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On the Trace Mappings for the Space $B_{p,\mu}(R^N)$

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1. Introduction

By $H^{\mu}(R^N)$ we shall understand the space of $u \in \mathscr{S}'(R^N)$ such that its Fourier transform $\hat{u}$ is a locally summable function satisfying

$$\|u\|_{\mu}^2 = \left(\frac{1}{2\pi}\right)^N \int_{\Xi^N} |\hat{u}(\xi)|^2 \mu^2(\xi) d\xi < \infty,$$

where R^N is an N-dimensional Euclidean space, Ξ^N its dual Euclidean space and μ is a temperate weight function in Ξ^N. In our previous paper [2] we have given a trace theorem for the space $H^{\mu}(R^N)$. Let $\mu = \mu(\xi', \tau)$, $\xi' = (\xi_1, \dots, \xi_n)$, $\tau = (\tau_1, \dots, \tau_m)$, $N = n + m$ and assume $\int_{\Xi^m} \frac{|\tau|^{2M}}{\mu^2(\xi', \tau)} d\tau < \infty$ for a non-negative integer M. Put $\nu_k(\xi') = \left\{\int_{\Xi^m} \frac{\tau^{2k}}{\mu^2(\xi', \tau)} d\tau\right\}^{-1/2}$ for $k = (k_1, \dots, k_m)$, k_j being a non-negative integer, such that $|k| \leqq M$. Then the mapping

$$H^{\mu}(R^N) \ni u \longrightarrow \{D_t^k u(x', 0)\} \in \prod_{|k| \leq M} H^{\nu_k}(R^n)$$

is an epimorphism if and only if there exists a positive constant C such that $\det|\kappa_{k+l}| \geqq C \prod_{|k| \leq M} \kappa_{2k}$ with $\kappa_k(\xi') = \int_{\Xi^m} \frac{\tau^k}{\mu^2(\xi', \tau)} d\tau$.

The purpose of this paper is to investigate the trace mappings for the space $B_{p,\mu}(R^N)$, $1 < p < \infty$, which consists of all distributions $u \in \mathscr{S}'(R^N)$ such that $\hat{u}$ is a function and

$$\|u\|_{p,\mu}^p = \left(\frac{1}{2\pi}\right)^N \int_{\Xi^N} |\hat{u}(\xi)|^p \mu^p(\xi) d\xi < \infty.$$

Here $B_{2,\mu}(R^N) = H^{\mu}(R^N)$. We shall give some sufficient conditions for the trace mapping of above type for $B_{p,\mu}(R^N)$ to be an epimorphism. We shall also investigate the trace mappings by making a comparison with the notions of multiplication of distributions and section of distributions.

2. Preliminaries

We shall use the same notations and terminologies as in our previous paper

[2]. For any points $x=(x_1,\dots,x_N)\in R^N$ and $\xi=(\xi_1,\dots,\xi_N)\in \Xi^N$ the scalar product is defined by $<x,\xi>=\sum_j x_j\xi_j$ and the lengths of x, ξ are defined by $|x|=(\sum_j|x_j|^2)^{1/2}$, $|\xi|=(\sum_j|\xi_j|^2)^{1/2}$. For an N-tuple $\alpha=(\alpha_1,\dots,\alpha_N)$ of non-negative integers α_j, we put $|\alpha|=\sum_j\alpha_j$ and $\alpha!=\alpha_1!\cdots\alpha_N!$. With $D=(D_1,\dots,D_N)$, $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$, we put $D^\alpha=D_1^{\alpha_1}\cdots D_N^{\alpha_N}$ and similarly $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_N^{\alpha_N}$. For a polynomial $P(\xi)=\sum_\alpha a_\alpha\xi^\alpha$ in ξ, we put $P(D)=\sum_\alpha a_\alpha D^\alpha$, $\bar{P}(\xi)=\sum_\alpha\bar{a}_\alpha\xi^\alpha$ and $P^{(\alpha)}=i^{|\alpha|}D^\alpha P$.

For any rapidly decreasing C^∞-function $\phi\in\mathscr{S}(R^N)$, its Fourier transform $\hat{\phi}$ is defined by the formula

$$\hat{\phi}(\xi)=\int_{R^N}\phi(x)e^{-i<x,\xi>}dx$$

and for any temperate distribution $u\in\mathscr{S}'(R^N)$, its Fourier transform $\hat{u}$ is defined by

$$<\hat{u},\phi>=<u,\hat{\phi}>,\quad {}^\forall\phi\in\mathscr{S}(R^N),$$

where $<\cdot,\cdot>$ means the dualform between $\mathscr{S}'(R^N)$ and $\mathscr{S}(R^N)$. We shall use the same notation also for the dualform between $\mathscr{D}'(R^N)$ and $\mathscr{D}(R^N)$.

A positive-valued continuous function μ defined in Ξ^N will be called a temperate weight function if there exist positive constants C and k such that

$$\mu(\xi+\eta)\leqq C(1+|\xi|^k)\mu(\eta),\quad {}^\forall\xi,\ \eta\in\Xi^N.$$

$\mu_1+\mu_2$, $\mu_1\mu_2$, $\frac{1}{\mu_1}$ are temperate weight functions with μ_1 and μ_2. By $B_{p,\mu}(R^N)$, $1<p<\infty$, we shall understand the space of $u\in\mathscr{S}'(R^N)$ such that $\hat{u}$ is a function satisfying $\|u\|_{p,\mu}<\infty$. The discussion on the space $B_{p,\mu}(R^N)$ has been in full detail in L. Hörmander [1] and in L. R. Volevich and B. P. Paneyakh [8]. $B_{p,\mu}(R^N)$ is a Banach space with the norm $\|u\|_{p,\mu}$ and $\mathscr{S}(R^N)\subset B_{p,\mu}(R^N)\subset\mathscr{S}'(R^N)$ in the topological sense. $\mathscr{D}(R^N)$ is dense in $B_{p,\mu}(R^N)$ [1, p. 37]. The strong dual of $B_{p,\mu}(R^N)$ is $B_{p',1/\mu}(R^N)$, $\frac{1}{p}+\frac{1}{p'}=1$ [1, p. 42] and we have

$$<w,\bar{u}>=\left(\frac{1}{2\pi}\right)^N\int_{\Xi^N}\hat{w}(\xi)\overline{\hat{u}(\xi)}d\xi$$

for any $u\in B_{p,\mu}(R^N)$ and $w\in B_{p',1/\mu}(R^N)$.

