hom}_{coal}(C, B)$. Therefore we see by the coassociativity of Δ_C

$$\begin{aligned}\psi_B(f\otimes g)\Delta_C &= m_4(1_B\otimes\sigma\otimes 1_B)(1_B\otimes c\otimes 1_B\otimes c)(\Delta\otimes\Delta)(f\otimes g)\Delta_C\\ &= m(\phi_B\otimes c)(f\otimes g\otimes g)(1_C\otimes\Delta_C)\Delta_C\\ &= m(\phi_B(f\otimes g)\otimes cg)(\Delta_C\otimes 1_C)\Delta_C\\ &= m(f*g*f^{-1}\otimes g^{-1})\Delta_C = [f, g],\end{aligned}$$

since we have $\phi_B(f\otimes g)\Delta_C=f*g*f^{-1}$ and $cg=g^{-1}$. In particular we have $[f, g](x)=\psi_B(f\otimes g)\Delta_C(x)=\sum_{(x)}\psi_B(f(x_{(1)})\otimes g(x_{(2)}))$ for any x in C with $\Delta_C(x)=\sum_{(x)}x_{(1)}\otimes x_{(2)}$. This means that $[f, g]$ is in $\mathrm{Hom}_{coal}(C, F)$ if f and g are in $\mathrm{Hom}_{coal}(C, D)$ and $\mathrm{Hom}_{coal}(C, E)$ respectively.
(ii)⇒(i). Put $C=D\otimes E$, and let p_D and p_E be the projections of C to D and E respectively. Then we see easily $(p_D\otimes p_E)\Delta_C=1_C$, and hence $\psi_B(i_D\otimes i_E)=\psi_B(i_D p_D\otimes i_E p_E)\Delta_C=[i_D p_D, i_E p_E]$. This means $\psi_B(D\otimes E)\subset F$ by the assertion (ii).
(ii)⇒(iii). We see easily $\phi_B(i_D\otimes i_E)=[i_D p_D, i_E p_E]*(i_E p_E)$ using the same notations as above, because we have $\phi_B(i_D\otimes i_E)=\phi_B(i_D p_D\otimes i_E p_E)\Delta_C=(i_D p_D)*(i_E p_E)*(i_D p_D)^{-1}$. Therefore the assertion *(iii)* follows from *(ii)*.
(iii)⇒(ii). If f and g are in $\mathrm{Hom}_{coal}(C, D)$ and $\mathrm{Hom}_{coal}(C, E)$ respectively, we see by *(iii)*

$$\begin{aligned}f*g*f^{-1} &= \phi_B(i_D\otimes i_E)(f\otimes g)\Delta_C = ((i_F h)*(i_E p_E))(f\otimes g)\Delta_C\\ &= (i_F h(f\otimes g)\Delta_C)*((i_E p_E)(f\otimes g)\Delta_C) = (i_F h(f\otimes g)\Delta_C)*g.\end{aligned}$$

Therefore $[f, g]=i_F h(f\otimes g)\Delta_C$ is in $\mathrm{Hom}_{coal}(C, F)$. q.e.d.

PROPOSITION 38. *Let $(B, m, i, \Delta, \varepsilon, c)$, D, E and ψ_B be as above. Then* *there exists the smallest Hopf subalgebra F of B containing* $\psi_B(D\otimes_k E)$.

PROOF. Since B is cocommutative, Δ and c are coalgebra homomorphisms as seen easily. Therefore ψ_B is also a coalgebra homomorphism and hence $\psi_B(D\otimes E)$ is a subcoalgebra of B. Moreover we see $c\psi_B(D\otimes E)=\psi_B(c(D)\otimes c(E))=\psi_B(D\otimes E)$. Then it is easy to see that the subalgebra F of B generated by $\psi_B(D\otimes E)$ is a Hopf subalgebra, and so F is the smallest Hopf subalgebra of B containing $\psi_B(D\otimes E)$. q.e.d.

Dualizing the above propositions, we have the following results on formal groups.

PROPOSITION 39. *If* $A/\mathfrak{a}$, $A/\mathfrak{b}$ *and* $A/\mathfrak{c}$ *are formal subgroups of a formal group* A *over* k, *the followings are equivalent*:

(*i*) $\psi_A(\mathfrak{c})\subset(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$.

(*ii*) *Let* R *be any noetherian complete local ring containing* k, *and let* f *and* g *be elements of* $\mathrm{Hom}_{loc}(A/\mathfrak{a}, R)$ *and* $\mathrm{Hom}_{loc}(A/\mathfrak{b}, R)$ *respectively, where we denote by* $\mathrm{Hom}_{loc}(S, R)$ *the set of all local* k*-homomorphisms of a local ring* S *to* R. *Then* $[f, g]=f*g*f^{-1}*g^{-1}$ *is in* $\mathrm{Hom}_{loc}(A/\mathfrak{c}, R)$.

(*iii*) *Let* ρ *be the natural homomorphism of* $\bar{A}'$ *to* $\bar{A}'/(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$ *and let* $\rho_{\mathfrak{b}}$ *and* $\rho_{\mathfrak{c}}$ *be those of* A *to* $A/\mathfrak{b}$ *and* $A/\mathfrak{c}$ *respectively. Let* $i_{\mathfrak{b}}$ *be the natural homomorphism of* $A/\mathfrak{b}$ *to* $\bar{A}'/(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$ *given by the injection of* A *to* $A\otimes_k A$ *mapping* a *to* $1\otimes a$. *Then we have* $\rho\phi_A=(h\rho_{\mathfrak{c}})*(i_{\mathfrak{b}}\rho_{\mathfrak{b}})$ *for some* h *in* $\mathrm{Hom}_{loc}(A/\mathfrak{c}, R)$.

PROPOSITION 40. *Let* A, $A/\mathfrak{a}$, $A/\mathfrak{b}$, $\bar{A}'$ *and* ψ_A *be as above. Then there exists the smallest formal subgroup* $A/\mathfrak{c}$ *of* A *satisfying* $\psi_A(\mathfrak{c})\subset(\mathfrak{a}\otimes A+A\otimes\mathfrak{b})\bar{A}'$.

Let D and E be Hopf subalgebras of a cocommutative Hopf algebra B over k. Then we denote by $[D, E]$ the Hopf subalgebra F obtained in Prop. 38 and call it *the commutator of D and E*. We see easily $[D, E]=[E, D]$. If $A/\mathfrak{a}$ and $A/\mathfrak{b}$ are formal subgroups of a formal group A over k, we can define similarly *the commutator* $[A/\mathfrak{a}, A/\mathfrak{b}]$ *of* $A/\mathfrak{a}$ *and* $A/\mathfrak{b}$ from Prop. 40. As for commutators of group subschemes we have the following

PROPOSITION 41. *Let* H, K *and* L *be group subschemes of a group scheme* G *over* k, *and let* i_H, i_K *and* i_L *be the natural immersions of* H, K *and* L *into* G *respectively. Then the followings are equivalent*:

(*i*) $\psi_G(i_H\times i_K)$ *decomposes through* L.

(*ii*) *Let* X *be any algebraic scheme over* k, *and let* f *and* g *be elements of* $\mathrm{Mor}(X, H)$ *and* $\mathrm{Mor}(X, K)$ *respectively. Then* $[f, g]=f*g*f^{-1}*g^{-1}$ *is in* $\mathrm{Mor}(X, L)$.

(*iii*) *There exists an element* h *in* $\mathrm{Mor}(H\times K, L)$ *satisfying* $\phi_G(i_H\times i_K)=(i_Lh)*(i_Kp_K)$, *where* p_K *is the projection of* $H\times K$ *to* K.

This is a group scheme version of Prop. 37 and the proof is exactly similar to that of Prop. 37. Therefore we omit the detail.

