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Partite-Claw-Decomposition of a Complete Multi-Partite Graph[*)]

Shinsei TAZAWA, Kazuhiko USHIO and Sumiyasu YAMAMOTO

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1. Introduction

A multi-partite graph, denoted by $G_m(n_1, n_2, \ldots, n_m)$, is a graph whose point set can be partitioned into m subsets $V_1, V_2, \ldots, V_m$ with $n_1, n_2, \ldots, n_m$ points each, such that every line joins different subsets. If it contains every line joining different subsets, then it is called a complete m-partite graph and is denoted by $K_m(n_1, n_2, \ldots, n_m)$. A complete graph K_m with m points may be regarded as a particular type of complete m-partite graph where $n_1 = n_2 = \cdots = n_m = 1$. A complete bipartite graph $K_2(1, c)$ or a tree with $c+1$ points and radius one is called a claw or a star of degree c.

A claw of degree c being a subgraph of a multi-partite graph will be called a *partite-claw* (PC) of degree c if no pair of points lies in the same set of points of the multi-partite graph.

A graph is called claw-decomposable if it can be decomposed into a union of line-disjoint claws of the same degree. The problem of claw-decomposability of a complete graph K_m has been raised and solved completely by Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [4]. The claw-decomposition of a complete graph provides us an optimal balanced file organization scheme of order two, called $\mathbf{HUBFS}_2$, for binary-valued records in a sense such that it has the least redundancy among all possible balanced binary-valued file organization schemes of order two having the same parameters, provided the distribution of records has the property of invariance with respect to the permutation of attributes [3].

An analogous theorem which states a necessary and sufficient condition for the claw-decomposability of a complete m-partite graph $K_m(n, n, \ldots, n)$ where $n_1 = n_2 = \cdots = n_m = n$ has been obtained by Ushio, Tazawa and Yamamoto [2].

In this paper, a theorem which states a necessary and sufficient condition for the decomposability of such a complete m-partite graph $K_m(n, n, \ldots, n)$ into a union of line-disjoint partite-claws of degree c, which will be called a PC-decomposition theorem of the m-partite graph, will be given. An algorithm for the de-

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composition will also be given. The PC-decomposition of a complete m-partite graph $K_m(n, n,\ldots, n)$ provides us an optimal balanced file organization scheme of order two for multiple-valued records with m attributes of n values each in a sense such that it has the least redundancy among all possible balanced schemes having the same parameters m, n and c, provided all types of records are equally probable [5].

2. Main theorem

With respect to the PC-decomposability of a complete m-partite graph, we have the following theorem which will be proved in the subsequent part of this paper.

THEOREM 2.1. *A complete m-partite graph $K_m(n, n,\ldots, n)$ with m sets of n points each can be decomposed into a union of line-disjoint $\binom{m}{2}n^2/c$ partite-claws of degree c each if and only if*

(i) $\binom{m}{2}n^2$ *is an integral multiple of c, and*

(ii) $m \geq c+1$ *if n is even and* $m \geq c+1+\frac{c-1}{n^2}$ *if n is odd.*

Note that in a particular case $n=1$, Theorem 2.1 turns out to be the theorem of the claw-decomposability of a complete graph K_m given in [4].

3. Proof of the necessity

In a trivial case $c=1$, the conditions (i) and (ii) are obviously necessary and sufficient. Thus we consider the case $c \geq 2$.

Suppose $K_m(n, n,\ldots, n)$ is PC-decomposable. Since the number of lines in $K_m(n, n,\ldots, n)$ must be a multiple of c, (i) is obviously necessary. Let y_i be the number of PC's of degree c whose root points are in the i th point set V_i of $K_m(n, n,\ldots, n)$. Then, the set of all lines incident with V_i can be partitioned into two sets X_i and X'_i with cardinalities $y_i c$ and $(m-1)n^2 - y_i c$, respectively, by classifying each line according as it belongs to one of the above PC's or not. Since no pair of lines in X'_i belongs to the same PC, we have

$$y_i + \{(m-1)n^2 - y_i c\} \leq \binom{m}{2} n^2/c\,.$$

Thus we have

$$y_i \geq \frac{n^2(m-1)\,(2c-m)}{2c(c-1)}\,. \tag{3.1}$$

On the other hand, since every line joining V_i and V_j must belong to a PC whose root is in either V_i or V_j, and since no pair of those lines belongs to the same PC, y_i and y_j must satisfy