Let $N=n+m$. We shall use the notations: $x=(x',t)\in R^N$, $x'=(x_1,\dots,x_n)$, $t=(t_1,\dots,t_m)$ and $\xi=(\xi',\tau)\in\Xi^N$, $\xi'=(\xi_1,\dots,\xi_n)$, $\tau=(\tau_1,\dots,\tau_m)$. For any temperate weight function μ in Ξ^N, the integral $\nu(\xi')=\int_{\Xi^m}\mu(\xi)d\tau$ diverges for every point $\xi'\in\Xi^n$, or converges for every point $\xi'\in\Xi^n$ and it is a temperate weight function in Ξ^n [8, p. 10]. For any $\phi\in\mathscr{S}(R^N)$ the partial Fourier transforms $\hat{\phi}_{x'}$

and $\hat{\phi}_t$ are defined by the formulas

$$\hat{\phi}_{x'}(\xi', t) = \int_{R^n} \phi(x', t) e^{-i<x',\xi'>} dx', \quad \hat{\phi}_t(x', \tau) = \int_{R^m} \phi(x', t) e^{-i<t,\tau>} dt$$

and for any $u \in \mathscr{S}'(R^N)$ its partial Fourier transforms are defined by

$$<\hat{u}_{x'}, \phi> = <u, \hat{\phi}_{x'}>, \quad <\hat{u}_t, \phi> = <u, \hat{\phi}_t>, \quad \forall \phi \in \mathscr{S}(R^N).$$

For any $u = u(x', t) \in \mathscr{D}(R^N)$ the trace $u(x', 0)$ on R^n clearly belongs to the space $\mathscr{D}(R^n) \subset \mathscr{D}'(R^n)$. $\mathscr{D}(R^N)$ is dense in $B_{p,\mu}(R^N)$. If the mapping $u \to u(x', 0)$ can be continuously extended to the mapping from $B_{p,\mu}(R^N)$ into $\mathscr{D}'(R^n)$ with weak topology, then the extended mapping is called the trace mapping on R^n and the image of $u \in B_{p,\mu}(R^N)$ is called the trace of u and denoted by $u(x', 0)$. From its very definition, the trace mapping is defined if and only if $\phi \otimes \delta \in B_{p',1/\mu}(R^N)$ for any $\phi \in \mathscr{D}(R^n)$, where δ is the Dirac measure in R^m. From this fact we see that the trace mapping is defined if and only if $1/\mu(0, \tau) \in L^{p'}(R^m)$. From the equations

$$u(x', 0) = \frac{1}{(2\pi)^N} \iint_{\Xi^N} \hat{u}(\xi', \tau) e^{i<x',\xi'>} d\xi' d\tau = \frac{1}{(2\pi)^m} \int_{\Xi^m} \hat{u}_t(x', \tau) d\tau$$

for any $u \in \mathscr{S}(R^N)$, we obtain

$$\widehat{u(x', 0)}(\xi') = \frac{1}{(2\pi)^m} \int_{\Xi^m} \hat{u}(\xi', \tau) d\tau,$$

which remains valid for any $u \in B_{p,\mu}(R^N)$.

3. Trace mappings

LEMMA 1. *For any non-trivial polynomial $P(\xi)$, the function $\tilde{P}_p$ defined by*

$$\tilde{P}_p(\xi) = (\sum_{|\alpha| \geq 0} |P^{(\alpha)}(\xi)|^p)^{1/p}$$

is a temperate weight function in Ξ^N.

PROOF. From Taylor's formula

$$P^{(\alpha)}(\xi + \eta) = \sum_{|\beta| \geq 0} \frac{\xi^\beta}{\beta!} P^{(\alpha+\beta)}(\eta),$$

it follows immediately that

$$\sum_{|\alpha| \geq 0} |P^{(\alpha)}(\xi + \eta)|^p \leq C(\sum_{|\alpha| \geq 0} |P^{(\alpha)}(\eta)|^p)(1 + |\xi|^{Mp}),$$

where M is the degree of P and C is a constant depending only on M and N. Thus we obtain

$$\tilde{P}_p(\xi+\eta) \leqq C'\tilde{P}_p(\eta)(1+|\xi|^M)$$

with a constant C'.

PROPOSITION 1. *Let P be a non-trivial polynomial of $\xi=(\xi', \tau)$ and μ a temperate weight function in Ξ^N. A necessary and sufficient condition that the trace mapping $B_{p,\mu}(R^N)\ni u\to[P(D)u](x',0)\in\mathscr{D}'(R^n)$ may be defined is that one of the following conditions is satisfied:*

(1) $\dfrac{1}{\mu_{\tilde{P},p'}(\xi')}=\left\{\displaystyle\int_{\Xi^m}\frac{\tilde{P}_{p'}^{p'}(\xi',\tau)}{\mu^{p'}(\xi',\tau)}d\tau\right\}^{1/p'}<\infty$ *for some point* $\xi'\in\Xi^n$.

(2) $\displaystyle\int_{\Xi^m}\frac{|P(\xi',\tau)|^{p'}}{\mu^{p'}(\xi',\tau)}d\tau<\infty$ *for every point* $\xi'\in\Xi^n$.

In this case $[P(D)u](x',0)\in B_{p,\mu_{\tilde{P},p'}}(R^n)$.

Furthermore, $[P(D)u](x',0)\in B_{p,\nu}(R^n)$ *for every* $u\in B_{p,\mu}(R^n)$ *if and only if one of the following conditions is satisfied:*

(3) $\nu(\xi')\leqq C_1\mu_{\tilde{P},p'}(\xi')$ *with a constant* C_1.

(4) $\nu^{p'}(\xi')\displaystyle\int_{\Xi^m}\frac{|P(\xi',\tau)|^{p'}}{\mu^{p'}(\xi',\tau)}d\tau\leqq C_2$ *with a constant* C_2.

PROOF. We suppose the trace mapping $B_{p,\mu}(R^N)\ni u\to[P(D)u](x',0)\in\mathscr{D}'(R^n)$ is defined. For any $\eta\in\Xi^N$ the mapping $u\to e^{i<x,\eta>}u$ is continuous from $B_{p,\mu}(R^N)$ into itself and we have $P(D)e^{i<x,\eta>}u=e^{i<x,\eta>}P(D+\eta)u$. For any $\phi\in\mathscr{D}(R^n)$ the map

$$u\longrightarrow<[P(D+\eta)u](x',0),\phi>=<u,\bar{P}(D+\eta)(\phi\otimes\delta)>$$

is a continuous linear form on $B_{p,\mu}(R^N)$ and therefore

$$\bar{P}(D+\eta)(\phi\otimes\delta)\in(B_{p,\mu}(R^N))'=B_{p',1/\mu}(R^N),$$

which implies

$$\frac{\bar{P}(\xi+\eta)\hat{\phi}(\xi')}{\mu(\xi)}=\frac{\hat{\phi}(\xi')}{\mu(\xi)}\sum_{|\alpha|\geq 0}\frac{\eta^\alpha}{\alpha!}\bar{P}^{(\alpha)}(\xi)\in L^{p'}(\Xi^N).$$

From the fact that $\{\eta^\alpha\}$ is linearly independent, it follows that $\hat{\phi}(\xi')\bar{P}^{(\alpha)}(\xi)/\mu(\xi)\in L^{p'}(\Xi^N)$ for each α and then

$$\int_{\Xi^n}|\hat{\phi}(\xi')|^{p'}d\xi'\int_{\Xi^m}\frac{\tilde{P}_{p'}^{p'}(\xi',\tau)}{\mu^{p'}(\xi',\tau)}d\tau<\infty.$$

As a result,

$$\int_{\Xi^m} \frac{\tilde{P}_{p'}^{p'}(\xi', \tau)}{\mu^{p'}(\xi', \tau)} d\tau < \infty \qquad \text{for a.e.} \quad \xi' \in \Xi^n.$$

Since μ and $\tilde{P}_{p'}$ are temperate weight functions, the integral exists for every $\xi' \in \Xi^n$ and it is a temperate weight function in Ξ^n.

