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Corrections to "Module Spectra over the Moore Spectrum"

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Theorem 4.5 in [1] is false. The reason is that the proof of Lemma 4.6 in [1] is incorrect, which was kindly noticed by Professor Z. Yosimura. From Example 6.8 (7) in [1], a counterexample for the theorem is constructed as follows: Let V and $X=C(g)$ be the spectra in the example. Then the sequence $[M\wedge X, X]_0^M \xrightarrow{(1\wedge i_g)^*} [M\wedge V, X]_2^M \xrightarrow{(1\wedge g)^*} [M\wedge V, X]_{-1}^M$ is $0\to Z_p\to 0$, which is not exact.

Since this theorem played an essential role in the proofs of Lemma 6.5 and Theorem 6.6 in [1], we must add some assumptions to these results as well as Theorem 4.5 to complete them. In another paper [2] we used [1, Th. 4.5] to simplify several proofs and we must also correct their proofs, cf. [2, Note on p. 446].

1. Corrections to Theorems 4.5 and 6.6 in [1].

1–1. Theorem 4.5 in [1] should be replaced by the following, and Lemma 4.6 in [1] should be deleted.

THEOREM 4.5′. *In a cofiber sequence*

$$\Sigma^k X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{\pi} \Sigma^{k+1} X,$$

assume that all spectra are associative M-module spectra and all maps are M-maps. Let Z be an M-module spectrum having the element in [1, Condition 7.1]. *Then the following sequences are exact*:

$$\cdots \longrightarrow [Z, X]_{j-k}^M \xrightarrow{f_*} [Z, Y]_j^M \xrightarrow{i_*} [Z, C(f)]_j^M \xrightarrow{\pi_*} [Z, X]_{j-k-1}^M \longrightarrow \cdots,$$

$$\cdots \longrightarrow [X, Z]_{j+k+1}^M \xrightarrow{\pi^*} [C(f), Z]_j^M \xrightarrow{i^*} [Y, Z]_j^M \xrightarrow{f^*} [X, Z]_{j+k}^M \longrightarrow \cdots.$$

PROOF. By the direct sum decompositions for $[Z,\ ]_*$ and $[\ ,Z]_*$ in [1, Th. 7.5], these exact sequences are easily derived from the usual ones of $[Z,\ ]_*$ and $[\ ,Z]_*$.

1–2. Lemma 6.5 in [1] should be replaced by the following

LEMMA 6.5′. *Let G be a finite Z_q-module and Y be an associative M_q-*

module spectrum such that G and $H_*(Y)$[1] *have no* 3*-torsion in case* $q\equiv\pm 3 \bmod 9$. *Let* $f: \Sigma^{k-1}M(G)\to Y$ *be an M-map, and consider the following conditions*:

(i) (=[1, (6.1)]) *For any prime p, the p-component of G is free over the p-component of* Z_q.

(ii) $[Y, M(G)]_{-k+4}=0$, $[Y, M(G)]_{-k+3}=0$.

(iii) $\theta: [Y, Y]_3\to[Y, Y]_4$ *is surjective.*

(iv) $[M(G), Y]^M_{k+2}=0$.

If (i), (ii) *and* (iii) *are satisfied, then* $C(f)$ *has an admissible and associative* M_q*-action. Further if* (iv) *is also satisfied, any admissible* M_q*-action on* $C(f)$ *is associative.*