THEOREM 7. *Let $(G, \mu, \varepsilon, \gamma)$ be a group scheme over k and let D and E be algebraic Hopf subalgebras of $\mathfrak{H}(G)$. Then the commutator $[D, E]$ is also algebraic.*

PROOF. Let $\mathcal{O}$ and $\mathcal{O}_n$ be the stalks of G at e and $G\times\cdots\times G$ (*n times*) at $e\times\cdots\times e$ respectively as in §4, and let Δ_n the comorphism of $\mathcal{O}=\mathcal{O}_1$ to $\mathcal{O}_n$ defined by the multiplication μ_n of $G\times\cdots\times G$ to G for $n\geq 2$. Moreover let H and K be connected group subschemes of G with the defining ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{O}$ respectively such that $D=\mathfrak{H}(H)$ and $E=\mathfrak{H}(K)$. If ψ^* is the comorphism of $\mathcal{O}$ to $\mathcal{O}_2$ defined by ψ_G, we put $\mathfrak{c}_1=\psi^{*-1}((\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}_2)$ and $\mathfrak{c}_2=\psi^{*-1}((\mathfrak{b}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a})\mathcal{O}_2)$. Putting $E=\{(i_1,\dots, i_n)|i_j=1 \text{ or } 2, n\geq 2\}$, we denote by $\mathfrak{d}_{i_1\cdots i_n}$ the ideal $\Delta_n^{-1}((\mathfrak{c}_{i_1}\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O}+\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes\mathfrak{c}_{i_n})\mathcal{O}_n)$ of $\mathcal{O}$ for any $(i_1,\dots, i_n)$ in E. Since the zero ideal of $\mathcal{O}/\mathfrak{a}_{\lambda_1}\otimes\cdots\otimes\mathcal{O}/\mathfrak{a}_{\lambda_s}$ with $\mathfrak{a}_{\lambda_j}=\mathfrak{a}$ or $\mathfrak{b}$ is a primary ideal as seen in the proof of Lemma 11, we see easily the zero ideal of $\mathcal{O}_2/\mathfrak{c}'_{i_1}\otimes\cdots\otimes\mathcal{O}_2/\mathfrak{c}'_{n_n}$ is also primary, where $\mathfrak{c}'_{i_j}=(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}_2$ or $(\mathfrak{b}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a})\mathcal{O}_2$ according to $i_j=1$ or 2. Since $\mathcal{O}/\mathfrak{c}_{i_1}\otimes\cdots\otimes\mathcal{O}/\mathfrak{c}_{i_n}$ is isomorphic to a subring of $\mathcal{O}_2/\mathfrak{c}'_{i_1}\otimes\cdots\otimes\mathcal{O}_2/\mathfrak{c}'_{i_n}$, the zero ideal of $\mathcal{O}/\mathfrak{c}_{i_1}\otimes\cdots\otimes\mathcal{O}/\mathfrak{c}_{i_n}$ is primary, i.e., $\mathfrak{c}_{i_1}\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O}+\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes\mathfrak{c}_{i_n}$ is a primary ideal of $\mathcal{O}\otimes\cdots\otimes\mathcal{O}$ (*n times*). Therefore we can apply Lemma 10 and the same argument as the proof of Lemma 11 shows the existence of an element $(j_1,\dots, j_m)$ in E such that $\mathfrak{d}_{j_1\cdots j_m}$ is contained in $\mathfrak{d}_{i_1\cdots i_n}$ for any $(i_1,\dots, i_n)$ in E. Put $\mathfrak{d}=\mathfrak{d}_{j_1\cdots j_m}$. Replacing $\mathfrak{a}_{i_j}$ and $\mathfrak{c}$ with $\mathfrak{c}_{i_j}$ and $\mathfrak{d}$ respectively in the proof of Th. 2, we see $\Delta_2(\mathfrak{d})\subset(\mathfrak{d}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{d})\mathcal{O}_2$. On the other hand we see easily $\gamma\psi_G=\psi_G S$, where S is the exchange of the factors of $G\times G$, and hence we have $\gamma^*(\mathfrak{c}_1)=\mathfrak{c}_2$ and $\gamma^*(\mathfrak{c}_2)=\mathfrak{c}_1$. This means $\gamma^{*-1}(\mathfrak{d}_{i_1\cdots i_n})=\mathfrak{d}_{i'_n\cdots i'_1}$ with $i'_j=1$ for $i_j=2$ and $i'_j=2$ for $i_j=1$. Then a similar argument to the proof of Th. 2 gives $\gamma^*(\mathfrak{d})=\mathfrak{d}$. Therefore by Th. 1 $\mathfrak{d}$ is the defining ideal of a connected group subscheme L of G in $\mathcal{O}$.

Now if we put $\mathfrak{c}'_{j_1\cdots j_m}=(\mathfrak{c}_{j_1}\otimes\mathcal{O}\otimes\cdots\otimes\mathcal{O}+\cdots+\mathcal{O}\otimes\cdots\otimes\mathcal{O}\otimes\mathfrak{c}_{j_m})\mathcal{O}_m$, Δ_m gives an injection of $\mathcal{O}/\mathfrak{d}$ into $\mathcal{O}_m/\mathfrak{c}_{j_1\cdots j_m}$ as seen easily. Therefore the transpose Δ_m^* of Δ_m maps the continuous dual coalgebra $C_{j_1\cdots j_m}$ of $\mathcal{O}/\mathfrak{c}'_{j\cdots j_m}$ onto the dual coalgebra $\mathfrak{H}(L)$ of $\mathcal{O}/\mathfrak{d}$. Since we see the homomorphism ψ_B of $B\otimes_k B$ to $B=\mathfrak{H}(G)$ is the restriction to $B\otimes_k B$ of the transpose of the comorphism ψ^*, we see easily, from Lemma 12 and the definition of $\mathfrak{d}$, that $\mathfrak{H}(L)$ is contained in $[\mathfrak{H}(H), \mathfrak{H}(K)]=[D, E]$. On the other hand we see $\mathfrak{d}\subset\mathfrak{c}_1\cap\mathfrak{c}_2$ by Lemma 10, and hence $\mathfrak{H}(L)$ contains $[\mathfrak{H}(H), \mathfrak{H}(K)]=[D, E]$ as seen easily from Lemma 12 and the definitions of $\mathfrak{c}_1$ and $\mathfrak{c}_2$. This means $\mathfrak{H}(L)=[D, E]$ and hence $[D, E]$ is algebraic. q.e.d.

Let H and K be connected group subschemes of a group scheme G over k. Then we denote by $[H, K]$ the connected group subscheme L of G satisfying

$\mathfrak{H}(L)=[\mathfrak{H}(H), \mathfrak{H}(K)]$, and we call $[H, K]$ *the commutator of* H *and* K. In particular if $H=K$, we call $[H, H]$ *the commutator of* H. It is easy to see from the proof of Th. 7 that $[H, K]$ is reduced if H and K are so. More generally we can see also that $\exp[H, K]\leq \mathrm{Max}(\exp H, \exp K)$ in the case of a positive characteristic.

PROPOSITION 42. *Let* G, H *and* K *be as above. Then* $[H, K]$ *is the smallest group subscheme* L *of* G *such that* H, K *and* L *satisfy the equivalent conditions in Proposition* 41.

PROOF. Put $L=[H, K]$ and let the notations be as those in the proof of Th. 7. Then $\mathfrak{c}_1$ contains $\mathfrak{d}=\mathfrak{d}_{j_1\cdots j_m}$ by Lemma 10. Since $\mathfrak{c}_1$ is a primary ideal of $\mathcal{O}$, there exists a unique irreducible closed subscheme X of G whose stalk at e is $\mathcal{O}/\mathfrak{c}_1$. On the other hand $H\times K$ is the unique irreducible closed subscheme of $G\times G$ whose stalk at $e\times e$ is $\mathcal{O}_2/(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}_2$ and $\mathfrak{c}_1$ is the inverse image of $(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{b})\mathcal{O}_2$ by ψ^*. Then we can see easily that the morphism $\psi_G(i_H\times i_K)$ decomposes through X. Since L is the unique irreducible closed subscheme of G having the stalk $\mathcal{O}/\mathfrak{d}$ at e, we see X is a subscheme of L. Therefore $\psi_G(i_H\times i_K)$ decomposes through L. Now let L' be a group subscheme of G satisfying the equivalent conditions in Prop. 41. Then we see easily $\mathfrak{H}(L')\supset\psi_{H(G)}(\mathfrak{H}(H)\otimes\mathfrak{H}(K))$, and hence we have $\mathfrak{H}(L')\supset[\mathfrak{H}(H), \mathfrak{H}(K)]=\mathfrak{H}(L)$. This means that L is a group subscheme of L'. q.e.d.

