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On p-Indicators in Ext(Q/Z, T)

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§1. Introduction

All groups considered in this paper are abelian groups and written additively. We mention basic notations and terminology here. Further details may be found in [1] and [2].

A group A is *divisible* if $nA=A$ for every integer n $(\neq 0)$. A is *reduced* if A contains no divisible subgroups $(\neq\{0\})$. A is said to be *p-divisible* if $pA=A$ for some prime p. Let p be a prime and σ be an ordinal. If $\sigma-1$ exists, $p^{\sigma}A=p(p^{\sigma-1}A)$; if σ is a limit ordinal, $p^{\sigma}A=\cap p^{\rho}A$ $(\rho<\sigma)$. The σ-th *Ulm subgroup*, denoted by A^{σ}, is $\cap_p p^{\omega\sigma}A$, where p runs over all primes.

There is a least ordinal λ such that $p^{\lambda}A$ is p-divisible. λ is called the *p-length* of A. If x is an element of A, $h_p(x)$ shall denote the *p-height* of x in A as follows: if $x\in p^{\sigma}A\backslash p^{\sigma+1}A$, $h_p(x)=\sigma$; if $x\in p^{\sigma}A=p^{\sigma+1}A$ for some σ, $h_p(x)=\infty$ where ∞ is considered to be larger than every occurring ordinal. Set $h_p(p^n x)=\sigma_n$ for $n=0, 1, 2,\cdots$. We call the sequence of ordinals and ∞'s $(\sigma_0, \sigma_1, \sigma_2,\cdots)$ the *p-indicator* of x. If $\sigma_n+1<\sigma_{n+1}$, then the p-indicator of x is said to have a *gap* between σ_n and σ_{n+1}. Let $p_1,\cdots, p_n,\cdots$ be the sequence of all primes. With a given element x, we associate the *height matrix*

$$\begin{pmatrix} \sigma_{10}\sigma_{11}\cdots \\ \cdots\cdots\cdots\cdots \\ \sigma_{n0}\sigma_{n1}\cdots \\ \cdots\cdots\cdots\cdots \end{pmatrix}$$

whose n-th row is the p_n-indicator of x.

A subgroup G of A is called *pure*, if $nG=G\cap nA$ holds for every integer n. G is called *isotype*, if $p^{\sigma}G=G\cap p^{\sigma}A$ for all ordinals σ and primes p. If this relation holds for some prime p, G is said to be *p-isotype* in A.

If a group A contains both nonzero elements of finite order and elements of infinite order, A is called *mixed*. The *torsion-free rank* of a group A is the cardinality of an independent subset of A which contains only elements of infinite order and which is maximal with respect to this property.

A group A is called *cotorsion* if every extension of A by a torsion-free group splits. A cotorsion group that is reduced and has no nonzero torsion-free direct summands is called *adjusted*. A group A is called *algebraically compact* if A is a direct summand in every group G that contains A as a pure subgroup.

Kaplansky introduced the notion of p-indicator in [3] where he called it Ulm-sequence. If A is a torsion-free group, then $p^{\omega}A$ is p-divisible and for given element a, $h_p(pa)=h_p(a)+1$ whenever $h_p(a)\neq\infty$. Therefore, the first column of the height matrix of a whose entries are non-negative integers and symbols ∞ determines the entire matrix at once. On the other hand, in a mixed group A, the structure of the torsion part of A influences what the p-indicators look like. Megibben studied mixed groups of torsion-free rank 1 in [4]. He gave an invariant, that is the height matrix, which determines the isomorphism classes of such groups in countable case. Fuchs generalized this existence theorem into uncountable case. However, there are some vague statements in his proof of this theorem in [2]. It seems that this problem is reduced to the study of p-indicators in the adjusted cotorsion group $\mathrm{Ext}(Q/Z, T)$ with given reduced torsion group T. Our first aim in this paper is to get a criterion for a sequence of ordinals and symbols ∞ to be the p-indicator of some $x\in\mathrm{Ext}(Q/Z, T)$ with given reduced p-primary group T. Then, we can generalize this result to the height matrix in $\mathrm{Ext}(Q/Z, T)$ with given reduced torsion group T immediately. Moreover we shall see that Theorem 103.3 in [2] lacks one more condition.