$$y_i + y_j \geq n^2$$

for every pair of i and j $(\neq i)$. Thus we have

$$y_i \geq \frac{n^2}{2} \tag{3.2}$$

for all i except at most one. Applying (3.1) and (3.2) to $\sum y_i$, and considering the integrity of y_i, we have

$$\binom{m}{2}n^2/c \geq \frac{n^2(m-1)(2c-m)}{2c(c-1)} + (m-1)\frac{n^2}{2} \qquad \text{for even } n\text{, and}$$

$$\geq \frac{n^2(m-1)(2c-m)}{2c(c-1)} + (m-1)\frac{n^2+1}{2} \qquad \text{for odd } n,$$

and hence we have the condition (ii).

4. Adjacency matrix and PC-decomposability

Before entering the proof of sufficiency of the conditions (i) and (ii) in Theorem 2.1, some auxiliary theorems will be given in this section.

Suppose an arbitrary direction of adjacency is assigned on every line joining a pair of points of $K_m(n, n,\dots, n)$. The number of possible ways of such assignment of direction is, of course, $2^{\binom{m}{2}n^2}$. To each way of the assignment, there corresponds a $0-1$ adjacency matrix

$$M = \|M_{ij}\| \tag{4.1}$$

of order mn composed of m^2 submatrices $M_{ij}=\|m_{ip,jq}\|$ of order n defined by

$$m_{ip,jq} = \begin{cases} 1 & \text{if } v_{ip} \text{ is adjacent to } v_{jq}, \\ 0 & \text{otherwise,} \end{cases}$$

where v_{ip} is the pth point in the ith set V_i and ip denotes the lexicographical number of v_{ip}, i.e., $ip=(i-1)n+p$. Clearly,

$$m_{ip,iq} = 0,\ m_{ip,jq} + m_{jq,ip} = 1 \tag{4.2}$$

hold for all p, q, i and j $(\neq i)$, i.e., $M_{ii}=0$ and $M_{ij}+M_{ji}^T=G_{n,n}$ $(i\neq j)$, where $G_{t,u}$ denotes a $t\times u$ matrix whose elements are all unity.

Conversely, if a $0-1$ matrix M of order mn satisfies (4.2), it is an adjacency matrix of $K_m(n, n,\dots, n)$ subject to a certain way of the assignment of direction.

THEOREM 4.1. *A complete m-partite graph $K_m(n, n, \ldots, n)$ can be decomposed into a union of line-disjoint* PC's *of degree c each, if and only if there exists a way of the assignment of direction to every line in such a manner that the corresponding adjacency matrix $M = \|M_{ij}\|$ of order mn satisfies the following two conditions*:

(a) *Every row sum of M is an integral multiple of c, i.e.,*

$$\sum_{j=1}^{m} \sum_{q=1}^{n} m_{ip,jq} = a_{ip}c .$$

(b) *Every partial row sum of M on the submatrix M_{ij} is bounded uniformly on j by* $\min(a_{ip}, n)$, *i.e.*, $\sum_q m_{ip,jq} \le \min(a_{ip}, n)$.

PROOF. Suppose $K_m(n, n, \ldots, n)$ is PC-decomposable and the direction of each line of $K_m(n, n, \ldots, n)$ is assigned in such a manner that a point corresponding to the root of a PC is adjacent to the other end points corresponding to its leaves. Let a_{ip} be the number of PC's which have the same root point v_{ip}. Then, since there are exactly $a_{ip}c$ points adjacent from v_{ip}, the condition (a) must hold for the adjacency matrix M. Moreover, as there are at most $\min(a_{ip}, n)$ leaf points in V_j, the condition (b) must hold for each submatrix M_{ij}.

Conversely, if (a) and (b) are satisfied by an adjacency matrix M, we can select a_{ip} sets of c ones standing on the ipth row of M in such a way that every set is composed of c ones selected at most once from the pth row of M_{ij}. This selection can be shown possible [1] and is achieved by an algorithm of the construction of $0-1$ matrix of size $a_{ip} \times m$ (cf. Corollary 1.3 and Theorem 1.1 in [4]) whose row and column sum vectors are $(c, c, \ldots, c)$ and $(s_1, s_2, \ldots, s_m)$, where $s_j = \sum_{q=1}^{n} m_{ip,jq} \le \min(a_{ip}, n)$. Since every selected set of c ones corresponds to a PC of degree c, $K_m(n, n, \ldots, n)$ is PC-decomposable.