The implication (1)⇒(2) is trivial.

Suppose (2) holds. From the equation

$$\widehat{[P(D)u](x', 0)}(\xi') = \frac{1}{(2\pi)^m} \int_{\Xi^m} P(\xi)\hat{u}(\xi) d\tau$$

for any $u \in \mathscr{D}(R^N)$ we have for any $\phi \in \mathscr{D}(R^n)$

$$|<[P(D)u](x', 0), \bar{\phi}>| = \frac{1}{(2\pi)^N} \left| \int_{\Xi^N} P(\xi)\hat{u}(\xi)\overline{\hat{\phi}(\xi')} d\xi \right|$$

$$\leqq \frac{1}{(2\pi)^N} \left[\int_{\Xi^n} |\hat{\phi}(\xi')|^{p'} \left\{ \int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right\} d\xi' \right]^{1/p'} \left[\int_{\Xi^N} |\hat{u}(\xi)|^p \mu^p(\xi) d\xi \right]^{1/p}.$$

Since $\mathscr{D}(R^N)$ is dense in $B_{p,\mu}(R^N)$, it suffices to show that $\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau$ is a slowly increasing function of ξ'. From Taylor's formula $P(\xi) = \sum_{|\alpha'| \geq 0} \frac{\xi'^{\alpha'}}{\alpha'!} P^{(\alpha')}(0, \tau)$ and (2) it follows that the integral $\int_{\Xi^m} \frac{|P^{(\alpha')}(0, \tau)|^{p'}}{\mu^{p'}(0, \tau)} d\tau$ exists. Since $\mu(0, \tau) \leqq C(1 + |\xi'|^k)\mu(\xi', \tau)$ with constants k and C we have

$$\int_{\Xi^m} \frac{|P^{(\alpha')}(0, \tau)|^{p'}}{\mu^{p'}(\xi', \tau)} d\tau \leqq C^{p'}(1 + |\xi'|^k)^{p'} \int_{\Xi^m} \frac{|P^{(\alpha')}(0, \tau)|^{p'}}{\mu^{p'}(0, \tau)} d\tau.$$

Thus the trace mapping $B_{p,\mu}(R^N) \ni u \to [P(D)u](x', 0) \in \mathscr{D}'(R^n)$ is defined.

Furthermore, we have for any $u \in \mathscr{D}(R^N)$

$$\|[P(D)u](x', 0)\|_{p,\mu_{\tilde{P},p'}}^p = \left(\frac{1}{2\pi}\right)^{n+pm} \int_{\Xi^n} \mu_{\tilde{P},p'}^p(\xi') \left| \int_{\Xi^m} P(\xi)\hat{u}(\xi) d\tau \right|^p d\xi'$$

$$\leqq \left(\frac{1}{2\pi}\right)^{n+pm} \int_{\Xi^n} \mu_{\tilde{P},p'}^p(\xi') \left(\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right)^{p/p'} \left(\int_{\Xi^m} |\hat{u}|^p \mu^p d\tau \right) d\xi'$$

$$\leqq \left(\frac{1}{2\pi}\right)^{n+pm} \int_{\Xi^n} \mu_{\tilde{P},p'}^p(\xi') \left(\int_{\Xi^m} \frac{\tilde{P}_{p'}^{p'}}{\mu^{p'}} d\tau \right)^{p/p'} \left(\int_{\Xi^m} |\hat{u}|^p \mu^p d\tau \right) d\xi'$$

$$= \left(\frac{1}{2\pi}\right)^{(p-1)m} \|u\|_{p,\mu}^p.$$

Since $\mathscr{D}(R^N)$ is dense in $B_{p,\mu}(R^N)$, $[P(D)u](x', 0) \in B_{p,\mu_{\tilde{P},p'}}(R^n)$ for any $u \in B_{p,\mu}(R^N)$.

Suppose the trace $[P(D)u](x', 0) \in B_{p,\nu}(R^n)$ for any $u \in B_{p,\mu}(R^N)$. As we have shown before $\hat{\phi}(\xi')\tilde{P}_{p'}(\xi')/\mu(\xi) \in L^{p'}(\Xi^N)$ for any $\phi \in B_{p',1/\nu}(R^n)$, that is,

$$\infty > \int_{\Xi^N} |\hat{\phi}(\xi')|^{p'} \frac{\tilde{P}_{p'}^{p'}(\xi)}{\mu^{p'}(\xi)} d\xi' d\tau$$

$$= \int_{\Xi^n} \frac{|\phi(\xi')|^{p'}}{\nu^{p'}(\xi')} \left(\nu^{p'}(\xi') \int_{\Xi^m} \frac{\tilde{P}_{p'}^{p'}(\xi)}{\mu^{p'}(\xi)} d\tau \right) d\xi'.$$

Thus we have for some constant $C_1 > 0$

$$\nu^{p'}(\xi') \int_{\Xi^m} \frac{\tilde{P}_{p'}^{p'}(\xi)}{\mu^{p'}(\xi)} d\tau \leq C_1^{p'} \qquad \text{a.e.} \quad \xi' \in \Xi^n.$$

Since ν, μ and $\tilde{P}_{p'}$ are temperate weight functions, the estimate remains valid for every $\xi' \in \Xi^n$ and therefore we have $\nu(\xi') \leqq C_1 \mu_{\tilde{P},p'}(\xi')$ for all $\xi' \in \Xi^n$.

Clearly (3) implies (4).

Suppose (4) holds. For any $u \in \mathscr{D}(R^N)$ we have

$$\|[P(D)u](x', 0)\|_{p,\nu}^p = \left(\frac{1}{2\pi}\right)^{n+pm} \int_{\Xi^n} \nu^p(\xi') \left| \int_{\Xi^m} P(\xi)\hat{u}(\xi) d\tau \right|^p d\xi'$$

$$\leqq \left(\frac{1}{2\pi}\right)^{n+pm} \int_{\Xi^n} \nu^p(\xi') \left(\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right)^{p/p'} \left(\int_{\Xi^m} |\hat{u}|^p \mu^p d\tau \right) d\xi'$$

$$\leqq \left(\frac{1}{2\pi}\right)^{n+pm} C_2^p \int_{\Xi^N} |\hat{u}|^p \mu^p d\xi$$

$$= \left(\frac{1}{2\pi}\right)^{m(p-1)} C_2^p \|u\|_{p,\mu}^p$$

and therefore

$$\|[P(D)u]\,(x', 0)\|_{p,\nu} \leqq \left(\frac{1}{2\pi}\right)^{m/p'} C_2 \|u\|_{p,\mu}.$$

Since $\mathscr{D}(R^N)$ is dense in $B_{p,\mu}(R^N)$, the estimate holds true for any $u \in B_{p,\mu}(R^N)$.