Now we need some results on relations between normal Hopf subalgebras and Hopf quotient algebras of a cocommutative Hopf algebra $(B, m, i, \Delta, \varepsilon, c)$ over k. First we have the following

LEMMA 22. *If* D *is a normal Hopf subalgebra of* B, *we have* $BD^\circ=D^\circ B$ *with* $D^\circ=D\cap(\ker\varepsilon)$. *In particular* BD° *is a Hopf ideal of* B.

PROOF. Let p_1 and p_2 be the projections of $B\otimes_k D$ to B and D as coalgebras respectively. Then there exist σ and τ in $\mathrm{Hom}_{coal}(B\otimes_k D, D)$ satisfying $p_1*p_2=\sigma*p_1$ and $p_2*p_1=p_1*\tau$ by the normality of D. In other words if Δ' is the comultiplication of $B\otimes_k D$, we have $m(p_1\otimes p_2)\Delta'=m(\sigma\otimes p_1)\Delta'$ and $m(p_2\otimes p_1)\Delta'=m(p_1\otimes\tau)\Delta'$. Let x and y be elements of B and D° respectively satisfying $\Delta(x)=\sum_{(x)}x_{(1)}\otimes x_{(2)}$ and $\Delta(y)=\sum_{(y)}y_{(1)}\otimes y_{(2)}$. Since we have $(p_1\otimes p_2)\Delta'=1_{B\otimes D}$ and $\sum_{(y)}\varepsilon(y_{(2)})y_{(1)}=y$, we see

$$\begin{aligned}xy &= m(x\otimes y)=m(p_1\otimes p_2)\Delta'(x\otimes y)=m(\sigma\otimes p_1)\Delta'(x\otimes y)\\ &= m(\sigma\otimes p_1)\Big(\sum_{(x),(y)}x_{(1)}\otimes y_{(1)}\otimes x_{(2)}\otimes y_{(2)}\Big)\\ &= \sum_{(x),(y)}\sigma(x_{(1)}\otimes y_{(1)})\varepsilon(y_{(2)})x_{(2)}\\ &= \sum_{(x)}\sigma(x_{(1)}\otimes y)x_{(2)}.\end{aligned}$$

Since y is in D°, we see easily that $\sigma(x_{(1)}\otimes y)$ is also in D°, and hence the right hand side of the above equality is in $D^\circ B$. Similarly we see $yx=\sum_{(x)}x_{(1)}\tau(x_{(2)}\otimes y)\in BD^\circ$. Therefore we have $BD^\circ=D^\circ B$. In particular BD° is a two sided ideal of B. Now since D° is a coideal of D, we may assume that one of $y_{(1)}$ and $y_{(2)}$ in $\Delta(y)=\sum_{(y)}y_{(1)}\otimes y_{(2)}$ belongs to D°. Then we have $\Delta(xy)=\Delta(x)\Delta(y)=(\sum_{(x)}x_{(1)}\otimes x_{(2)})(\sum_{(y)}y_{(1)}\otimes y_{(2)})=\sum_{(x),(y)}x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)}$, and hence $\Delta(xy)$ is contained in $BD^\circ\otimes B+B\otimes BD^\circ$. Therefore BD° is a coideal of B. Moreover we see $c(BD^\circ)=c(D^\circ)c(B)=D^\circ B=BD^\circ$. This means that BD° is a Hopf ideal of B. q.e.d.

If D is a normal Hopf subalgebra of a cocommutative Hopf algebra B over k, B/BD° is a Hopf quotient algebra of B. We call it *the Hopf quotient algebra of B by D.* We denote it by B/D. If ρ_D is the natural homomorphism of B to $B/D=B/BD^\circ$, ρ_D is a surjective Hopf algebra homomorphism.

LEMMA 23. *Let C and C' be cocommutative coalgebras over k, and let f be a surjective coalgebra homomorphism of C to C'. Then if C is colocal, so is C'.*

PROOF. First assume that C is of finite dimension. Then the dual algebra C^* of C is an artinian local ring containing k. Then transpose f^* of f is an injective k-algebra homomorphism and C^* may be considered as a finite C'^*-module. Therefore C'^* is also an artinian local ring and hence C' is colocal. In general case if C' is not colocal, there exists two minimal subcoalgebras D_1 and D_2 of C'. Let x_1 and x_2 be non-zero elements of D_1 and D_2 respectively, and let y_1 and y_2 be elements of C such that $f(y_1)=x_1$ and $f(y_2)=x_2$. Then there is a finite dimensional subcoalgebra D of C containing y_1 and y_2 as well known. Then $f(D)$ is a subcoalgebra of C' containing x_1 and x_2, and hence $f(D)$ contains D_1 and D_2. However $f(D)$ is colocal as seen in the above, because D is of finite dimension. This is a contradiction. q.e.d.

LEMMA 24. *Let B be a cocommutative Hopf algebra of finite dimension over k, and let D be a normal Hopf subalgebra of B. Then D is the h-kernel of the canonical homomorphism ρ_D of B to B/D.*

This is Lemma 16.0.3 in [10], and we omit the proof.

PROPOSITION 43. *Let B be the Hopf algebra $\mathfrak{H}(A)$ attached to a formal group A over k. If D is a normal Hopf subalgebra of B, the h-kernel of the canonical homomorphism ρ_D of B to B/D is D.*

PROOF. First assume that the characteristic of k is $p>0$. Let $\mathfrak{m}$ be the maximal ideal of A and denote by $\mathfrak{m}^{(n)}$ the ideal of A generated by the elements a^{p^n} with a in $\mathfrak{m}$. Then we see easily $A/\mathfrak{m}^{(n)}$ is a formal subgroup of A. Put B_n

$=\mathfrak{H}(A/\mathfrak{m}^{(n)})$ and consider it as a Hopf subalgebra of B. Since $\bigcap_{n=1}^{\infty}\mathfrak{m}^{(n)}=0$, we see $B=\bigcup_{n=1}^{\infty}B_n$. Moreover B_n has a finite dimension over k for any n, because $\mathfrak{m}^{(n)}$ is an $\mathfrak{m}$-primary ideal. Let D_i be the intersection of D and B_i for each i. Then D_i is a Hopf subalgebra of B_i and it is easy to see that D_i is normal in B_i. If ρ_i is the canonical homomorphism of B_i to B_i/D_i, we see h-ker $\rho_i=D_i$ by Lemma 24. Now let E be the h-kernel of ρ_D. Then since $\rho_D(D)=k\subset B/D$, E contains D. Put $E_i=E\cap B_i$. Since $D=\bigcup_{i=1}^{\infty}D_i$, we see $D^\circ=\bigcup_{i=1}^{\infty}D_i^\circ$ and hence $BD^\circ=\bigcup_{i=1}^{\infty}B_iD_i^\circ$. On the other hand we see $E\subset k\oplus BD^\circ$ from $\rho_D(E)=k$, where we identify k with the image of k by the identity of B. Since $\dim_k E_i<\infty$, it is easily seen that each E_i is contained in $k\oplus B_jD_j^\circ$ for some j depending on i. Therefore E_i is contained in D_j, because $D_j=h$-ker ρ_j is the largest Hopf subalgebra of B_j contained in $k\oplus B_jD_j^\circ$. This means $D\supset D_j\supset E_i$, and hence D contains $E=\bigcup_{i=1}^{\infty}E_i$. In other words $D=E$ is the h-kernel of ρ_D.