Theorem 4.1 shows that if an adjacency matrix M of order mn which satisfies the conditions (a) and (b) can be constructed for an appropriately given set of nonnegative integers a_{ip} $(i = 1, 2, \ldots, m;\ p = 1, 2, \ldots, n)$ satisfying $\sum_{i=1}^{m} \sum_{p=1}^{n} a_{ip} = \binom{m}{2} n^2/c$, then $K_m(n, n, \ldots, n)$ can be decomposed into a union of line-disjoint PC's of degree c each. To this end, the following theorem is useful in constructing an adjacency matrix M satisfying the conditions (a) and (b), and, consequently, for the proof of the sufficiency of Theorem 2.1.

THEOREM 4.2. *Given a set of nonnegative integers a_{ip}'s satisfying $\sum_{i=1}^{m} \sum_{p=1}^{n} a_{ip} = \binom{m}{2} n^2/c$, if*

1) *an $m \times m$ nonnegative integral matrix $X = \|x_{ij}\|$ satisfying $\sum_{j=1}^{m} x_{ij} = c \sum_{p=1}^{n} a_{ip}$, $x_{ii} = 0$ and $x_{ij} + x_{ji} = n^2$ $(i \ne j)$ can be constructed,*

2) *an* $n\times m$ *nonnegative integral matrix* $Y_i=\|y_{ip,j}\|$ *satisfying*

$$\sum_{j=1}^{m} y_{ip,j} = a_{ip}c,\ \sum_{p=1}^{n} y_{ip,j} = x_{ij},\quad \text{and} \tag{4.3}$$

$$0 \le y_{ip,j} \le \min(a_{ip}, n) \tag{4.4}$$

can be constructed for every i, and finally,

3) *an* $n\times n$ $0-1$ *matrix* $M^*_{ij}=\|m_{ip,jq}\|$ *satisfying*

$$\sum_{q=1}^{n} m_{ip,jq} = y_{ip,j} \text{ and } \sum_{p=1}^{n} m_{ip,jq} = n - y_{jq,i} \tag{4.5}$$

can be constructed for every pair of i and j satisfying $1\le i<j\le m$, *then* $K_m(n, n,\dots, n)$ *can be decomposed into a union of line-disjoint* PC's *of degree c each.*

PROOF. Consider an $mn\times mn$ $0-1$ matrix $M=\|M_{ij}\|$ defined by

$$M_{ij} = \begin{cases} M^*_{ij} & \text{for } i<j,\\ 0 & \text{for } i=j,\\ G_{n,n} - M^{*T}_{ji} & \text{for } i>j,\end{cases}$$

then M is an adjacency matrix of $K_m(n, n,\dots, n)$. Moreover, since the pth row sum of the component matrix M_{ij} is $y_{ip,j}$ for every j, M satisfies the conditions (a) and (b) of Theorem 4.1.

5. Proof of the sufficiency

In a trivial case $c=1$, the conditions (i) and (ii) are obviously sufficient. Thus we consider the case $c\ge 2$.

For a set of parameters m, n and c satisfying the condition (i) of Theorem 2.1, put

$$a_{ip} = \begin{cases} a+1 & \text{for } p=1, 2,\dots, t_i,\\ a & \text{for } p=t_i+1, t_i+2,\dots, n,\end{cases} \tag{5.1}$$

for every $i=1, 2,\dots, m$, where a, d and s are nonnegative integers satisfying

$$\binom{m}{2}n^2/c = mna + md + s,\quad 0\le d<n,\quad 0\le s<m,$$

and $t_i=d+1$ or d according as $i\in\{1, 2,\dots, s\}$ or $i\in\{s+1, s+2,\dots, m\}$. Since $\sum_{i=1}^{m}\sum_{p=1}^{n} a_{ip}=\binom{m}{2}n^2/c$, the sufficiency of the remaining condition (ii) imposed on the parameters in Theorem 2.1 will be proved by showing that the matrices X,

Y_i and M^*_{ij} stated in Theorem 4.2 can be constructed for the particular set of a_{ip} given in (5.1).