PROOF. Let m_C be any admissible M_q-action on $C(f)$. We consider the exact sequences involving $[C(f), C(f)]_2$ derived from the cofibering $Y\xrightarrow{i}C(f)\xrightarrow{\pi}\Sigma^k M(G)$. Since $i^*a(m_C)=0$ and $\pi_*a(m_C)=0$ by [1, Cor. 5.9] and the associativity of $M(G)$, and since $\pi^*: [M(G), M(G)]_2\to[C(f), M(G)]_{-k+2}$ is one-to-one by the assumption (ii), we can put $a(m_C)=i\xi\pi$ for $\xi\in[M(G), Y]_{k+2}$. By [1, Th. 5.10, Lemma 6.1], we see $i\theta(\xi)\pi=\pm\theta(a(m_C))=0$, and so $\theta(\xi)=f\xi_1+\xi_2 f$ for some $\xi_1\in[M(G), M(G)]_4$, $\xi_2\in[Y, Y]_4$. By (i), [1, Prop. 7.2] and $[M(G), M(G)]_5=0$, Theorem 7.5 in [1] implies $\xi_1\in \operatorname{Im}\theta$. Also, $\xi_2\in\operatorname{Im}\theta$ by (iii), and hence $a(m_C)=i\theta(\eta)\pi$ for some $\eta\in[M(G), Y]_{k+1}$ by [1, Th. 7.5] with $X=M(G)$. Then, for $d=(-1)^k i\eta\pi$, $m'_C=m_C+d(\pi\wedge 1_C)$ is admissible and associative. If (iv) is satisfied, $\theta(\eta)=0$ and m_C is associative.

1–3. Theorem 6.6 in [1] should be replaced by the following

THEOREM 6.6′. *Let X be an* M_q*-module spectrum, and in case* $q\equiv\pm 3 \bmod 9$ *assume that* $H_*(X)$ *has no* 3*-torsion. If X satisfies the following conditions* (i)–(iii), *then X admits an associative* M_q*-action.*

(i) *If* $1\leqq|i-j|\leqq 5$, $\#H_i(X)$ *is relatively prime to* $\#H_j(X)$, *where* $\#G$ *denotes the order of a finite group G.*

(ii) *For any prime p, the p-component of* $H_i(X)$ *is free over the p-component of* Z_q.

(iii) *Let r* (*resp. s*) *be the minimal* (*resp. maximal*) *degree k for which* $H_k(X)\neq 0$. *If* $r\leqq j<i<s$ *and* $\mathrm{GCD}(\#H_i(X), \#H_j(X))>1$, *then* $[M(H_i(X)), M(H_j(X))]^M_l$ *vanishes for* $l=i-j+2$ *and* $i-j+5$.

PROOF. We denote simply $H_k(X)$ by H_k. There is a filtration $\{X_k\}_{r\leqq k\leqq s}$ of X together with cofiber sequences ($X_{r-1}=*$, $X_s=X$):

$$(1)_k \qquad \Sigma^{k-1}M(H_k)\xrightarrow{f_k}X_{k-1}\xrightarrow{i_k}X_k\longrightarrow\Sigma^k M(H_k)$$

for $r\leqq k\leqq s$. By (i), $[X_{k-1}, M(H_k)]_{-k+1}=0$, and so there are M-actions m_k

1) For a spectrum X, $H_*(X)$ denotes the reduced homology group of X.

on X_k such that all maps in $(1)_k$ are M-maps, i.e., $(1)_k$ is admissible. We consider the following statements:

(A_k) *m_k is associative.*

(A'_k) *There is an admissible and associative M-action on X_k (which may be different from m_k).*

(B_k) θ: $[X_k, X_k]_3 \longrightarrow [X_k, X_k]_4$ *is surjective.*

(C_k) $[M(H_k), X_{k-1}]^M_{k+2} = 0$.

(C'_k) $[M(H_k), X_{k-1}]^M_{k+5} = 0$.

By Lemma 6.5′ of above, we have immediately

(2) (A_{k-1}) *and* (B_{k-1}) *imply* (A'_k),

(3) (A_{k-1}), (B_{k-1}) *and* (C_k) *imply* (A_k).