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§2. Basic lemmas and propositions

First, we present some basic lemmas and propositions.

LEMMA 1. *Let A be a mixed group and G be a subgroup whose torsion part coincides with that of A. G is an isotype subgroup of A if and only if A/G is torsion-free.*

PROOF. The sufficiency follows from 103.1 in [2].

For the necessity, suppose that A/G is not torsion-free. There exists an element $x\in A$ such that $px\in G$ for some prime p. The relation $pG=G\cap pA$ implies $px=py$ for some $y\in G$. This means that $x-y\in T$, where T denotes the torsion part of A. From the fact that $T\subset G$, it follows that $x\in G$.

LEMMA 2. *If G is a subgroup of A such that A/G is torsion-free, then for any ordinal σ and any prime p, $p^{\sigma}G$ is an isotype subgroup of $p^{\sigma}A$.*

PROOF. $p^{\sigma}A/p^{\sigma}G=p^{\sigma}A/G\cap p^{\sigma}A$ since G is an isotype subgroup of A by Lemma 1. Since A/G is torsion-free, $p^{\sigma}A/G\cap p^{\sigma}A$ is torsion-free. Hence, again by Lemma 1, $p^{\sigma}G$ is an isotype subgroup of $p^{\sigma}A$.

LEMMA 3. *Let T be a reduced torsion group and let A be a reduced non-splitting mixed group of torsion-free rank* 1 *with the torsion part T. A can be* *embedded in* $\mathrm{Ext}(Q/Z, T)$ *such that the diagram*

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & T & \longrightarrow & \mathrm{Ext}(Q/Z, T) & \longrightarrow & \mathrm{Ext}(Q, T) & \longrightarrow & 0 \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & A & \longrightarrow & \mathrm{Ext}(Q/Z, T) & & & & \end{array}$$

commutes. A can be embedded as an isotype subgroup in $\mathrm{Ext}(Q/Z, T)$ *if and only if* $A/T\cong Q$.

PROOF. First part of Lemma 3 follows from 103.2 in [2].

We notice that $\mathrm{Ext}(Q/Z, T)/A=\mathrm{Ext}(Q, T)/(A/T)$. Suppose $A/T\cong Q$. A/T is a direct summand of a torsion-free divisible group $\mathrm{Ext}(Q, T)$. Hence $\mathrm{Ext}(Q/Z, T)/A$ is torsion-free, that is, A is isotype in $\mathrm{Ext}(Q/Z, T)$. Conversely, suppose that A is isotype in $\mathrm{Ext}(Q/Z, T)$. This means that A/T is pure in the divisible group $\mathrm{Ext}(Q/T)$. Hence A/T is a divisible torsion-free group of torsion-free rank 1, i.e., $A/T\cong Q$.

LEMMA 4. *Let A be a reduced mixed group with torsion part T. A can be embedded in* $\mathrm{Ext}(Q/Z, A)\cong\mathrm{Ext}(Q/Z, T)\oplus\mathrm{Ext}(Q/Z, A/T)$ *as an isotype subgroup.*

PROOF. See [4].

LEMMA 5. *Let A be a reduced mixed group with torsion part T. If A contains an element x whose p-height is not less than ω and not equal to ∞, then there exists an element x' in* $\mathrm{Ext}(Q/Z, T)$ *whose p-indicator coincides with that of x.*

PROOF. By Lemma 4, we can write $x=x'+x''$, where $x'\in\mathrm{Ext}(Q/Z, T)$ and $x''\in\mathrm{Ext}(Q/Z, A/T)$. Since $\mathrm{Ext}(Q/Z, A/T)$ is torsion-free, $h_p(x'')$ is finite or ∞. From our assumption $h_p(x)\geqq\omega$, it follows that $h_p(x'')=\infty$. Therefore the p-indicator of x' coincides with that of x.