5.1. Construction of X

It is sufficient to construct an $m \times m$ nonnegative integral matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{5.2}$$

composed of four submatrices X_{kl} $(k, l = 1, 2)$ which satisfy

$$\begin{gathered} X_{11} + X_{11}^T = n^2(G_{s,s} - I_s), \quad X_{22} + X_{22}^T = n^2(G_{m-s,m-s} - I_{m-s}), \\ X_{12} + X_{21}^T = n^2 G_{s,m-s}, \end{gathered} \tag{5.3}$$

and

$$[X_{11}\ X_{12}]\mathbf{j}_m = c(na + d + 1)\mathbf{j}_s, \quad [X_{21}\ X_{22}]\mathbf{j}_m = c(na + d)\mathbf{j}_{m-s}, \tag{5.4}$$

where $\mathbf{j}_t$ denotes a t-vector whose components are all unity and I_t denotes the identity matrix of order t.

(a) Case n is even. Let

$$X^* = \begin{bmatrix} X_{11} & X_{12}^* \\ X_{21}^* & X_{22} \end{bmatrix} = \begin{bmatrix} \frac{n^2}{2}(G_{s,s} - I_s) & \frac{n^2}{2} G_{s,m-s} \\ \frac{n^2}{2} G_{m-s,s} & \frac{n^2}{2}(G_{m-s,m-s} - I_{m-s}) \end{bmatrix},$$

then X^* satisfies (5.3) and its upper and lower halves of row sum vector are $[X_{11}\ X_{12}^*]\mathbf{j}_m = (m-1)\frac{n^2}{2}\mathbf{j}_s$ and $[X_{21}^*\ X_{22}]\mathbf{j}_m = (m-1)\frac{n^2}{2}\mathbf{j}_{m-s}$. Put $x = c(na+d+1) - \frac{(m-1)n^2}{2} = (m-s)\frac{c}{m}$ and $y = \frac{(m-1)n^2}{2} - c(na+d) = s\frac{c}{m}$, and let $\boldsymbol{\alpha}^T = (x, x, \dots, x)$ and $\boldsymbol{\beta}^T = (y, y, \dots, y)$. Then since $0 \le x < m-s$ and $0 \le y < s$, a 0–1 matrix B of size $s \times (m-s)$ which satisfies $B\mathbf{j}_{m-s} = \boldsymbol{\alpha}$, $B^T\mathbf{j}_s = \boldsymbol{\beta}$ can be constructed [1; 4]. Thus X^* can be adjusted to X which satisfies (5.4) in addition to (5.3) after replacing X_{12}^* and X_{21}^* by $X_{12} = X_{12}^* + B$ and $X_{21} = X_{21}^* - B^T$, respectively.

(b) Case n is odd. Consider

$$X^* = \begin{bmatrix} X_{11} & X_{12}^* \\ X_{21}^* & X_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n^2-1}{2}(G_{s,s} - I_s) + T_s & \frac{n^2+1}{2} G_{s,m-s} \\ \frac{n^2-1}{2} G_{m-s,s} & \frac{n^2-1}{2}(G_{m-s,m-s} - I_{m-s}) + T_{m-s} \end{bmatrix},$$

where the form of both T_s and T_{m-s} is either

$$T_{2h+1} = \begin{bmatrix} 0\ 1 \cdot\cdot\cdot 1\ 0 \cdot\cdot\cdot 0 \\ \cdot\ \cdot\ \cdot \qquad \cdot\ \cdot \qquad \cdot \\ \cdot \quad \cdot\ \cdot \qquad \cdot\ \cdot \quad \cdot \\ \cdot \qquad \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot \\ \cdot \qquad\quad \cdot\ \cdot \qquad \cdot\ 0 \\ 0 \cdot\cdot\cdot\cdot\cdot 0\ 1 \cdot\cdot\cdot 1 \\ 1\ \cdot \qquad\quad \cdot\ \cdot \qquad \cdot \\ \cdot\ \cdot\ \cdot \qquad \cdot\ \cdot \quad \cdot \\ \cdot \quad \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot \\ \cdot \qquad \cdot\ \cdot \qquad \cdot\ 1 \\ 1 \cdot\cdot\cdot 1\ 0 \cdot\cdot\cdot\cdot\cdot 0 \end{bmatrix} \qquad T_{2h+1}\mathbf{j}_{2h+1} = \begin{pmatrix} h \\ h \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ h \end{pmatrix}$$