If $(A_r),\dots,(A_{k-1})$ are valid, we can apply $[M(H_k),\ ]^M_*$ to the sequences $(1)_l$, $r+1\leqq l\leqq k-1$, to obtain exact sequences in Theorem 4.5′ of above. By the assumption (iii), we have

(4) *For $k<s$, $(A_r),\dots,(A_{k-1})$ imply (C_k) and (C'_k).*

Since $[X_{k-1}, M(H_k)]_j=0$ for $j=-k+3$, $-k+4$ by (i), $(i_k)_*$: $[X_{k-1}, X_{k-1}]_j \to [X_{k-1}, X_k]_j$ is surjective for $j=3$, 4, and so (B_{k-1}) implies that θ is surjective on $[X_{k-1}, X_k]_3$. A similar discussion to the proof of $\xi_1\in\operatorname{Im}\theta$ in Lemma 6.5′ shows that θ is injective on $[M(H_k), M(H_k)]_2$ and surjective on $[M(H_k), M(H_k)]_3$. Similarly, (A_{k-1}) and (C'_k) imply that θ is surjective on $[M(H_k), X_{k-1}]_{k+3}$. By these facts and iterated use of the five lemma, we obtain

(5) (A_{k-1}), (B_{k-1}), (C_k) *and* (C'_k) *imply* (B_k).

From (2)–(5), we obtain, by the induction on k, the statements (A_k) for $k<s$ and finally (A'_s).

1–4. As a special case of Theorem 6.6′, we have the following

COROLLARY. *Let X be an M_q-module spectrum whose homology group $H_i(X)$ vanishes unless $i=r$ and s for which $H_i(X)$ satisfies the condition* (ii) *in the above theorem and has no* 3*-torsion in case $q\equiv\pm3 \bmod 9$. Then X admits an associative M_q-action.*

PROOF. If $|s-r|>5$, this is immediate from the above theorem. In the remaining case, without loss of generality, we can assume $5\geqq s>r=0$. Then X is a mapping cone of an M-map $f\in[M(G), M(H)]^M_{s-1}$, where $G=H_s(X)$ and $H=H_0(X)$, and $[M(G), M(H)]^M_{s-1}$ is nontrivial only if $s=4$ or 5 and both G and

H have 3-torsions. Then, it is sufficient to consider the case when $G=H=Z_{3^a}$ $(a\geqq 2)$ where $q=3^a q'$ with $q'\not\equiv 0 \bmod 3$. In this case, f is a multiple of $\alpha_1(3)\wedge 1_{MG}$ or $\lambda\alpha\rho$ according as $s=4$ or 5, where $\alpha\in[M_3, M_3]_4=Z_3$ is a generator and $\alpha_1(3)=\pi\alpha i$. The associativity of $C(\alpha_1(3)\wedge 1)$ and $C(\lambda\alpha\rho)$ are easily derived from that of $M(G)$ and $C(\alpha)$[1)], respectively.

2. Corrections to some proofs in [2]

We have used Theorem 4.5 in [1] to prove the following results of [2]: Proposition 1.3, Theorem 2.1, Proposition 3.9, Lemma 4.2, Theorem 4.3. We can prove the first two by using Theorem 4.5′ of above, and the last three without using exact sequences of $[\ ,\ ]^M_*$.

2–1. In the proof of [2, Prop. 1.3], we can take m' and n' so that $\theta_1(m')=0$ and $\theta_1(n')=0$ by using Theorem 4.5′, since the spectra M and K in [2] satisfy [1, Condition 7.1]. Here we use the same notations θ_1 and θ_2 as in [2, (1.5), Lemma 2.2]. We must show that $\theta_2(m')=0$ and $\theta_2(n')=0$ are satisfied.