LEMMA 6. *Let T be a reduced torsion group, T_p be its p-primary component and S be the complementary direct summand. In the direct decomposition* $\mathrm{Ext}(Q/Z, T)\cong\mathrm{Ext}(Q/Z, T_p)\oplus\mathrm{Ext}(Q/Z, S)$, $\mathrm{Ext}(Q/Z, S)$ *is p-divisible, and for any prime q that is different from p,* $\mathrm{Ext}(Q/Z, T_p)$ *is q-divisible. Each factor in this direct decomposition is reduced.*

Let T be a reduced torsion group and x be an element of $\mathrm{Ext}(Q/Z, T)$. Then x_p shall denote the projection of x into $\mathrm{Ext}(Q/Z, T_p)$.

PROPOSITION 1. *Let T be a reduced torsion group. The p-height of an element x in* $\mathrm{Ext}(Q/Z, T)$ *is equal to that of x_p in* $\mathrm{Ext}(Q/Z, T_p)$. *The p-length of* $\mathrm{Ext}(Q/Z, T)$ *is equal to that of* $\mathrm{Ext}(Q/Z, T_p)$. *Especially, $h_p(x)=\infty$ if and only if $x_p=0$.*

PROOF. This is an immediate consequence of Lemma 6.

COROLLARY. *$h_p(p^n x)=\infty$ for some n if and only if $x_p \in T_p$.*

We refer to Prop. 56.4 and Prop. 56.5 in [2] as Lemma 7 since they are most useful for our study.

LEMMA 7. $\mathrm{Ext}(Q/Z, T)^\sigma \cong \mathrm{Ext}(Q/Z, T^\sigma) \oplus \mathrm{Hom}(Q/Z, H_\sigma)$, *where* 1) *if $\sigma-1$ exists, $H_\sigma \cong \hat{T}_{\sigma-1}/T_{\sigma-1}$ [$\hat{T}_{\sigma-1}$ is the p-adic completion of $T_{\sigma-1}$ and $T_{\sigma-1} \cong T^{\sigma-1}/T^\sigma$],* and 2) *otherwise, H_σ is isomorphic to the quotient group of the inverse limit $L_\sigma = \varprojlim T/T_p^\rho$ $(\rho<\sigma)$ taken modulo T/T^σ.*

The following proposition will make the condition (iv) in [2, Th. 103.3] clear.

PROPOSITION 2. *Let T be a reduced torsion group with p-length $\lambda \geqq \omega$, and T_p be the p-primary component of T. Let σ be an ordinal such that $\lambda = \omega\sigma+n$, where n is a non-negative integer. The p-length of* $\mathrm{Ext}(Q/Z, T)$ *is λ or $\lambda+\omega$. The p-length of* $\mathrm{Ext}(Q/Z, T)$ *is λ if and only if $T_p^{\sigma-1}/T_p^\sigma$ is torsion-complete or T_p/T_p^σ is the torsion part of $L_{p\sigma} = \varprojlim T_p/T_p^\rho$ $(\rho<\sigma)$ according as σ is an isolated or a limit ordinal.*

PROOF. In view of Lemma 6, we may confine ourselves to the p-length of $\mathrm{Ext}(Q/Z, T_p)$.

It is evident that the p-length of $\mathrm{Ext}(Q/Z, T_p)$ is not less than λ. Since the torsion part of $p^\lambda \mathrm{Ext}(Q/Z, T_p)$ is $p^\lambda T_p$ $(=0)$, $p^\lambda \mathrm{Ext}(Q/Z, T_p)$ is torsion-free. Hence, for any non-negative integer n, $p^{\lambda-n}\mathrm{Ext}(Q/Z, T_p) \neq 0$ if $p^\lambda \mathrm{Ext}(Q/Z, T_p) \neq 0$. Since $p^\lambda \mathrm{Ext}(Q/Z, T_p)$ is torsion-free, $p^{\lambda+\omega}\mathrm{Ext}(Q/Z, T_p)$ is p-divisible. And actually, $p^{\lambda+\omega}\mathrm{Ext}(Q/Z, T_p)=0$ since $\mathrm{Ext}(Q/Z, T_p)$ is reduced and $p^{\lambda+\omega}\mathrm{Ext}(Q/Z, T_p)$ is divisible.