or

$$T_{2h} = \begin{bmatrix} 0\ 1 \cdot\cdot\cdot 1\ 0 \cdot\cdot 0 \\ \cdot\ \cdot\ \cdot \qquad \cdot\ \cdot \quad \cdot \\ \cdot \quad \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot \\ \cdot \qquad \cdot\ \cdot \qquad \cdot\ 0 \\ \cdot \qquad\quad \cdot\ \cdot \qquad 1 \\ 0 \cdot\cdot\cdot\cdot\cdot 0\ 1 \cdot\cdot 1 \\ 1\ \cdot \qquad\quad \cdot\ \cdot \quad \cdot \\ \cdot\ \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot \\ \cdot \quad \cdot\ \cdot \qquad \cdot\ 1 \\ 1 \cdot\cdot 1\ 0 \cdot\cdot\cdot\cdot 0 \end{bmatrix} \qquad T_{2h}\mathbf{j}_{2h} = \begin{pmatrix} h \\ \cdot \\ \cdot \\ \cdot \\ h \\ h-1 \\ \cdot \\ \cdot \\ \cdot \\ h-1 \end{pmatrix} \begin{matrix} \left.\right\} h \\ \\ \left.\right\} h \end{matrix}$$

according as s and $m-s$ are odd or even, then X^* satisfies (5.3). Put

$$x = c(na+d+1) - (m-1)\frac{n^2+1}{2} + \frac{s-1}{2}$$

$$\text{or} \quad (m-1)\frac{n^2+1}{2} - \frac{s-1}{2} - c(na+d+1)$$

$$y = (m-1)\frac{n^2-1}{2} + \frac{m-s-1}{2} - c(na+d)$$

$$\text{or} \quad c(na+d) - (m-1)\frac{n^2-1}{2} - \frac{m-s-1}{2}$$

according as $m \leq 2c$ or $m \geq 2c+1$, and let

$$\boldsymbol{\alpha}^T = \begin{cases} (x, x, \ldots, x) & \text{for odd } s, \\ \Big(\underbrace{x-\frac{1}{2}, \ldots, x-\frac{1}{2}}_{s/2}, \underbrace{x+\frac{1}{2}, \ldots, x+\frac{1}{2}}_{s/2}\Big) & \text{for even } s \text{ and } m \leq 2c, \\ \Big(\underbrace{x+\frac{1}{2}, \ldots, x+\frac{1}{2}}_{s/2}, \underbrace{x-\frac{1}{2}, \ldots, x-\frac{1}{2}}_{s/2}\Big) & \text{for even } s \text{ and } m \geq 2c+1, \end{cases}$$

$$\beta^T = \begin{cases} (y, y, \ldots, y) & \text{for odd } m-s\,, \\ \Big(\underbrace{y+\frac{1}{2}, \ldots, y+\frac{1}{2}}_{(m-s)/2}, \underbrace{y-\frac{1}{2}, \ldots, y-\frac{1}{2}}_{(m-s)/2}\Big) & \text{for even } m-s \text{ and } m \le 2c\,, \\ \Big(\underbrace{y-\frac{1}{2}, \ldots, y-\frac{1}{2}}_{(m-s)/2}, \underbrace{y+\frac{1}{2}, \ldots, y+\frac{1}{2}}_{(m-s)/2}\Big) & \text{for even } m-s \text{ and } m \ge 2c+1\,, \end{cases}$$

then, since $m > c$, it can be shown that a $0-1$ matrix B of size $s \times (m-s)$ satisfying $B\mathbf{j}_{m-s} = \boldsymbol{\alpha}$ and $B^T\mathbf{j}_s = \boldsymbol{\beta}$ can be constructed [1; 4]. Put

$$\begin{cases} X_{12} = X_{12}^* + B \\ X_{21} = X_{21}^* - B^T \end{cases} \qquad \text{for } c + 1 + \frac{c-1}{n^2} \le m \le 2c, \text{ and}$$

$$\begin{cases} X_{12} = X_{12}^* - B \\ X_{21} = X_{21}^* + B^T \end{cases} \qquad \text{for} \quad m \ge 2c + 1\,,$$

then X^* can be adjusted to X in order to satisfy (5.4) in addition to (5.3).