The difference element between the M-actions $K\wedge\mathring{K}$ (resp. $M\wedge\mathring{K}$) and $\mathring{K}\wedge K$ (resp. $\mathring{M}\wedge K$) is given by $d=(1\wedge m)(T\wedge 1)(n\wedge 1)$ (resp. $d'=(1_M\wedge m)\cdot(T_M\wedge 1)(n_M\wedge 1)$), and we have $d'=-nm$, $d(i'\wedge 1)=-(i'\wedge 1)nm$, $(\pi'\wedge 1)d=nm(\pi'\wedge 1)$ and $\theta_1(d)=0$. By [1, Th. 2.2], $\theta_2(m')=-m'dn'(\pi'\wedge 1)$. Therefore $m'dn'$ lies in the group $\mathscr{A}_{pq+2}(M\wedge K)\cap\operatorname{Ker}\theta_1\cap\operatorname{Ker}\theta_2=A$. For any $\xi\in\mathscr{A}_{pq+1}(M\wedge K)\cap\operatorname{Ker}\theta_1=A'$, it holds $(m'+\xi(\pi'\wedge 1))d(n'-(i'\wedge 1)\xi)-m'dn'=\xi nm+nm\xi$, and easy calculations show the equality $A=\{\xi nm+nm\xi \mid \xi\in A'\}$. Thus, there is an $m'\in\operatorname{Ker}\theta$, for which $m'dn'=0$ holds. This implies $\theta_2(m')=0$ and $\theta_2(n')=-(i'\wedge 1)m'dn'=0$ as desired.

REMARK. The difference element d of above is equal to $-(i'\wedge 1)nmm'+n'nm(\pi'\wedge 1)=-\nu_2\mu_1+\nu_4\mu_3$. The elements m' and n' of the proposition are unique because of $\mathscr{A}_{pq+1}(M\wedge K)\cap\operatorname{Ker}\theta_1\cap\operatorname{Ker}\theta_2=0$.

2–2. To prove Theorem 2.1 in [2], we used [1, Th. 4.5] on page 435, lines 20–22 and page 439, lines 20–22. But the discussions there are valid if we use Theorem 4.5′ instead of [1, Th. 4.5].

2–3. In the proof of [2, Prop. 3.9], the assertion that g is an M-map is not valid. We have $g=g_1+g_2\delta_M$ for some $g_i\in[M, K]^M_*$ by [2, Prop. 1.8], and each g_i lies in $i'^*\mathscr{C}_*(K)$. Hence $f=h\delta'$ for some $h\in\mathscr{C}_*(K)$, and the proof of [2, Prop. 3.9] is now done as it is.

1) $C(\alpha)$ is the spectrum $V(1)$ at 3 of Toda. He showed the non-associativity of $C(\alpha)$ as M_3-module spectrum. But the formula [1, (5.7)] assures that it is associative as M_q-module spectrum if $q\equiv 0 \bmod 9$.

2–4. In the proof of [2, Lemma 4.2], "$g_s = h_s + h'_s\delta'$ for h_s, $h'_s \in \mathscr{C}_*(K)$" on page 444, line 15 should be replaced by

$$\text{``}g_s \equiv h_s \bmod \operatorname{Im} \delta^* + \operatorname{Im}(\delta')^* \qquad \text{for} \quad h_s \in \mathscr{C}_*(K)\text{''}.$$

2–5. In the proof of [2, Th. 4.3], the sentence "Hence, by (4.4) (i)$\cdots g_s\lambda_K = \lambda_K(f_s)^p$." on page 444, lines -7 to -6 should be replaced by

"Hence, by (4.4) (i), there are maps g_s such that $g_s\lambda_K = \lambda_K(f_s)^p$ and that $C(g_s)$ is a mapping cone of some map $\sum^{-1}C((f_s)^p) \to C((f_s)^p)$. Since $C((f_s)^p)$ is an M-module spectrum, $C(g_s)$ is an M'-module spectrum by [15, Cor. 3.6]. The map g_s is the M'-map by [15, Lemma 4.7]".

References

[1] S. Oka, *Module spectra over the Moore spectrum,* Hiroshima Math. J. 7 (1977), 93–118.

[2] S. Oka, *Realizing some cyclic BP_*-modules and applications to stable homotopy of spheres,* Hiroshima Math, J., 7 (1977), 427–447.

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