For the second part of this proposition, from Lemma 7 it follows that $\mathrm{Ext}(Q/Z, T_p)^\sigma \cong \mathrm{Ext}(Q/Z, T_p^\sigma) \oplus \mathrm{Hom}(Q/Z, H_{p\sigma})$, where $H_{p\sigma} \cong \hat{T}_{p,\sigma-1}/T_{p,\sigma-1}$ or $H_{p\sigma}$ is isomorphic to the quotient group of the inverse limit $L_{p\sigma} = \varprojlim T_p/T_p^\rho$ $(\rho<\sigma)$ taken modulo T_p/T_p^σ. Multiplying both sides of the above relation by p_n, we obtain $p^n \mathrm{Ext}(Q/Z, T_p)^\sigma = p^\lambda \mathrm{Ext}(Q/Z, T_p) \cong p^n \mathrm{Hom}(Q/Z, H_{p\sigma})$ because $p^n T_p^\sigma = 0$ leads to $p^n \mathrm{Ext}(Q/Z, T_p^\sigma)=0$. Since $\mathrm{Hom}(Q/Z, H_{p\sigma})$ is torsion-free, $p^\lambda \mathrm{Ext}(Q/Z, T_p)=0$ if and only if $\mathrm{Hom}(Q/Z, H_{p\sigma})=0$. And, $\mathrm{Hom}(Q/Z, H_{p\sigma})=0$ if and only if a divisible group $H_{p\sigma}$ is torsion-free. What we have stated in the last part of Proposition 2 is the necessary and sufficient condition for $H_{p\sigma}$ to be torsion-free.

COROLLARY. *Let A be a group with torsion part T and λ be the p-length of T. Then the p-length of A is λ or $\lambda+\omega$. Besides, if $\lambda \geqq \omega$ and if T satisfies the same condition as in Prop. 2, the p-length of A is λ.*

PROOF. From Lemma 4, it follows that $\lambda \leqq$ the p-length of $A \leqq \lambda+\omega$. Suppose for any $x \in p^{\lambda}A$, $p^n x = p^{n+1}y$ for some $y \in p^{\lambda}A$. $p^n(x-py)=0$ implies $x-py \in p^{\lambda}A \cap T = p^{\lambda}T$. Since $p^{\lambda}T$ is p-divisible, $x \in p^{\lambda+1}A$. Therefore the p-length of A is λ or $\lambda+\omega$.

The last part of this corollary follows from Lemma 2, Lemma 4 and Proposition 2.

§ 3. Gap-free indicators

We study p-indicators by classifying them in several cases.

LEMMA 8. *Let T be a reduced torsion group. Then T is dense in* Ext *$(Q/Z, T)$ with respect to Z-adic topology.*

PROOF. From the divisibility of Ext$(Q/Z, T)/T$, our assertion follows.

Especially, if T is a torsion complete p-group, Ext$(Q/Z, T)$ is the p-adic completion of T. Hence, if T is separable but not torsion complete p-group, the p-adic completion of T is equal to Ext$(Q/Z, \bar{T})$, where $\bar{T}$ is the torsion completion of T.

THEOREM 1. *Let T be a reduced torsion group with p-length $\lambda \geqq \omega$. There is no element of p-indicator $(k, k+1, k+2, \cdots)$ in* Ext$(Q/Z, T)$*, where k is a non-negative integer.*

PROOF. By Proposition 1, we may assume T is a p-group. Suppose x has the p-indicator $(k, k+1, \cdots)$. Then there exists a sequence of torsion elements $x_1, x_2, \cdots$, where $h_p(x-x_n) \geqq n$ by Lemma 8. Let i be an integer such that $p^i x_{k+1}=0$. Then $h_p(p^i x) \geqq k+i+1$. This contradicts the fact that $h_p(p^i x) = k+i$.

REMARK 1. Fuchs [2, pp. 198–199] gave an element of p-indicator $(l_0, \cdots, l_k, \cdots)$ in Ext$(Q/Z, \bar{B})$. However, if we set $l_{k+1}=l_k+1$ for every k, then Ext$(Q/Z, \bar{B})$ can not contain any element of p-indicator $(l_0, \cdots, l_k, \cdots)$. But there exists a splitting group of torsion-free rank 1 which contains such an element, that is, a direct sum of reduced p-group T and p-adic integers.