5.2. Construction of Y_i

For a pair of given integers u_1 and u_2, let

$$\text{(5.5)} \qquad \begin{aligned} v_1 &= (a+1)c - (m-1)u_1\,, \\ v_2 &= ac - (m-1)u_2\,, \end{aligned}$$

$$\text{(5.6)} \qquad r_{ip} = \begin{cases} v_1 & p = 1, 2, \ldots, t_i\,, \\ v_2 & p = t_i + 1, t_i + 2, \ldots, n\,, \end{cases}$$

$$\text{(5.7)} \qquad s_{ij} = \begin{cases} x_{ij} - u_1 t_i - u_2(n - t_i) & j = 1, 2, \ldots, m\ (j \ne i)\,, \\ x_{ii} = 0 & j = i\,, \end{cases}$$

for every $i = 1, 2, \ldots, m$, and denote $\mathbf{r}_i^T = (r_{i1}, r_{i2}, \ldots, r_{in})$, $\mathbf{s}_i^T = (s_{i1}, s_{i2}, \ldots, s_{im})$, where t_i and x_{ij} are the same given in Section 5.1.

LEMMA 5.1. *If a $0-1$ matrix Z_i of size $n \times m$ satisfying*

$$\text{(5.8)} \qquad Z_i\mathbf{j}_m = \mathbf{r}_i \quad \textit{and} \quad Z_i^T\mathbf{j}_n = \mathbf{s}_i$$

can be constructed for appropriately chosen integers u_1 and u_2 satisfying

$$\text{(5.9)} \qquad u_1 + 1 \le \min(a+1, n) \quad \textit{and} \quad u_2 + 1 \le \min(a, n)$$

for every $i=1, 2,\ldots, m$, *then the matrices* $Y_1, Y_2,\ldots, Y_m$ *which satisfy the condition* 2) *of Theorem* 4.2, *can be constructed.*

PROOF. Let

$$Y_i^* = \begin{bmatrix} Y_{1i}^* \\ Y_{2i}^* \end{bmatrix} \qquad \text{for } i=1,2,\ldots, m$$

and put $Y_i = Y_i^* + Z_i$, where $Y_{1i}^* = u_1 G_{t_i,m}^{(i)}$, $Y_{2i}^* = u_2 G_{n-t_i,m}^{(i)}$ and $G_{h,m}^{(i)}$ denotes an $h \times m$ matrix whose elements in the ith column are all zero and others are all unity. Then, since the upper and lower halves of row sum vector of Y_i^* are $Y_{1i}^* \mathbf{j}_m = (m-1)u_1 \mathbf{j}_{t_i}$ and $Y_{2i}^* \mathbf{j}_m = (m-1)u_2 \mathbf{j}_{n-t_i}$, respectively, and the column sum vector of Y_i^* is $Y_i^{*T} \mathbf{j}_n = \{u_1 t_i + u_2(n-t_i)\} \mathbf{j}_m^{(i)}$, and since (5.5) and (5.8) hold for Z_i, Y_i satisfies (4.3), where $\mathbf{j}_m^{(i)}$ denotes an m-vector whose i th component is zero and others are all unity. Moreover, since each element in the first t_i rows of Y_i is at most u_1+1 and each element in the last $n-t_i$ rows of Y_i is at most u_2+1, (5.9) shows that Y_i satisfies (4.4).

The construction of Y_i is, therefore, reduced to the construction of Z_i which will be seen in the following:

(a) Case n is even. Put $u_1 = \frac{n}{2}$ and $u_2 = \frac{n}{2} - 1$, then u_1 and u_2 satisfy (5.9), since $a \geq \frac{n}{2}$ holds by the condition (ii) of the sufficiency. Since s_{ij} takes either $n-d-1$ or $n-d$ except $s_{ii}=0$, and since $0 < v_1 \leq m-1$, $0 < v_2 \leq m-1$ hold by condition (ii), it can be verified easily that a $0-1$ matrix Z_i satisfying (5.8) can be constructed for every $i=1, 2,\ldots, m$ [1; 4].