PROPOSITION 2. *Suppose the integral*

$$\frac{1}{\mu_{\tilde{P},p'}(\xi')} = \left\{ \int_{\Xi^m} \frac{\tilde{P}_{p'}^{p'}(\xi', \tau)}{\mu^{p'}(\xi', \tau)} d\tau \right\}^{1/p'}$$

is finite. A necessary and sufficient condition that the trace mapping

$$\mathscr{T}\colon B_{p,\mu}(R^N) \ni u \longrightarrow [P(D)u](x', 0) \in B_{p,\mu_{\tilde{P},p'}}(R^n)$$

may be an epimorphism is that one of the following conditions is satisfied:

(1) *The range of the transposed map ${}^t\mathscr{T}$ is closed in $B_{p',1/\mu}(R^N)$.*

(2) $v_{p'}(\xi')=\left\{\int_{\Xi^m}\frac{|P(\xi',\tau)|^{p'}}{\mu^{p'}(\xi',\tau)}d\tau\right\}^{-1/p'}$ *is a temperate weight function in* Ξ^n.

(3) *If* $f(\xi')\bar{P}(\xi)/\mu(\xi)\in L^{p'}(\Xi^N)$ *with a locally integrable function* $f(\xi')$, *then* $f/\mu_{\bar{P},p'}\in L^{p'}(\Xi^n)$.

In this case, $v_{p'}\sim\mu_{\bar{P},p'}$, *namely, there exist two positive constants* C_1, C_2 *such that* $C_1v_{p'}\leqq\mu_{\bar{P},p'}\leqq C_2v_{p'}$.

PROOF. For any $f\in\mathscr{D}(R^N)$ and $v\in\mathscr{D}(R^n)$ we have

$$<\mathscr{T}f,\bar{v}>=\left(\frac{1}{2\pi}\right)^n\int_{\Xi^n}\widehat{[P(D)f](x',0)}(\xi')\overline{\hat{v}(\xi')}d\xi'$$

$$=\left(\frac{1}{2\pi}\right)^N\int_{\Xi^N}P(\xi)\hat{f}(\xi)\overline{\hat{v}(\xi')}d\xi$$

and

$$<\overline{{}^t\mathscr{T}v},f>=\left(\frac{1}{2\pi}\right)^N\int_{\Xi^N}\overline{\widehat{{}^t\mathscr{T}v}(\xi)}\hat{f}(\xi)d\xi.$$

Since $\mathscr{D}(R^n)$ is dense in $B_{p',1/\mu_{\bar{P},p'}}(R^n)$, we have $\widehat{{}^t\mathscr{T}v}(\xi)=\hat{v}(\xi')\bar{P}(\xi)$ for any $v\in B_{p',1/\mu_{\bar{P},p'}}(R^n)$. If ${}^t\mathscr{T}v=0$, then

$$\int_{\Xi^n}|\hat{v}(\xi')|^{p'}\left(\int_{\Xi^m}\frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)}d\tau\right)d\xi'=0$$

and therefore $\hat{v}(\xi')=0$ a.e. in Ξ^n, which implies $v=0$. Thus the map $\mathscr{T}$ is an epimorphism if and only if the range of ${}^t\mathscr{T}$ is closed in $B_{p',1/\mu}(R^N)$.

Suppose (1) holds. Then there exists a constant C such that

$$\|v\|_{p',1/\mu_{\bar{P},p'}}\leqq C\|{}^t\mathscr{T}v\|_{p',1/\mu}$$

for any $v\in B_{p',1/\mu_{\bar{P},p'}}(R^n)$, and hence

$$\int_{\Xi^m}|\hat{v}(\xi')|^{p'}\frac{1}{\mu_{\bar{P},p'}^{p'}(\xi')}d\xi'\leqq C^{p'}\int_{\Xi^n}|\hat{v}(\xi')|^{p'}\left(\int_{\Xi^m}\frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)}d\tau\right)d\xi',$$

which implies $1/\mu_{\bar{P},p'}\leqq C/v_{p'}(\xi')\leqq C/\mu_{\bar{P},p'}$. Thus $v_{p'}$ is a temperate weight function in Ξ^n and $v_{p'}\sim\mu_{\bar{P},p'}$.

Suppose (2) holds. We shall first note that $v_{p'}\sim\mu_{\bar{P},p'}$. For any $\eta\in\Xi^N$ with $|\eta|\leqq 1$ there exist two positive constants C_1, C_2 such that

$$\frac{C_1}{v_{p'}^{p'}(\xi')}\geqq\frac{1}{v_{p'}^{p'}(\xi'+\eta')}=\int_{\Xi^m}\frac{|P(\xi+\eta)|^{p'}}{\mu^{p'}(\xi+\eta)}d\tau\geqq C_2\int_{\Xi^m}\frac{|P(\xi+\eta)|^{p'}}{\mu^{p'}(\xi)}d\tau.$$

By Taylor's formula $P(\xi+\eta)=\sum_{|\alpha|\geq 0}\frac{\eta^\alpha}{\alpha!}P^{(\alpha)}(\xi)$ we can find a positive constant

C_3 such that

$$\frac{1}{v_{p'}^{p'}(\xi')} \geqq C_3 \int_{\Xi^m} \frac{|P^{(\alpha)}(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau$$

and therefore $v_{p'} \sim \mu_{\tilde{P},p'}$. Suppose $f(\xi')\bar{P}(\xi)/\mu(\xi) \in L^{p'}(\Xi^N)$ with $f(\xi') \in L^1_{loc}(\Xi^n)$. From the relations

$$\int_{\Xi^n} |f(\xi')|^{p'} \frac{1}{v_{p'}^{p'}(\xi')} d\xi' = \int_{\Xi^N} |f(\xi')|^{p'} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\xi < \infty$$

and $v_{p'} \sim \mu_{\tilde{P},p'}$ we obtain $f/\mu_{\tilde{P},p'} \in L^{p'}(\Xi^n)$.

Suppose (3) holds. We shall first show that $\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau > 0$ for any $\xi' \in \Xi^n$. Let ξ_0' be any point of Ξ^n and B be a closed unit ball with center at ξ_0'. Let E be the set of $f \in L^1_{loc}(\Xi^n)$ such that $\operatorname{supp} f \subset B$ and

$$\int_{\Xi^n} |f(\xi')|^{p'} \left(\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right) d\xi' < \infty.$$

Then E is a Banach space with the norm $\|f\|_E$:

$$\|f\|_E^{p'} = \left\{ \int_B |f(\xi')| \, d\xi' \right\}^{p'} + \int_B |f(\xi')|^{p'} \left(\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right) d\xi'.$$

Let $f \in E$. Then $f \in L^1_{loc}(\Xi^n)$ and $f(\xi')\bar{P}(\xi)/\mu(\xi) \in L^{p'}(\Xi^N)$ and therefore $f/\mu_{\tilde{P},p'} \in L^{p'}(\Xi^n)$ by (3). By the closed graph theorem the map $f \to f/\mu_{\tilde{P},p'}$ is continuous from E into $L^{p'}(\Xi^n)$. Let B_ε be a closed ball with center at $\xi_0' \in \Xi^n$ and radius ε, $0 < \varepsilon < 1$. If we take the characteristic function f on B_ε, then there exists a positive constant C independent of ε (depending on ξ_0) such that

$$C \leqq |B_\varepsilon|^{p'-1} + \frac{1}{|B_\varepsilon|} \int_{B\varepsilon} \left(\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right) d\xi',$$

where $|B_\varepsilon|$ stands for the Lebesgue measure of B_ε. Passing to the limit $\varepsilon \to 0$, we have

$$0 < C \leqq \int_{\Xi^m} \frac{|P(\xi_0', \tau)|^{p'}}{\mu^{p'}(\xi_0', \tau)} d\tau.$$