Next assume that the characteristic of k is zero. If E is the h-kernel of ρ_D, we see $E\supset D$ as above. Now B, D and E are reduced Hopf algebras. Let $\{l_1,\dots,l_n\}$ be a basis of the space $\mathfrak{L}(B)$ of primitive elements of B over k such that $\{l_r,\dots,l_n\}$ and $\{l_s,\dots,l_n\}$ are bases for $\mathfrak{L}(E)$ and $\mathfrak{L}(D)$ over k respectively. If we put $l_i^{(t)}=l_i^t/t!$, we see $\{l_i^{(t)}|t>0\}$ is a sequence of divided powers of l_i for each i. Then $\{l_1^{(e_1)}\cdots l_n^{(e_n)}|e_i\geq 0\}$ and $\{l_s^{(e_s)}\cdots l_n^{(e_n)}|e_i\geq 0\}$ are bases for B and D over k respectively by Th. 3 in [9]. Then it is easy to see that $\{l_s^{(e_s)}\cdots l_n^{(e_n)}|e_i\geq 0,\ e_s+e_{s+1}+\cdots+e_n>0\}$ is a basis for D° over k. Since D° is a two sided ideal of D, we see easily that $S=\{l_1^{(e_1)}\cdots l_s^{(e_s)}\cdots l_n^{(e_n)}|e_i\geq 0,\ e_s+e_{s+1}+\cdots+e_n>0\}$ is a basis for BD° over k. If $E\supsetneqq D$, we have $r<s$. On the other hand E° is contained in BD°, because $E\subset k\oplus BD^\circ$. In particular $l_r=l_r^{(1)}$ is in BD°. But this is impossible, because $\{l_r^{(1)}\}\cup S$ is linearly independent over k. Therefore we have $E=D$. q. e. d.

COROLLARY. *Let A, $B=\mathfrak{H}(A)$ and D be as above, and let F by any cocommutative coalgebra over k. Then the following sequence of groups is exact*:

$$\{1\}\longrightarrow \mathrm{Hom}_{coal}(F, D)\xrightarrow{i_{D*}}\mathrm{Hom}_{coal}(F, B)\xrightarrow{\rho_{D*}}\mathrm{Hom}_{coal}(F, B/D),$$

where i_{D*} and ρ_{D*} are the group homomorphisms naturally obtained from i_D and ρ_D respectively.

PROOF. Let f be an element of $\mathrm{Hom}_{coal}(F, D)$. Since $f(F)\subset D\subset k\oplus BD^\circ$, we see $\rho_D(f(F))=k$. This means that $\rho_{D*}i_{D*}(f)$ is the neutral element of $\mathrm{Hom}_{coal}(F, B/D)$. Conversely let g be an element of $\mathrm{Hom}_{coal}(F, B)$ such that $\rho_{D*}(g)$ is the neutral element of $\mathrm{Hom}_{coal}(F, B/D)$. Therefore $\rho_D(g(F))$ is equal to k, i.e., $g(F)$ is contained in $k\oplus BD^\circ$. Since D is the maximal subcoalgebra of

B contained in $k\oplus BD^{\circ}$ by Prop. 43 and 2, $g(F)$ is a subcoalgebra of D. This means that g is in the image of i_{D*}. The injectivity of i_{D*} is trivial. q.e.d.

PROPOSITION 44. *Let B be the Hopf algebra $\mathfrak{H}(A)$ attached to a formal group A over k, and let D and E be Hopf subalgebras of B. Let i_D and i_E be the canonical injections of D and E into B respectively, and denote by p_E the projection of $D\otimes_k E$ to E as coalgebras. Then if F is a normal Hopf subalgebra of B, the followings are equivalent:*
(i) $[D, E]\subset F$.
(ii) $\rho_F(\phi_B(i_D\otimes i_E))=\rho_F(i_E p_E)$.

PROOF. *(i)⇒(ii).* From Prop. 37 there is an element h in $\mathrm{Hom}_{coal}(D\otimes E, F)$ such that $\phi_B(i_D\otimes i_E)=(i_F h)*(i_E p_E)$. Therefore we see by Cor. to Prop. 43 $\rho_F(\phi_B(i_D\otimes i_E))=\rho_F((i_F h)*(i_E p_E))=\rho_F(i_E p_E)$.
(ii)⇒(i). From the equality (ii) $\rho_F(\phi_B(i_D\otimes i_E)*(i_E p_E)^{-1})$ is the neutral element of $\mathrm{Hom}_{coal}(D\otimes E, B/D)$. By Cor. to Prop. 43 there exists an element h in $\mathrm{Hom}_{coal}(D\otimes E, F)$ satisfying $\phi_B(i_D\otimes i_E)*(i_E p_E)^{-1}=i_F h$. This means $[D, E]\subset F$ by Prop. 37. q.e.d.

PROPOSITION 45. *Let $(B, m, i, \Delta, \varepsilon, c)$ be the Hopf algebra $\mathfrak{H}(G)$ attached to a group scheme G over k. Let E and F be Hopf subalgebras of B, and assume that F is normal in B. Then there exists the largest Hopf subalgebra D of B such that $[D, E]\subset F$ and $D\subset \mathrm{N}_B(E)$. Moreover this Hopf subalgebra D is algebraic.*

PROOF. Let notations be the same as in Prop. 44. If we put $H=\mathrm{Tr}_{\mathrm{Ad}}(E, BF^{\circ}\cap E)$ and $D=\mathfrak{H}(H)$, we see, by Th. 6, $\sum_{(x)} x_{(1)}Ec(x_{(2)})\subset BF^{\circ}\cap E$ for any x in D° with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. Therefore we see easily for any x in D and y in E

$$(*)\qquad \sum_{(x)} x_{(1)}yc(x_{(2)})-\varepsilon(x)y\in BF^{\circ}\cap E \quad \text{with} \quad \Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}.$$

This means $(\phi_B-i_E p_E)(D\otimes E)\subset BF^{\circ}\cap E$ and hence we see $\rho_F(\phi_B(i_D\otimes i_E))=\rho_F i_E p_E$. Then we have $[D, E]\subset F$ by Prop. 44. The formula $(*)$ shows also $\phi_B(x\otimes y)\in E$ for any x in D and y in E, i.e., D is contained in $\mathrm{N}_B(E)$. Now let D' be a Hopf subalgebra of B satisfying $[D', E]\subset F$ and $D'\subset \mathrm{N}_B(E)$. From Prop. 44 and $[D', E]\subset F$, we see $\sum_{(x)} x_{(1)}yc(x_{(2)})-\varepsilon(x)y\in BF^{\circ}$ for any x in D' and y in E with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. On the other hand we see $\sum_{(x)} x_{(1)}yc(x_{(2)})\in E$ for the above x and y by Prop. 18 and $D'\subset \mathrm{N}_B(E)$, and hence we have $\sum_{(x)} x_{(1)}yc(x_{(2)})\in BF^{\circ}\cap E$ for any x in D'° and y in E with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$. This means by Th. 6 that D' is a Hopf subalgebra of $D=\mathfrak{H}(H)$. q.e.d.

COROLLARY. *Let G, B, E and F be as above. Then if a Hopf subalgebra*

D of B satisfies $[D, E]\subset F$ and $D\subset \mathrm{N}_B(E)$, the algebraic hull $\mathscr{A}(D)$ of D satisfies also $[\mathscr{A}(D), E]\subset F$ and $\mathscr{A}(D)\subset \mathrm{N}_B(E)$.