(b) Case n is odd. When $n=1$, it is seen easily that the matrix X itself given in Section 5.1 is an adjacency matrix M satisfying the conditions (a) and (b) in Theorem 4.1. Thus we consider the case $n \geq 3$.

Now put $r = md + s$, and three subcases with respect to r will be examined separately.

(1°) Case $0 \leq r < mn\left(1 - \frac{m-1}{2c}\right)$. Put $u_1 = \frac{n+1}{2}$ and $u_2 = \frac{n-1}{2}$, then u_1 and u_2 satisfy (5.9), since $a \geq \frac{n+1}{2}$ can be obtained from the inequality $\frac{r}{mn} < 1 - \frac{m-1}{2c}$ and the condition (ii). In this case, s_{ij} takes either $\frac{1}{2}(n-2d-3)$, $\frac{1}{2}(n-2d-1)$ or $\frac{1}{2}(n-2d+1)$ except $s_{ii}=0$, and satisfies the inequality $0 \leq s_{ij} < n$, since $d \leq \frac{n-3}{2}$ can be derived as follows:

$$d \leq \frac{r}{m} < n - \frac{n(m-1)}{2c} \leq \frac{n-1}{2} + a - \frac{n(m-1)}{2c} \leq \frac{n-1}{2}.$$

Moreover, v_1 and v_2 satisfy $0 < v_1 < m-1$ and $0 < v_2 < m-1$ by the inequality

$\frac{r}{mn}<1-\frac{m-1}{2c}$ and the condition (ii). Thus it can be verified that a $0-1$ matrix Z_i satisfying (5.8) can be constructed for every $i=1, 2,\dots, m$ [1; 4].

(2°) Case $mn\left(1-\frac{m-1}{2c}\right)\leq r\leq mn\frac{m-1}{2c}$. Put $u_1=u_2=\frac{n-1}{2}$, then u_1 and u_2 satisfy (5.9), since $a\geq\frac{n+1}{2}$ can be obtained from the inequality $r\leq mn\frac{m-1}{2c}$ and the condition (ii). In this case, $0<v_1\leq m-1$ and $0<v_2\leq m-1$ hold.

For the case $m\geq 2c+1$, s_{ij} takes either $\frac{n-1}{2}$ or $\frac{n+1}{2}$ except $s_{ii}=0$. Thus a $0-1$ matrix Z_i can be constructed for every $i=1, 2,\dots, m$ [1; 4].

For the case $c+1+\frac{c-1}{n^2}\leq m\leq 2c$. For every $i\in\{1, 2,\dots, s\}$, s_{ij} takes either $\frac{n-1}{2}$, $\frac{n+1}{2}$ or $\frac{n+3}{2}$ except $s_{ii}=0$ and the frequency of the third value is not greater than $x+\frac{1}{2}=(m-s)\frac{2c-m}{2m}+\frac{1}{2}$, where x is the same given in Section 5.1(b). It can be verified that a $0-1$ matrix Z_i can be constructed [1; 4]. For every $i\in\{s+1, s+2,\dots, m\}$, s_{ij} takes either $\frac{n-3}{2}$, $\frac{n-1}{2}$ or $\frac{n+1}{2}$ except $s_{ii}=0$ and the frequency of the third value is not greater than $\frac{m-s}{2}$. It can be verified that a $0-1$ matrix Z_i can be constructed [1; 4].

(3°) Case $mn\frac{m-1}{2c}<r<mn$. Put $u_1=\frac{n-1}{2}$ and $u_2=\frac{n-3}{2}$, then u_1 and u_2 satisfy (5.9), since $a\geq\frac{n-1}{2}$ holds by the condition (ii). In this case, s_{ij} takes either $\frac{1}{2}(3n-2d-3)$, $\frac{1}{2}(3n-2d-1)$ or $\frac{1}{2}(3n-2d+1)$ except $s_{ii}=0$ and satisfies the inequality $0\leq s_{ij}\leq n$. The latter can be verified by showing that $d\geq\frac{n+1}{2}$.