Let us now show that the range of ${}^t\mathscr{T}$ is closed in $B_{p',1/\mu}(R^N)$. Let $\{v^j(\xi')\}$ be any sequence of $B_{p',1/\mu_{\tilde{P},p'}}(R^n)$ such that ${}^t\mathscr{T}v^j$ tends to u in $B_{p',1/\mu}(R^N)$. Then $\widehat{{}^t\mathscr{T}v^j} = \hat{v}^j(\xi')\bar{P}(\xi)/\mu(\xi)$ tends to $\hat{u}/\mu$ in $L^{p'}(\Xi^N)$ and therefore $\hat{v}^j(\xi') \left\{ \int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau \right\}^{1/p'}$ is a Cauchy sequence in $L^{p'}(\Xi^n)$. From the fact that $\int_{\Xi^m} \frac{|P(\xi)|^{p'}}{\mu^{p'}(\xi)} d\tau > 0$ for $\xi' \in \Xi^n$, $\hat{v}^j(\xi')$ tends to $f(\xi')$ in $L^1_{loc}(\Xi^n)$ and we can write $\hat{u} = f(\xi')\bar{P}(\xi)$. From

$\hat{u}/\mu = f(\xi')\bar{P}(\xi)/\mu(\xi) \in L^{p'}(\Xi^N)$ and (3) we obtain $f/\mu_{\tilde{P},p'} \in L^{p'}(\Xi^n)$, that is, $f \in B_{p',1/\mu_{\tilde{P},p'}}(R^n)$. Thus u belongs to the range of ${}^t\mathcal{T}$, which completes the proof.

If $P(D)$ is a polynomial in D_t and

$$\frac{1}{\nu_{p'}(\xi')} = \left\{\int_{\Xi^m} \frac{|P(\tau)|^{p'}}{\mu^{p'}(\xi', \tau)} d\tau\right\}^{1/p'} < \infty,$$

then $\nu_{p'}$ is a temperate weight function in Ξ^n. By virtue of Proposition 2 the trace mapping $u \to [P(D_t)u](x', 0)$ from $B_{p,\mu}(R^N)$ into $B_{p,\nu_{p'}}(R^n)$ is an epimorphism in this case.

4. Trace theorems

Let $\mu = \mu(\xi', \tau)$ be a temperate weight function defined in Ξ^N. We assume that for some non-negative integer M

$$\int_{\Xi^m} \frac{|\tau|^{p'M}}{\mu^{p'}(\xi', \tau)} d\tau < \infty.$$

For any $k = (k_1, \dots, k_m)$, k_j being a non-negative integer, such that $|k| \leqq M$ we put

$$\nu_{k,p'}(\xi') = \left\{\int_{\Xi^m} \frac{|\tau^k|^{p'}}{\mu^{p'}(\xi', \tau)} d\tau\right\}^{-1/p'}$$

and consider the trace mapping $\mathcal{T}$:

$$B_{p,\mu}(R^N) \ni u \longrightarrow \{D_t^k u(x', 0)\} \in \prod_{|k| \leq M} B_{p,\nu_{k,p'}}(R^n).$$

For $p=2$ we have already obtained the following theorem and its corollary in our previous paper [2].

THEOREM 1. *A necessary and sufficient condition that the mapping* $\mathcal{T}$: $B_{2,\mu}(R^N) \ni u \to \{D_t^k u(x', 0)\} \in \prod_{|k| \leq M} B_{2,\nu_{k,2}}(R^n)$ *may be an epimorphism is that one of the following conditions is satisfied*:

(1) *The range of the transposed mapping* ${}^t\mathcal{T}$ *is closed in* $B_{2,1/\mu}(R^N)$.

(2) *There exists a positive constant* C *such that* $\det|\kappa_{k+l}| \geqq C \sum_{|k| \leq M} \kappa_{2k}$, *where* $\kappa_k(\xi') = \int_{\Xi^m} \frac{\tau^k}{\mu^2(\xi', \tau)} d\tau$.

(3) *If* $u \in B_{2,1/\mu}(R^N)$ *and* $\hat{u}(\xi) = \sum_{|k| \leq M} f_k(\xi')\tau^k$, *then* $f_k/\nu_{k,2} \in L^2(\Xi^n)$ *for* $|k| \leqq M$.

(4) *If* $u \in B_{2,1/\mu}(R^N)$ *and* $\hat{u}(\xi) = \sum_{|k| \leq M} f_k(\xi')\tau^k$, *then*

$$\hat{u}(\xi', \tau_1, \dots, \tau_{j-1}, \frac{\tau_j}{2}, \tau_{j+1}, \dots, \tau_m)/\mu(\xi) \in L^2(\Xi^N)$$

for $j=1, 2,\ldots, m$.

COROLLARY. *If* $\mu(\xi', \tau_1,\ldots, \tau_{j-1}, 2\tau_j, \tau_{j+1},\ldots, \tau_m) \geqq C\mu(\xi)$, *C being a positive constant, for* $j=1, 2,\ldots, m$, *then the mapping* $\mathcal{T}$ *considered in the theorem is an epimorphism.*

For any p with $1<p<\infty$ we have

THEOREM 2. *The trace mapping* $\mathcal{T}: B_{p,\mu}(R^N) \ni u \to \{D_t^k u(x', 0)\} \in \prod_{|k|\leq M} B_{p,\nu_{k,p'}}(R^n)$ *is an epimorphism if and only if the range of the transposed mapping* ${}^t\mathcal{T}$ *is closed in* $B_{p',1/\mu}(R^N)$.

PROOF. We first note that the transposed image ${}^t\mathcal{T}\vec{v}$ of $\vec{v}=\{v_k\} \in \prod_{|k|\leq M} B_{p',1/\nu_{k,p'}}(R^n)$ has the form

$$\widehat{{}^t\mathcal{T}\vec{v}}(\xi) = \sum_{|k|\leq M} \hat{v}_k(\xi')\tau^k.$$

It is sufficient to show this relation when $v_k \in \mathcal{D}(R^n)$. For any $f \in \mathcal{D}(R^N)$ we have

$$\begin{aligned} <\mathcal{T}f, \bar{v}> &= \left(\frac{1}{2\pi}\right)^n \sum \int_{\Xi^n} \widehat{D_t^k f(x', 0)}(\xi') \overline{\hat{v}_k(\xi')} d\xi' \\ &= \left(\frac{1}{2\pi}\right)^N \sum \int_{\Xi^N} \widehat{D_t^k f}(\xi) \overline{\hat{v}_k(\xi')} d\xi \\ &= \left(\frac{1}{2\pi}\right)^N \sum \int_{\Xi^N} \hat{f}(\xi)\tau^k \overline{\hat{v}_k(\xi')} d\xi \end{aligned}$$

and therefore $\widehat{{}^t\mathcal{T}\vec{v}}(\xi) = \sum_{|k|\leq M} \hat{v}_k(\xi')\tau^k$. By this relation we see that the transposed map ${}^t\mathcal{T}$ is injective. Thus the map $\mathcal{T}$ is an epimorphism if and only if the range of ${}^t\mathcal{T}$ is closed in $B_{p',1/\mu}(R^N)$.