LEMMA 25. *Let $(B, m, i, \Delta, \varepsilon, c)$ be a cocommutative Hopf algebra over k, and let D and E be normal Hopf subalgebras of B. Then the commutator $[D, E]$ of D and E is also normal in B.*

PROOF. Let ϕ_B and ψ_B be as before. Then we can see easily

$$(*) \qquad \phi_B(1_B\otimes m) = m(\phi_B\otimes\phi_B)(1_B\otimes\sigma\otimes 1_B)(\Delta\otimes 1_B\otimes 1_B) \qquad \text{and}$$

$$(**) \qquad \phi_B(1_B\otimes\psi_B) = \psi_B(\phi_B\otimes\phi_B)(1_B\otimes\sigma\otimes 1_B)(\Delta\otimes 1_B\otimes 1_B)$$

in a similar way to the proof of Lemma 18, where σ is the exchange of the factors of $B\otimes_k B$. From $(**)$ we see

$$\phi_B(B\otimes\psi_B(D\otimes E)) \subset \psi_B(\phi_B(B\otimes D)\otimes\phi_B(B\otimes E)).$$

Since D and E are normal in B, we have $\phi_B(B\otimes D)\subset D$ and $\phi_B(B\otimes E)\subset E$. Therefore we see

$$(***) \qquad \phi_B(B\otimes\psi_B(D\otimes E)) \subset \psi_B(D\otimes E) \subset [D, E].$$

Now $[D, E]$ is the subalgebra of B generated by $\psi_B(D\otimes E)$ as seen in the proof of Prop. 38, we see by $(*)$ and $(***)$

$$\phi_B(B\otimes[D, E]) \subset [D, E].$$

This means that $[D, E]$ is normal in B. q.e.d.

THEOREM 8. *Let B be the Hopf algebra $\mathfrak{H}(G)$ attached to a group scheme G over k, and let D and E be Hopf subalgebras of B. Then we have $\mathscr{A}([D, E])\subset[\mathscr{A}(D), \mathscr{A}(E)]$. Moreover if $D\subset\mathrm{N}_B(E)$ and $E\subset\mathrm{N}_B(D)$, we have $[D, E]=\mathscr{A}([D, E])=[\mathscr{A}(D), \mathscr{A}(E)]$.*

PROOF. Since $D\subset\mathscr{A}(D)$ and $E\subset\mathscr{A}(E)$, we see that $[D, E]$ is contained in $[\mathscr{A}(D), \mathscr{A}(E)]$ which is algebraic by Th. 7. Therefore we see $\mathscr{A}([D, E])\subset[\mathscr{A}(D), \mathscr{A}(E)]$. Let H and K be connected group subschemes of G such that $\mathscr{A}(D)=\mathfrak{H}(H)$ and $\mathscr{A}(E)=\mathfrak{H}(K)$, and put $G_1=J(H, K)$ and $B_1=\mathfrak{H}(G_1)$. If $D\subset\mathrm{N}_B(E)$, we see $\mathscr{A}(D)\subset\mathrm{N}_B(E)$ by Prop. 26. Similarly we have $\mathscr{A}(E)\subset\mathrm{N}_B(E)$ from $E\subset\mathrm{N}_B(E)$. On the other hand we see $B_1=\mathfrak{H}(G_1)=J(\mathfrak{H}(H), \mathfrak{H}(K))=J(\mathscr{A}(D), \mathscr{A}(E))$ by the definition of G_1 and Th. 2. Therefore we have $B_1\subset\mathrm{N}_B(E)$. Similarly if $E\subset\mathrm{N}_B(D)$, we have $B_1\subset\mathrm{N}_B(D)$. This means that we may assume $B=\mathrm{N}_B(D)=\mathrm{N}_B(E)$ replacing G and B with G_1 and B_1 respectively to prove the last assertion. Then $F=[D, E]$ is also normal in B by Lemma 25, and hence we see $[\mathscr{A}(D), E]\subset F=[D, E]$ by Cor. to Prop. 45. By Cor. to Prop. 27 $\mathscr{A}(D)$ is also normal in B. Therefore we see $[\mathscr{A}(E), \mathscr{A}(D)] \subset [E, \mathscr{A}(D)]$ replacing D and E by E and $A(D)$ in the above. This means $[\mathscr{A}(D), \mathscr{A}(E)] = [\mathscr{A}(E), \mathscr{A}(D)] \subset [E, \mathscr{A}(D)] = [\mathscr{A}(D), E] \subset [D, E]$, and hence we have $[\mathscr{A}(D), \mathscr{A}(E)] = \mathscr{A}([D, E]) = [D, E]$. q.e.d.

COROLLARY. *Let G and B be as above. Then we have followings*:
(*i*) *If D is any Hopf subalgebra of B*, $[D, D]$ *is equal to* $[\mathscr{A}(D), \mathscr{A}(D)] = \mathscr{A}([D, D])$. *In particular* $[D, D]$ *is algebraic.*[8]
(*ii*) *If E and F are normal Hopf subalgebras of B*, $[E, F]$ *is equal to* $\mathscr{A}([E, F]) = [\mathscr{A}(E), \mathscr{A}(F)]$ *and* $[E, F]$ *is algebraic.*

§ 10. Lie algebras attached to group schemes

The aim of this section is to show some results[9] on Lie algebras attached to group schemes over an algebraically closed field of characteristic zero using our results on Hopf algebras attached to group schemes. Therefore we assume that *k is always an algebraically closed field of characteristic zero* in the following.

Let G be a group scheme over k and let $\mathfrak{H}(G)$ be the Hopf algebra attached to G. Then we denote by $\mathfrak{L}(G)$ the space of primitive elements of $\mathfrak{H}(G)$ and call it *the Lie algebra attached to G*. As seen in § 4 $\mathfrak{L}(G)$ is a finite dimensional Lie algebra over k whose Lie product $[x, y]$ is defined by $xy - yx = m(x \otimes y) - m(y \otimes x)$ for any x and y in $\mathfrak{L}(G)$ where m is the multiplication of $\mathfrak{H}(G)$. Moreover if $\mathfrak{m}$ is the maximal ideal of the stalk $\mathcal{O}$ of G at the neutral point, $\mathfrak{L}(G)$ may be identified with the dual space of $\mathfrak{m}/\mathfrak{m}^2$ as seen easily. If H is a group subscheme of G, we may identify $\mathfrak{L}(H)$ naturally with a Lie subalgebra of $\mathfrak{L}(G)$. Now we have the following

PROPOSITION 46. *Let G be a group scheme over k. Then there is one to one correspondence between the set of Hopf subalgebras D of* $\mathfrak{H}(G)$ *and the set of Lie subalgebras M of* $\mathfrak{L}(G)$ *such that M is the space* $\mathfrak{L}(D)$ *of primitive elements of D.*

PROOF. Identifying k with its image in $\mathfrak{H}(G)$ by the identity i of $\mathfrak{H}(G)$, we may assume that $\mathfrak{H}(G)$ contains k. If M is a Lie subalgebra of $\mathfrak{L}(G)$, we see easily that $k \oplus M$ is a subcoalgebra of $\mathfrak{H}(G)$. Then the subalgebra D of $\mathfrak{H}(G)$ generated by $k \oplus M$ is a Hopf subalgebra of $\mathfrak{H}(G)$ as seen easily. Since the space $\mathfrak{L}(D)$ of primitive elements of D is equal to $D \cap \mathfrak{L}(G)$, we see $\mathfrak{L}(D) \supset M$. Let $\{x_1, \ldots, {}_r\}$ be a basis for M over k. Therefore we have

8) J. Dieudonné gave a proof of this result in the special case where G is an affine algebraic group and D is reduced, and M. Takeuchi announced in the foot notes of [11] that he obtained this result.

9) See § 7 in [1], and Th. 13, 14 and 15 in [2] should be referred.

$$x_i x_j - x_j x_i = [x_i, x_j] = \sum_{n=1}^{r} \alpha_{ijh} x_h \quad \text{with} \quad \alpha_{ijh} \in k$$

and hence any monomial $x_{i_1} \cdots x_{i_s}$ $(1 \leq i_j \leq r)$ can be written as a linear combination of the monomials $x_{j_1} \cdots x_{j_n}$ with $j_1 \leq j_2 \leq \cdots \leq j_n$ as seen easily by induction on s. This means by Th. 3 in [9] that M coincides with $\mathfrak{L}(D)$, because $\{x_i^s/s! | s \geq 0\}$ is a sequence of divided powers of x_i for each i. Conversely if D is any Hopf subalgebra of $\mathfrak{H}(G)$, D is generated by $k \oplus \mathfrak{L}(D)$ as a k-algebra by Th. 3 in [9]. Therefore our assertion is true. q.e.d.