In the case $a\geq\frac{n+1}{2}$, we have $d\geq\frac{n+1}{2}$, since $d=\frac{r}{m}-\frac{s}{m}>\frac{(m-1)n}{2c}-\frac{s}{m}\geq a-\frac{s}{m}>\frac{n-1}{2}$. In the case $a=\frac{n-1}{2}$, we also have $d\geq\frac{n+1}{2}$, since we have

$$d=\frac{r}{m}-\frac{s}{m}=\frac{(m-1)n^2}{2c}-\frac{n(n-1)}{2}-\frac{s}{m}$$

$$\geq\frac{n^2}{2}\left(1+\frac{c-1}{n^2c}\right)-\frac{n(n-1)}{2}-\frac{m-1}{m}=\frac{n-1}{2}+\frac{2c-m}{2mc}>\frac{n-1}{2}$$

by using condition (ii), $s\leq m-1$ and the fact that $m<2c$ holds in the case (3°). The inequalities $0<v_1<m-1$ and $0<v_2<m-1$ can also be verified easily in this case. Thus it can be verified that a $0-1$ matrix Z_i can be constructed [1; 4].

5.3. Construction of M_{ij}^*

(a) Case n is even. In the case $n \geq 4$, $y_{ip,j}$ and $n - y_{jq,i}$ takes either $\frac{n}{2}+1$, $\frac{n}{2}$ or $\frac{n}{2}-1$ for every pair of i and j $(\neq i)$. Thus it can be shown easily from [1; 4] that a $0-1$ matrix M_{ij}^* of order n satisfying (4.5) can be constructed for every pair of i and j satisfying $1 \leq i < j \leq m$.

In the case $n=2$, a little consideration is needed. When $d=0$, the inequality $ac - x \geq [s/2]$ holds, where x is the same in Section 5.1(a) and $[t]$ is the greatest integer not exceeding t. Thus, we can construct a matrix Y_i in a way such that $y_{i2,j}=1$ for j $(1 \leq j \leq s)$ satisfying $j-1 \equiv i, i+1, \ldots, i+[s/2]-1 \bmod s$ for every $i \in \{1, 2, \ldots, s\}$. When $d=1$, the inequality $ac \geq [(m-s)/2]$ holds. Thus, we can construct a matrix Y_i in a way such that $y_{i2,j}=1$ for j $(s+1 \leq j \leq m)$ satisfying $j-1 \equiv i, i+1, \ldots, i+[(m-s)/2]-1 \bmod m-s$ for every $i \in \{s+1, \ldots, m\}$. Hence, we can exclude the case in which both $(y_{i1,j}, y_{i2,j})$ and $(2-y_{j1,i}, 2-y_{j2,i})$ are either (0, 2) or (2, 0). We can construct M_{ij}^* for all $i<j$.

(b) Case n is odd. In the case $0 \leq r < mn\left(1 - \frac{m-1}{2c}\right)$, $y_{ip,j}$ takes either $\frac{n+3}{2}$, $\frac{n+1}{2}$ or $\frac{n-1}{2}$ and $n - y_{jq,i}$ takes either $\frac{n+1}{2}$, $\frac{n-1}{2}$ or $\frac{n-3}{2}$. In the case $mn\left(1 - \frac{m-1}{2c}\right) \leq r \leq mn\frac{m-1}{2c}$, $y_{ip,j}$ and $n - y_{jq,i}$ takes either $\frac{n+1}{2}$ or $\frac{n-1}{2}$. In the case $mn\frac{m-1}{2c} < r < mn$, $y_{ip,j}$ takes either $\frac{n+1}{2}$, $\frac{n-1}{2}$ or $\frac{n-3}{2}$ and $n - y_{jq,i}$ takes either $\frac{n+3}{2}$, $\frac{n+1}{2}$ or $\frac{n-1}{2}$. In any one of the above cases, it can be shown easily from [1; 4] that a $0-1$ matrix M_{ij}^* of order n satisfying (4.5) can be constructed for every pair i and j satisfying $1 \leq i < j \leq m$.

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Hiroshima College of Economics,
Hiroshima 731–01, *JAPAN,*
Niihama, Technical College,
Niihama, Ehime 792, *JAPAN*
and
Department of Mathematics,
Faculty of Science,
Hiroshima University,
Hiroshima 730, *JAPAN*