THEOREM 3. *The following conditions are equivalent*:

(1) *If* $u \in B_{p',1/\mu}(R^N)$ *and* $\hat{u}(\xi) = \sum_{|k|\leq M} f_k(\xi')\tau^k$, *then* $f_k/\nu_{k,p'} \in L^{p'}(\Xi^n)$ *for any k with* $|k| \leqq M$.

(2) *If* $u \in B_{p',1/\mu}(R^N)$ *and* $\hat{u}(\xi) = \sum_{|k|\leq M} f_k(\xi')\tau^k$, *then*

$$\hat{u}\left(\xi', \tau_1,\ldots, \tau_{j-1}, \frac{\tau_j}{2}, \tau_{j+1},\ldots, \tau_m\right)/\mu \in L^{p'}(\Xi^N)$$

for every $j=1, 2,\ldots, m$.

(3) *If* $u \in B_{p',1/\mu}(R^N)$ *and* $\hat{u}(\xi) = \sum_{|k|\leq M} f_k(\xi')\tau^k$, *then*

$$\hat{u}\left(\xi', \frac{\tau_1}{2^{i_1}},\ldots, \frac{\tau_m}{2^{i_m}}\right) \in L^{p'}(\Xi^N)$$

for any non-negative integers i_j.

In this case, the trace mapping $\mathscr{T}: B_{p,\mu}(R^N) \ni u \to \{D_t^k u(x', 0)\} \in \prod_{|k| \leq M} B_{p,\nu_{k,p'}}(R^n)$ *is an epimorphism.*

PROOF. (1)⇒(2). Suppose (1) holds. Then $f_k/\nu_{k,p'} \in L^{p'}(\Xi^n)$, that is, $f_k(\xi')\tau^k/\mu(\xi) \in L^{p'}(\Xi^N)$, and then $f_k(\xi')\tau^k/2^{k_j}\mu(\xi) \in L^{p'}(\Xi^N)$. Thus we have

$$\hat{u}\left(\xi', \tau_1, \dots, \frac{\tau_j}{2}, \dots, \tau_m\right)\Big/\mu = \sum f_k(\xi')\tau^k/2^{k_j}\mu(\xi) \in L^{p'}(\Xi^N).$$

(2)⇒(3). Suppose (2) holds. Then $\hat{u}\left(\xi', \tau_1, \dots, \frac{\tau_j}{2}, \dots, \tau_m\right)/\mu \in L^{p'}(\Xi^N)$ and $\hat{u}\left(\xi', \tau_1, \dots, \frac{\tau_j}{2}, \dots, \tau_m\right) = \sum_k f_k(\xi')\tau^k/2^{k_j}\mu \in L^{p'}(\Xi^N)$, which implies $\hat{u}\left(\xi', \tau_1, \dots, \frac{\tau_j}{2^2}, \dots, \tau_m\right)/\mu \in L^{p'}(\Xi^N)$. Repeating this procedure, we have (3).

(3)⇒(1). Suppose (3) holds. Then $\hat{u}\left(\xi', \frac{\tau_1}{2^{i_1}}, \dots, \frac{\tau_m}{2^{i_m}}\right)/\mu \in L^{p'}(\Xi^N)$ for any non-negative integers i_j and

$$\hat{u}\left(\xi', \frac{\tau_1}{2^{i_1}}, \dots, \frac{\tau_m}{2^{i_m}}\right)\Big/\mu(\xi) = \sum f_k(\xi')\left(\frac{\tau_1}{2^{i_1}}\right)^{k_1} \cdots \left(\frac{\tau_m}{2^{i_m}}\right)^{k_m}\Big/\mu(\xi) \in L^{p'}(\Xi^N).$$

For any fixed $i_1, \dots, i_{m-1}$ if we take $i_m = 0, 1, \dots, M$, then we have

$$\sum_{k_1 + \dots + k_{m-1} \leq M - k_m} f_k(\xi')\left(\frac{\tau_1}{2^{i_1}}\right)^{k_1} \cdots \left(\frac{\tau_{m-1}}{2^{i_{m-1}}}\right)^{k_{m-1}} \tau_m^{k_m}\Big/\mu(\xi) \in L^{p'}(\Xi^N)$$

for each $k_m = 0, 1, \dots, M$. Repeating this procedure, we have $f_k(\xi')\tau^k/\mu \in L^{p'}(\Xi^N)$ for any k with $|k| \leq M$, which means $f_k/\nu_{k,p'} \in L^{p'}(\Xi^n)$.

We shall next show that the range of ${}^t\mathscr{T}$ is closed in $B_{p',1/\mu}(R^N)$ when (1) holds true. Let $\hat{v}^j = \{v_k^j\} \in H' = \prod_{|k| \leq M} B_{p',1/\nu_{k,p'}}(R^n)$ and suppose ${}^t\mathscr{T}\hat{v}^j$ converges to u in $B_{p',1/\mu}(R^N)$. Namely $\sum_{|k| \leq M} \hat{v}_k^j(\xi')\tau^k/\mu$ converges to $\hat{u}/\mu$ in $L^{p'}(\Xi^N)$. From the fact that μ is a positive-valued continuous function, we see that $\sum_k \hat{v}_k^j(\xi')\tau^k$ converges to $\hat{u}$ in $L_{loc}^1(\Xi^N)$. Thus we can write $\hat{u} = \sum_{|k| \leq M} f_k(\xi')\tau^k$ with $f_k \in L_{loc}^1(\Xi^n)$. By the condition (1) $f_k/\nu_{k,p'} \in L^{p'}(\Xi^n)$ for $|k| \leq M$. If we take $v_k = \mathscr{F}^{-1}(f_k)$, then $v_k \in B_{p',1/\nu_{k,p'}}(R^n)$ and $\hat{u} = \sum_{|k| \leq M} \hat{v}_k(\xi')\tau^k$. Thus the range of ${}^t\mathscr{T}$ is closed in $B_{p',1/\mu}(R^N)$. By virtue of Theorem 2 the map $\mathscr{T}$ is an epimorphism.

COROLLARY. *If* $\mu(\xi', \tau_1, \dots, \tau_{j-1}, 2\tau_j, \tau_{j+1}, \dots, \tau_m) \geq C\mu(\xi)$ *with a positive constant* C *for* $j = 1, 2, \dots, m$, *then the trace mapping* $\mathscr{T}: B_{p,\mu}(R^N) \ni u \to \{D_t^k u(x', 0)\} \in \prod_{|k| \leq M} B_{p,\nu_{k,p'}}(R^n)$ *is an epimorphism.*

PROOF. $B_{p,\mu}(R^N) \subset B_{p,\nu}(R^N)$ if and only if there exists a positive constant C

such that $\nu \leqq C\mu$. Hence the condition $\mu(\xi', \tau_1,..., 2\tau_j,..., \tau_m) \geqq C\mu(\xi)$ means that $u\left(\xi', \tau_1,..., \frac{\tau_j}{2},..., \tau_m\right)/\mu \in L^{p'}(\Xi^N)$ for any $u \in B_{p',1/\mu}(R^N)$. Since the condition (2) of Theorem 3 is satisfied, the map $\mathscr{T}$ is an epimorphism.