COROLLARY. *Let H_1 and H_2 be connected group subschemes of a group scheme G over k. Then if $\mathfrak{L}(H_1) = \mathfrak{L}(H_2)$, we have $H_1 = H_2$.*

PROOF. As seen in § 1, H_1 is equal to H_2 if and only if $\mathfrak{H}(H_1)$ is equal to $\mathfrak{H}(H_2)$. But by Prop. 46 the last assertion is equivalent to $\mathfrak{L}(H_1) = \mathfrak{L}(H_2)$. q.e.d.

Let M_1 and M_2 be Lie subalgebras of a Lie algebra L over k. Then there exists the smallest Lie subalgebra M of L containing M_1 and M_2. We denote M by $J(M_1, M_2)$ and call it *the join* of M_1 and M_2. On the other hand if we put $I(M_1, M_2) = M_1 \cap M_2$, $I(M_1, M_2)$ is the largest Lie subalgebra of L contained in M_1 and M_2. We call $I(M_1, M_2)$ *the intersection* of M_1 and M_2.

PROPOSITION 47. *Let D_1 and D_2 be Hopf subalgebras of the Hopf algebra $\mathfrak{H}(G)$ attached to a group scheme G over k. Then we have $J(\mathfrak{L}(D_1), \mathfrak{L}(D_2)) = \mathfrak{L}(J(D_1, D_2))$ and $I(\mathfrak{L}(D_1), \mathfrak{L}(D_2)) = \mathfrak{L}(I(D_1, D_2))$.*

PROOF. Since $J(D_1, D_2)$ contains D_1 and D_2, $\mathfrak{L}(J(D_1, D_2))$ contains $\mathfrak{L}(D_1)$ and $\mathfrak{L}(D_2)$. Therefore we see $\mathfrak{L}(J(D_1, D_2)) \supset J(\mathfrak{L}(D_1), \mathfrak{L}(D_2))$. Let D' be the Hopf subalgebra of $\mathfrak{H}(G)$ such that $\mathfrak{L}(D') = J(\mathfrak{L}(D_1), \mathfrak{L}(D_2))$. Since $\mathfrak{L}(D')$ contains $\mathfrak{L}(D_1)$ and $\mathfrak{L}(D_2)$, we see $D' \supset D_1$ and $D' \supset D_2$ from the proof of Prop. 46. Therefore D' contains $J(D_1, D_2)$, and hence $\mathfrak{L}(D') = J(\mathfrak{L}(D_1), \mathfrak{L}(D_2))$ contains $\mathfrak{L}(J(D_1, D_2))$. This means $\mathfrak{L}(J(D_1, D_2)) = J(\mathfrak{L}(D_1), \mathfrak{L}(D_2))$. Similarly we see $\mathfrak{L}(I(D_1, D_2)) = I(\mathfrak{L}(D_1), \mathfrak{L}(D_2))$ but we omit the detail. q.e.d.

Let G be a group scheme over k and let M be a Lie subalgebra of $\mathfrak{L}(G)$. Then we say that M is *algebraic* if M is equal to $\mathfrak{L}(H)$ for a group subscheme H of G. For an arbitrary M there exists the smallest algebraic Lie subalgebra $\mathscr{A}(M)$ of $\mathfrak{L}(G)$ containing M by Prop. 46 and 5. We call $\mathscr{A}(M)$ *the algebraic hull* of M. The following proposition is a direct consequence of Th. 2, Prop. 10 and Prop. 47.

PROPOSITION 48. *If M_1 and M_2 are algebraic Lie subalgebras of the Lie algebra $\mathfrak{L}(G)$ attached to a group scheme G over k, so are $J(M_1, M_2)$ and $I(M_1, M_2)$.*

Now let V be a vector space of dimension n over k and let ϕ be a rational representation of G to $GL_V=(GL_n, id(t_{ij}, v_i))$ where $\{v_i\}$ is a basis for V over k. Then we have the following

LEMMA 26. *If ρ and ϕ_* are the canonical representation of $\mathfrak{H}(GL_n)$ to $M_n(k)$ with respect to $\{t_{ij}\}$ and the tangential homomorphism attached to ϕ respectively, then $\rho\phi_*|_{\mathfrak{L}(G)}$ is a Lie algebra homomorphism of $\mathfrak{L}(G)$ to $M_n(k)$. In particular $\rho|_{\mathfrak{L}(GL_n)}$ gives an isomorphism between $\mathfrak{L}(GL_n)$ and $M_n(k)$ as Lie algebras over k.*

PROOF. Since it is easy to see that $\phi_*|_{\mathfrak{L}(G)}$ is a Lie algebra homomorphism of $\mathfrak{L}(G)$ to $\mathfrak{L}(GL_n)$, it suffices to show that $\rho|_{\mathfrak{L}(GL_n)}$ is an isomorphism between $\mathfrak{L}(GL_n)$ and $M_n(k)$. If we put $s_{ij}=t_{ij}-\delta_{ij}$ for $1\leq i, j\leq n$, $\{s_{ij}\}$ is a regular system of parameters of the stalk of GL_n at the neutral point. Then if $\{l_{a_{11}\cdots a_{nn}}|a_{ij}\geq 0\}$ is the canonical basis for $\mathfrak{H}(GL_n)$ over k with respect to $\{s_{11},\dots, s_{nn}\}$, we see easily that $\{l_{a_{11}\cdots a_{nn}}|a_{11}+\cdots+a_{nn}=1\}$ is a basis for $\mathfrak{L}(GL_n)$ over k. Since we have $<l_{0\cdots 0\overset{ij}{\smile}\!\!\!\!1\, 00\cdots 0}, s_{uv}>=\delta_{iu}\delta_{jv}$, ρ maps $\mathfrak{L}(GL_n)$ onto $M_n(k)$. From $\dim_k \mathfrak{L}(GL_n) =\dim_k M_n(k)=n^2$, $\rho|_{\mathfrak{L}(GL_n)}$ gives an isomorphism between $\mathfrak{L}(GL_n)$ and $M_n(k)$. q.e.d.

PROPOSITION 49. *Let V, G, $GL_V=(GL_n, id(t_{ij}, v_i))$, ϕ and ρ be as above, and consider V as an $\mathfrak{L}(G)$-module by $\rho\phi_*$. Then the following conditions on a vector subspace W of V are equivalent:*
(i) W is a $G(k)$-submodule of V.
(ii) W is an $\mathfrak{L}(G)$-submodule of V.
(iii) W is an $\mathfrak{H}(G)$-submodule of V.

PROOF. *(i)*⇔*(iii)*. Since G is reduced, we saw already this in Cor. to Prop. 15.
(iii)⇔*(ii)*. Since $\mathfrak{H}(G)$ is generated by $k\oplus\mathfrak{L}(G)$ as an algebra over k, W is an $\mathfrak{H}(G)$-submodule of V if and only if it is an $L(G)$-submodule of V. q.e.d.

Next we want to show some corresponding results on Lie subalgebras of $\mathfrak{L}(G)$ to those on Hopf subalgebras of $\mathfrak{H}(G)$ obtained in §§7, 8 and 9. For this purpose we need the following lemmas.

LEMMA 27. *Let $\mathfrak{H}(G)=(B, m, i, \Delta, \varepsilon, c)$ be the Hopf algebra attached to a group scheme G over k. Let U and W be subspaces of B satisfying $U\supset W$. Then if D is a Hopf subalgebra of B, the followings are equivalent:*
(i) $\sum_{(x)} x_{(1)}yc(x_{(2)})$ with $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ is in W for any x in $D^\circ=D\cap(\mathrm{Ker}\,\varepsilon)$ and y in U.
(ii) $xy-yx$ is in W for any x in $\mathfrak{L}(D)$ and y in U.