THEOREM 2. *Let T be a reduced torsion group with p-length $\lambda \geqq \omega+\omega$. Let σ be an ordinal such that $\omega\sigma+n<\lambda$ for every non-negative integer n. Then there exists an element x with gap-free p-indicator $(\omega\sigma, \omega\sigma+1, \omega\sigma+2, \cdots)$ in*

$\mathrm{Ext}(Q/Z, T)$ *if and only if* $T_p^{\sigma-1}/T_p^{\sigma}$ *is not torsion-complete or* T_p/T_p^{σ} *is not the torsion part of* $\varprojlim T_p/T_p^{\rho}$ $(\rho\to\sigma)$ *according as* σ *is an isolated or a limit ordinal.*

PROOF. We assume T is a p-primary group in view of Prop. 1. Suppose that there exists an element x in $\mathrm{Ext}(Q/Z, T)$ with the p-indicator $(\omega\sigma, \omega\sigma+1, \omega\sigma+2,\cdots)$. Set $x=x_1+x_2$ in the direct decomposition $p^{\omega\sigma}\mathrm{Ext}(Q/Z, T)\cong \mathrm{Ext}(Q/Z, T^{\sigma})\oplus\mathrm{Hom}(Q/Z, H_{\sigma})$ in Lemma 7. At least one of x_1 and x_2 has p-height $\sigma\omega$ since $h_p(x_1)\geqq\omega\sigma$ and $h_p(x_2)\geqq\omega\sigma$. It is easily seen that one of them has the gap-free indicator $(\omega\sigma, \omega\sigma+1, \omega\sigma+2,\cdots)$. Suppose x_1 has this indicator. $x_1\in\mathrm{Ext}(Q/Z, T^{\sigma})$ and $h_p(p^n x_1)=\omega\sigma+n$ imply $p^n x_1\in p^n\mathrm{Ext}(Q/Z, T^{\sigma})\backslash p^{n+1}\mathrm{Ext}(Q/Z, T^{\sigma})$. In other words, $\mathrm{Ext}(Q/Z, T^{\sigma})$ contains an element x_1 of gap-free p-indicator $(0, 1, 2,\cdots)$. This contradicts Theorem 1. Hence x_2 has the same indicator as x has. $x_2\in\mathrm{Hom}(Q/Z, H_{\sigma})$ and $h_p(p^n x_2)=\omega\sigma+n$ imply $p^n x_2\in p^n\mathrm{Hom}(Q/Z, H_{\sigma})\backslash p^{n+1}\mathrm{Hom}(Q/Z, H_{\sigma})$. Hence $\mathrm{Hom}(Q/Z, H_{\sigma})$ is not p-divisible. That is, $\mathrm{Hom}(Q/Z, H_{\sigma})\neq 0$ since T is assumed to be a reduced p-group. Conversely, if $\mathrm{Hom}(Q/Z, H_{\sigma})\neq 0$, it contains an element x of p-indicator $(0, 1, 2,\cdots)$ since $\mathrm{Hom}(Q/Z, H_{\sigma})$ is torsion-free. Considering $\mathrm{Hom}(Q/Z, H_{\sigma})$ as a direct summand of $p^{\omega\sigma}\mathrm{Ext}(Q/Z, T)$, we see that x has the p-indicator $(\omega\sigma, \omega\sigma+1, \omega\sigma+2,\cdots)$ in $\mathrm{Ext}(Q/Z, T)$. In accordance with the definition of H_{σ}, we get the conclusion immediately.

COROLLARY. *Let* T, λ *and* σ *be the same as in Theorem* 2. *Suppose that* A *is a reduced group with torsion part* T. *Then there is no element in* A *whose* p-*indicator is* $(\omega\sigma, \omega\sigma+1, \omega\sigma+2,\cdots)$, *unless the condition in the above theorem is satisfied.*

PROOF. This follows immediately from Lemma 5.

REMARK 2. Fuchs [2, p. 202] gave an element $g\in\mathrm{Ext}(Q/Z, T)$ with p-indicator $(\sigma_k, \sigma_k+1, \sigma_k+2,\cdots)$. However, it is impossible unless the condition in the above theorem is satisfied. In view of the above corollary, if T does not satisfy the condition in Theorem 2, any group with torsion part T cannot contain such an element.

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