PROPOSITION 3. *Let $\{f_k\}$ be an arbitrary element of $\prod_{|k| \leq M} B_{p,\nu_{k,p'}}(R^n)$ and suppose for each k there exist a positive valued continuous function λ_k on Ξ^n and a slowly increasing continuous function Φ_k on Ξ^m such that*

$$\mu(\xi', \lambda_k(\xi')\tau) \leqq \lambda_k^{|k|+m/p'}(\xi')\nu_{k,p'}(\xi')\Phi_k(\tau).$$

Let $\psi \in \mathscr{D}(R^m)$ such that $\psi = 1$ in a neighbourhood of 0. If we put

$$\hat{u}_{x'}(\xi', t) = \sum_{|k| \leq M} \hat{f}_k(\xi') \frac{(it)^k}{k!} \psi(\lambda_k(\xi')t),$$

then $u \in B_{p,\mu}(R^N)$ and $D_t^k u(x', 0) = f_k(x')$ for $|k| \leqq M$.

PROOF. By the equations

$$\begin{aligned}\hat{u}(\xi) &= \sum_{|k| \leq M} \frac{(-i)^{|k|}}{k!} \hat{f}_k(\xi') D_\tau^k \int_{\Xi^m} \psi(\lambda_k t) e^{-i<t,\tau>} dt \\ &= \sum_{|k| \leq M} \frac{(-i)^{|k|}}{k!} \hat{f}_k(\xi') \frac{1}{\lambda_k^{|k|+m}} D_\tau^k \hat{\psi}\left(\frac{\tau}{\lambda_k}\right)\end{aligned}$$

we have

$$\begin{aligned}&\int_{\Xi^N} |\hat{u}|^p \mu^p d\xi \\ &\leqq C \sum_{|k| \leq M} \frac{1}{(k!)^p} \int_{\Xi^n} |\hat{f}_k(\xi')|^p \frac{d\xi'}{\lambda_k^{p(|k|+m)-m}} \int_{\Xi^m} |D_\tau^k \hat{\psi}(\tau)|^p \mu^p(\xi', \lambda_k \tau) d\tau \\ &\leqq C \sum_{|k| \leq M} \frac{1}{(k!)^p} \int_{\Xi^n} |\hat{f}_k(\xi')|^p \nu_{k,p'}^p(\xi') d\xi' \int_{\Xi^m} |D_\tau^k \hat{\psi}(\tau)|^p |\Phi_k(\tau)|^p d\tau < \infty.\end{aligned}$$

Thus $u \in B_{p,\mu}(R^N)$ and clearly $D_t^k u(x', 0) = f_k(x')$ for $|k| \leqq M$.

EXAMPLE. Let μ be written in the form

$$\mu(\xi) = \mu_1(\xi') + |\tau|^a \mu_2(\xi'),$$

where μ_1 and μ_2 are temperate weight functions defined in Ξ^n and a is a real number with $a > m/p'$. Let M be the largest integer such that $M < a - m/p'$. Then we have

$$\nu_{k,p'} \sim \mu_1^{1-(|k|+m/p')/a} \mu_2^{(|k|+m/p')/a} \qquad \text{for} \quad |k| \leqq M.$$

If we take $\lambda_k = (\mu_1/\mu_2)^{1/a}$ and $\Phi_k(\tau) = 1 + |\tau|^a$ for $|k| < a - m/p'$, then

$$\mu(\xi', \lambda_k(\xi')\tau) \leqq C\lambda_k^{|k|+m/p'}(\xi')\nu_{k,p'}(\xi')\Phi_k(\tau).$$

In fact, we have

$$\frac{1}{\nu_{k,p'}(\xi')} = \left\{\int_{\Xi^m} \frac{|\tau^k|^{p'}}{(\mu_1(\xi') + |\tau|^a\mu_2(\xi'))^{p'}} d\tau\right\}^{1/p'}$$

$$= \frac{1}{\mu_1^{1-(|k|+m/p')/a}\mu_2^{(|k|+m/p')/a}} \left\{\int_{\Xi^m} \frac{|\tau^k|^{p'}}{(1 + |\tau|^a)^{p'}} d\tau\right\}^{1/p'}$$

and

$$\mu(\xi', \lambda_k(\xi')\tau) = \mu_1(\xi')(1 + |\tau|^a) \sim \lambda_k^{|k|+m/p'}(\xi')\nu_{k,p'}(\xi')\Phi_k(\tau).$$

Then Proposition 3 is applicable to this case; that is, for any given $\vec{f} = \{f_k\} \in \prod_{|k|\leq M} B_{p,\nu_{k,p'}}(R^n)$ if we take

$$\hat{u}_{x'}(\xi', t) = \sum_{|k|\leq M} \hat{f}_k(\xi') \frac{(it)^k}{k!} \psi\left(\left(\frac{\mu_1}{\mu_2}\right)^{1/a} t\right)$$

with $\psi \in \mathscr{D}(R^m)$ such that $\psi = 1$ in a neighbourhood of 0, then $u \in B_{p,\mu}(R^N)$ and $D_t^k u(x', 0) = f_k(x')$ for $|k| \leqq M$.

5. The relation between trace mappings and other notions

Let us recall the concept of multiplication of distributions. Let $u, v \in \mathscr{D}'(R^N)$. If the distributional limit $\lim_{j\to\infty}(u*\rho_j)v$ exists for every δ-sequence $\{\rho_j\}$ on R^N, then the limit is uniquely defined and it is called the multiplicative product of u and v in the strict sense and denoted by $u \cdot v$ [7]. In this case the distributional limit $\lim_{j\to\infty} u(v*\rho_j)$ exists for every δ-sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^N)$ and $\lim_{j\to\infty}(u*\rho_j)v = \lim_{j\to\infty} u(v*\rho_j)$.

For any $\phi \in \mathscr{D}(R^N)$ such that $\phi \geqq 0$ and $\int_{R^N} \phi\, dx = 1$ we put $\phi_\varepsilon(x) = \frac{1}{\varepsilon^N}\phi\left(\frac{x}{\varepsilon}\right)$ for small $\varepsilon > 0$. If the distributional limit $\lim_{\varepsilon\downarrow 0}(u*\phi_\varepsilon)v$ exists for any ϕ and does not depend on the choice of ϕ, then the limit is called the multiplicative product of u and v and denoted by uv. The product uv is also defined as the distributional limit of $(u*\rho_j)v$ for any restricted δ-sequence $\{\rho_j\}$, which is a sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^N)$, such that (i) $\operatorname{supp}\rho_j \to \{0\}$ as $j \to \infty$, (ii) $\int_{R^N} \rho_j dx \to 1$ as $j \to \infty$ and (iii) $\int_{R^N} |x|^{|k|}|D^k\rho_j| dx \leqq M_k$, M_k being a constant independent of ρ_j [6, p. 91].