PROOF. $(i)\Rightarrow(ii)$. Since we have $\Delta(x)=x\otimes 1+1\otimes x$ and $c(x)=-x$ for x in $\mathfrak{L}(D)$, we see that $xy-yx=\sum_{(x)} x_{(1)}yc(x_{(2)})$ is in W for x in $\mathfrak{L}(D)$ and y in U.
$(ii)\Rightarrow(i)$. If x is in $\mathfrak{L}(D)$, we have $xy-yx=\sum_{(x)} x_{(1)}yc(x_{(2)})$ for any y in B as seen in the above. Therefore we have $\sum_{(x)} x_{(1)}yc(x_{(2)})\in W$ for any x in $\mathfrak{L}(D)$ and y in U. Let x and x' be in D° and assume that $\sum_{(x)} x_{(1)}Uc(x_{(2)})\subset W$ and $\sum_{(x')} x'_{(1)}Uc(x'_{(2)})\subset W$. Then we see $\sum_{(xx')} (xx')_{(1)}\ Uc\ ((xx')_{(2)})\subset W$ with $\Delta(xx')=\sum_{(xx')} (xx')_{(1)}\otimes(xx')_{(2)}$. In fact let ϕ_B be the k-linear map of $B\otimes_k B$ to B given in the beginning of § 7. In the same way as the proof of Lemma 18 we see $\phi_B(m\otimes 1_B)=\phi_B(1_B\otimes\phi_B)$, and hence we have $\phi_B(xx'\otimes y)=\phi_B(x\otimes\phi_B(x'\otimes y))$ for y in B. In particular if y is in U, we see $\phi_B(xx'\otimes y)=\sum_{(xx')} (xx')_{(1)}yc((xx')_{(2)})\in W$ by our assumption. Since $\mathfrak{L}(D)$ generates D° as a k-algebra by Th. 3 in [9], we see that the assertion (i) is true if (ii) is so. q. e. d.

LEMMA 28. *Let G, B, D, U and W be as above and assume that U is a Hopf subalgebra of B. Moreover putting $W^{\circ}=W\cap(\ker\varepsilon)$, assume that $W^{\circ}U$ and UW° are contained in W. Then the followings are equivalent*:

(i) $\sum_{(x)} x_{(1)}yc(x_{(2)})$ *with* $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ *is in W for any x in $D^{\circ}=D\cap(\ker\varepsilon)$ and y in U.*

(ii) $xy-yx$ is in W for any x in $\mathfrak{L}(D)$ and y in $\mathfrak{L}(U)$.

PROOF. $(i)\Rightarrow(ii)$. This is a direct consequence of Lemma 27.
$(ii)\Rightarrow(i)$. Let x be in $\mathfrak{L}(D)$, and let y and z be in U. Then we see $xyz-yzx=(xy-yx)z+y(xz-zx)$. If $xy-yx$ and $xz-zx$ are in W, then they are in W° as seen easily. Therefore we see that $x(yz)-(yz)x$ is in W by our assumption. Since U° is generated by $\mathfrak{L}(U)$ as a k-algebra, $xy-yx$ is in W for any x in $\mathfrak{L}(D)$ and any y in U° from the assertion (ii). On the other hand if y is in $i(k)$, we have $xy-yx=0$. Therefore we see $xy-yx-W$ for any x in $\mathfrak{L}(D)$ and any y in $U=i(k)\oplus U^{\circ}$, and hence the assertion follows from Lemma 27. q. e. d.

LEMMA 29. *Let G, B, U and W be as in Lemma 27. Then there exists a connected group subscheme H of G satisfying the followings*:
(i) $xy-yx$ is in W for any x in $\mathfrak{L}(H)$ and any y in U.
(ii) If M is any Lie subalgebra of $\mathfrak{L}(G)=\mathfrak{L}(B)$ such that $xy-yx$ is in W for any x in M and any y in U, then M is a Lie subalgebra of $\mathfrak{L}(H)$.

PROOF. Put $H=\mathrm{Tr}_{\mathrm{Ad}}(U, W)$. Then this is a direct consequence of Th. 6, Prop. 46 and Lemma 27. q. e. d.

Let L be a Lie algebra over k, and let M_1 and M_2 be Lie subalgebras of L.

Then we say that M_1 *normalizes* (resp. *centralizes*) M_2, if we have $[x, y] \in M_2$ (resp. $[x, y]=0$) for any x in M_1 and y in M_2. If we put $\mathrm{N}_L(M_2)=\{x \in L | [x, y] \in M_2$ for any y in $M_2\}$, we see easily from the Jocobi identity for the Lie product of L that $\mathrm{N}_L(M_2)$ is the largest Lie subalgebra of L normalizing M_2. We call $\mathrm{N}_L(M_2)$ *the normalizer* of M_2 in L. If we have $L=\mathrm{N}_L(M_1)$, we say that M_1 is *normal* in L.[10] Similarly if we put $C_L(M_2)=\{x \in L | [x, y]=0$ for any y in $M_2\}$, we see that $C_L(M_2)$ is the largest Lie subalgebra of L centralizing M_2. We call $C_L(M_2)$ *the centralizer* of M_2 in L.

PROPOSITION 50. *Let G be a group scheme over k, and let D and E be Hopf subalgebras of the Hopf algebra $\mathfrak{H}(G)$ attached to G. Then D normalizes (resp. centralizes) E if and only if $\mathfrak{L}(D)$ normalizes (resp. centralizes) $\mathfrak{L}(E)$. In particular we have $\mathfrak{L}(\mathrm{N}_{\mathfrak{H}(G)}(E))=\mathrm{N}_{\mathfrak{L}(G)}(\mathfrak{L}(E))$ and $\mathfrak{L}(\mathrm{C}_{\mathfrak{H}(G)}(E))=\mathrm{C}_{\mathfrak{L}(G)}(\mathfrak{L}(E))$.*

PROOF. If we put $U=W=E$ in Lemma 28, we see easily that $\mathfrak{L}(D) \subset \mathrm{N}_{\mathfrak{L}(G)}(\mathfrak{L}(E))$ if and only if $D \subset \mathrm{N}_{\mathfrak{H}(G)}(E)$. Similarly if we put $U=E$ and $W=0$, we see that $\mathfrak{L}(D) \subset \mathrm{C}_{\mathfrak{L}(G)}(\mathfrak{L}(E))$ if and only if $D \subset \mathrm{C}_{\mathfrak{H}(G)}(E)$ by Prop. 28, (ii)$'$. The last assertion follows easily from the above. q.e.d.

COROLLARY. *Let K be a connected group subscheme of a group scheme G over k. Then $\mathrm{N}_{\mathfrak{L}(G)}(\mathfrak{L}(K))$ and $\mathrm{C}_{\mathfrak{L}(G)}(\mathfrak{L}(K))$ are algebraic.*

This is a direct consequence of Prop. 25, Prop. 35 and Prop. 50.

Similarly we can give the results on Lie subalgebras of the Lie algebra $\mathfrak{L}(G)$ attached to a group scheme G over k corresponding to Cor. 2 to Prop. 25, Prop. 26, Prop. 27, Cor. to Prop. 27, Cor 2 to Prop. 35, and Prop. 36 by replacing Hopf subalgebras with Lie subalgebras, but we omit the detail.

Next we shall give some results on commutators of Lie subalgebras of $\mathfrak{L}(G)$ corresponding to § 9.

LEMMA 30. *Let M_1 and M_2 be Lie subalgebras of a Lie algebra L over k such that $M_1 \subset \mathrm{N}_L(M_2)$ and $M_2 \subset \mathrm{N}_L(M_1)$. Then the subspace M of L generated by $[x, y]$ for x in M_1 and y in M_2 is a Lie subalgebra of L.*

PROOF. Since we have $M_1 \subset \mathrm{N}_L(M_2)$ and $M_2 \subset \mathrm{N}_L(M_1)$, we see $M \subset M_1 \cap M_2$. Therefore $[x, y]$ is contained in M for any x and y in M, and hence M is a Lie subalgebra of L. q.e.d.

In the following we denote by $[V_1, V_2]$ the subspace of a Lie algebra L over k generated by $[x, y]$ for x in a subspace V_1 and y in another one V_2. We call

10) In the theory of Lie algebras a normal Lie subalgebra of a Lie algebra L is called *an ideal* of L.

$[V_1, V_2]$ *the commutator* of V_1 and V_2.

PROPOSITION 51. *Let D be a normal Hopf subalgebra of the Hopf algebra $B=\mathfrak{H}(G)$ attached to a group scheme G over k. Then the sequence*

$$0 \longrightarrow \mathfrak{L}(D) \xrightarrow{i_{D^*}} \mathfrak{L}(B) \xrightarrow{\rho_{D^*}} \mathfrak{L}(B/D) \longrightarrow 0$$

of Lie algebras is exact. In particular we have $\mathfrak{L}(D)=\mathfrak{L}(B)\cap BD^\circ$, where D° is the kernel of the coidentity of D.

PROOF. Since the characteristic of k is zero, the proof of Prop. 14.11 shows $\mathfrak{L}(B)\cap BD^\circ=\mathfrak{L}(D)$. Since $k\oplus\mathfrak{L}(B)$ generates B as a k-algebra, $\rho_{D*}(k\oplus\mathfrak{L}(B)) =k\oplus\rho_{D*}(\mathfrak{L}(B))$ generates B/D and $\rho_{D*}(\mathfrak{L}(B))$ is a Lie subalgebra of $\mathfrak{L}(D/B)$. This means, by Th. 3 in [9], $\rho_{D*}(\mathfrak{L}(B))=\mathfrak{L}(B/D)$, and our assertion is true as easily seen. q.e.d.

PROPOSITION 52. *Let M_1 and M_2 be Lie subalgebras of the Lie algebra $\mathfrak{L}(G)$ attached to a group scheme G over k. Then if M_2 is normal in $\mathfrak{L}(G)$, there exists the largest Lie subalgebra M of $\mathfrak{L}(G)$ such that $M\subset \mathrm{N}_{\mathfrak{L}(G)}(M_1)$ and $[M, M_1]\subset M_2$. Moreover M is algebraic.*

PROOF. If D_1 and D_2 are the Hopf subalgebras of $\mathfrak{H}(G)=B$ such that $\mathfrak{L}(D_1) =M_1$ and $\mathfrak{L}(D_2)=M_2$, we put $H=\mathrm{Tr}_{\mathrm{Ad}}(D_1, BD_2^\circ\cap D_1)$. Then by Lemma 28 and Th. 6 we see $[\mathfrak{L}(H), \mathfrak{L}(D_1)]=[\mathfrak{L}(H), M_1]\subset BD_2^\circ\cap D_1$. Since $\mathfrak{L}(G)$ contains $[\mathfrak{L}(H), \mathfrak{L}(D_1)]$, we see $[\mathfrak{L}(H), M_1]\subset M_2$ by Prop. 51, and also $[\mathfrak{L}(H), M_1]\subset M_1$ from $\mathfrak{L}(G)\cap D_1=\mathfrak{L}(D_1)=M_1$. Conversely if M is any Lie subalgebra of $\mathfrak{L}(G)$ satisfying $M\subset \mathrm{N}_{\mathfrak{L}(G)}(M_1)$ and $[M, M_1]\subset M_2$, let D be the Hopf subalgebra of $B=\mathfrak{H}(G)$ such that $\mathfrak{L}(D)=M$. Then since we have $[M, M_1]\subset M_1\subset D_1$ and $[M, M_1]\subset M_2\subset BD_2^\circ$, we see from Lemma 28 and Th. 6 $D\subset\mathfrak{H}(H)$. Therefore we see $M=\mathfrak{L}(D)\subset\mathfrak{L}(\mathfrak{H}(H))=\mathfrak{L}(H)$. q.e.d.

COROLLARY. *Let G, M_1 and M_2 be as above. Then if a Lie subalgebra M of $\mathfrak{L}(G)$ satisfies $[M, M_1]\subset M_2$ and $M\subset N_{\mathfrak{L}(G)}(M_1)$, so does the algebraic hull $\mathscr{A}(M)$ of M.*

THEOREM 9. *Let H and K be connected group subschemes of a group scheme G over k such that H normalizes K and that K normalizes H. Then we have $\mathfrak{L}([H, K])=[\mathfrak{L}(H), \mathfrak{L}(K)]$.*

PROOF. By Cor. 2 to Prop. 25 and Prop. 50 we see $\mathfrak{L}(H)\subset \mathrm{N}_{\mathfrak{L}(G)}(\mathfrak{L}(K))$ and $\mathfrak{L}(K)\subset \mathrm{N}_{\mathfrak{L}(G)}(\mathfrak{L}(H))$, and hence $[\mathfrak{L}(H), \mathfrak{L}(K)]$ is a Lie subalgebra of $\mathfrak{L}(G)$ by Lemma 30. On the other hand we have $\mathfrak{H}(J(H, K))=J(\mathfrak{H}(H), \mathfrak{H}(K))$ by Th. 2. Replacing G with $J(H, K)$, we may assume that H and K are normal in G. Then we see easily from the Jacobi identity for the Lie product of $\mathfrak{L}(G)$ that

$[\mathfrak{L}(H), \mathfrak{L}(K)]$ is normal in $\mathfrak{L}(G)$. Let F be the Hopf subalgebra of $B=\mathfrak{H}(G)$ satisfying $\mathfrak{L}(F)=[\mathfrak{L}(H), \mathfrak{L}(K)]$ and put $E=\mathfrak{H}(K)$. Then if we put $H_1=\mathrm{Tr}_{\mathrm{Ad}}(E, BF^\circ \cap E)$, we see $\mathfrak{L}(H_1)\supset\mathfrak{L}(H)$ from the proof of Prop. 52, and hence $\mathfrak{H}(H_1)\supset\mathfrak{H}(H)$. This means $[\mathfrak{H}(H), E]=[\mathfrak{H}(H), \mathfrak{H}(K)]\subset F$ from the proof of Prop. 45. Therefore we see $\mathfrak{L}([\mathfrak{H}(H), \mathfrak{H}(K)])\subset\mathfrak{L}(F)=[\mathfrak{L}(H), \mathfrak{L}(K)]$. Now if $\psi_{\mathfrak{H}(G)}$ is the k-linear map of $\mathfrak{H}(G)\otimes_k\mathfrak{H}(G)$ to $\mathfrak{H}(G)$ given in the beginning of §9, we see easily $\psi_{\mathfrak{H}(G)}(x\otimes y)=xy-yx=[x, y]$ for x and y in $\mathfrak{L}(G)$. Therefore we see $\mathfrak{L}([\mathfrak{H}(H), \mathfrak{H}(K)])\supset[\mathfrak{L}(H), \mathfrak{L}(K)]$ from the definition of the commutator of $\mathfrak{H}(H)$ and $\mathfrak{H}(K)$. Since we have $\mathfrak{H}([H, K])=[\mathfrak{H}(H), \mathfrak{H}(K)]$ by Th. 7, we see $[\mathfrak{L}(H), \mathfrak{L}(K)]=\mathfrak{L}([H, K])$. q.e.d.

THEOREM 10. *Let M_1 and M_2 be Lie subalgebras of the Lie algebra $\mathfrak{L}(G)$ attached to a group scheme G over k satisfying $M_1\subset \mathrm{N}_{\mathfrak{L}(G)}(M_2)$ and $M_2\subset \mathrm{N}_{\mathfrak{L}(G)}(M_1)$. Then we have $[M_1, M_2]=\mathscr{A}([M_1, M_2])=[\mathscr{A}(M_1), \mathscr{A}(M_2)]$.*

The proof of this theorem can be given in an exactly similar way to that of Th. 8 using Cor. to Prop. 52, Th. 9 and Prop. 48 instead of Cor. to Prop. 45, Th. 7 and Th. 2, but we omit the detail.

COROLLARY. *Let G and $\mathfrak{L}(G)$ be as above. Then we have the followings:*
(i) If M is any Lie subalgebra of $\mathfrak{L}(G)$, $[M, M]$ is equal to $[\mathscr{A}(M), \mathscr{A}(M)]=\mathscr{A}([M, M])$.
(ii) If M_1 and M_2 are normal Lie subalgebras of $\mathfrak{L}(G)$, $[M_1, M_2]$ is equal to $\mathscr{A}([M_1, M_2])=[\mathscr{A}(M_1), \mathscr{A}(M_2)]$.

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