Let $u \in \mathscr{D}'(R^N)$ and $\psi \in \mathscr{D}(R_{x'}^n)$. By $<u, \psi>_{x'}$ we shall denote the distribution $\in \mathscr{D}'(R_t^m)$ defined by

$$\mathscr{D}(R_t^m) \ni \chi \longrightarrow <u, \psi \otimes \chi>

and similarly, for any $\chi \in \mathscr{D}(R_t^n)$ we shall denote by $<u, \chi>_t$ the distribution $\in \mathscr{D}'(R_{x'}^n)$ defined by

$$\mathscr{D}(R_{x'}^n) \ni \psi \longrightarrow <u, \psi \otimes \chi>.$$

Let $w \in \mathscr{D}'(R_t^m)$ and $u \in \mathscr{D}'(R^N)$. If the product $(1 \otimes w)u$ exists, then it is called the partial product of w and u and denoted by wu [3, p. 170]. Let δ be the Dirac measure on R_t^m. Then the partial product δu exists if and only if $\delta<u, \psi>_{x'}$ exists in $\mathscr{D}'(R_t^m)$ for any $\psi \in \mathscr{D}(R_{x'}^n)$. In this case $<\delta u, \psi>_{x'} = \delta<u, \psi>_{x'}$. Also δu is defined as the unique limit $\lim_{j\to\infty} (\delta * \rho_j)u = \lim_{j\to\infty} \rho_j u$ or equivalently $\lim_{j\to\infty} \delta(u *_t \rho_j)$ for any restricted δ-sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R_t^m)$, where $*_t$ means the partial convolution with respect to the variable t.

In accordance with S. Łojasiewicz [5, p. 15] we say that $u \in \mathscr{D}'(R^N)$ has the section $\in \mathscr{D}'(R^n)$ for $t=0$ if the distributional limit $\lim_{\varepsilon\downarrow 0} u(x', \varepsilon t)$ exists and does not depend on t, namely, $\lim_{\varepsilon\downarrow 0} <u, \phi_\varepsilon>_t$ exists in $\mathscr{D}'(R_{x'}^n)$ for any $\phi \in \mathscr{D}(R_t^m)$ with $\phi(t) \geqq 0$, $\int \phi(t)dt = 1$, and it is independent of ϕ. By the equation $<u, \phi_\varepsilon>_t \otimes \delta = \delta(u *_t \check{\phi}_\varepsilon)$ we see that u has the section for $t=0$ if and only if the partial product δu exists [4, p. 406]. In this case, if $\alpha \in \mathscr{D}'(R_{x'}^n)$ is the section of u, then $\alpha = \lim_{\varepsilon\downarrow 0} <u, \phi_\varepsilon>_t$, $\lim_{\varepsilon\downarrow 0} u(x', \varepsilon t) = \alpha \otimes 1_t$ and $\delta u = \alpha \otimes \delta$.

If $\lim_{j\to\infty} <u, \rho_j>_t$ exists in $\mathscr{D}'(R_{x'}^n)$ for any δ-sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R_t^m)$, we shall say that u has the section for $t=0$ in the strict sense. Then, by the equation $<u, \rho_j>_t \otimes \delta = \delta(u *_t \check{\rho}_j)$ we see that the partial product $\delta \cdot u$ exists if and only if u has the section for $t=0$ in the strict sense.

THEOREM 4. *For the space $B_{p,\mu}(R^N)$ the following statements are equivalent:*

(1) *The trace mapping $B_{p,\mu}(R^N) \ni u \to u(x', 0) \in \mathscr{D}'(R^n)$ is defined.*

(2) *The section for $t=0$ exists for every $u \in B_{p,\mu}(R^N)$.*

(2)′ *The condition* (2) *holds in the strict sense.*

(3) *The partial product δu exists for every $u \in B_{p,\mu}(R^N)$, where δ is the Dirac measure in R_t^m.*

(3)′ *The partial product $\delta \cdot u$ exists for every $u \in B_{p,\mu}(R^N)$.*

(4) *The distributional limit $\lim_{j\to\infty} (1 \otimes \delta)(u * \rho_j)$ exists for a fixed restricted δ-sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^N)$, for every $u \in B_{p,\mu}(R^N)$.*

(5) *The distributional limit $\lim_{j\to\infty} \rho_j u$ exists for a fixed restricted δ-sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R_t^m)$, for every $u \in B_{p,\mu}(R^N)$.*

PROOF. We have already noted the equivalences (2)⇔(3) and (2)′⇔(3)′. The implications (3)′⇒(3), (3)′⇒(4), (5) are trivial. It suffices to show the implications (1)⇒(3)′, (4)⇒(1) and (5)⇒(1).

(1)⇒(3)′. Suppose (1) holds. Then $\psi \otimes \delta \in B_{p',1/\mu}(R^N)$ for every $\psi \in \mathscr{D}(R^n)$.

Let $u \in B_{p,\mu}(R^N)$. Then $u*\rho_j$ converges to u in $B_{p,\mu}(R^N)$ for any δ-sequence $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^N)$. In fact, we have

$$\int_{\Xi^N} |(u*\rho_j - u)^\wedge|^p \mu^p d\xi = \int_{\Xi^N} |\hat{u}(\hat{\rho}_j - 1)|^p \mu^p d\xi$$

$$\leqq \sup |\hat{\rho}_j - 1|^p \int_{\Xi^N} |\hat{u}|^p \mu^p d\xi$$

and $\hat{\rho}_j$ converges to 1 boundedly and uniformly on every compact subset of Ξ^N when $j \to \infty$. For any $\phi \in \mathscr{D}(R^N)$ we have

$$<(1 \otimes \delta)(u*\rho_j), \phi>_{\mathscr{D}',\mathscr{D}} = <u*\rho_j, \phi(x', 0) \otimes \delta>_{B_{p,\mu}, B_{p',1/\mu}},$$

which implies the existence of $\lim_{j\to\infty} (1\otimes\delta)(u*\rho_j)$. Thus the partial product $\delta \cdot u$ exists.

(4)⇒(1). Let $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^N)$, be a fixed restricted δ-sequence. For each j the map

$$B_{p,\mu}(R^N) \ni u \longrightarrow (1 \otimes \delta)(u*\rho_j) = (u*\rho_j)(x', 0) \otimes \delta \in \mathscr{D}'(R^N)$$

is continuous, for the map $u \to u*\rho_j$ is continuous from $\mathscr{D}'(R^N)$ into $\mathscr{E}(R^N)$. By the Banach-Steinhaus theorem the map

$$B_{p,\mu}(R^N) \ni u \longrightarrow \lim_{j\to\infty} (1 \otimes \delta)(u*\rho_j) \in \mathscr{D}'(R^N)$$

is also continuous. Thus, for any $\phi \in \mathscr{D}(R^N)$ there exists $w_\phi \in B_{p',1/\mu}(R^N)$ such that

$$<\lim_{j\to\infty} (1 \otimes \delta)(u*\rho_j), \phi>_{\mathscr{D}',\mathscr{D}} = <u, w_\phi>_{B_{p',\mu}, B_{p',1/\mu}}.$$

If we take $u = \alpha \in \mathscr{D}(R^N)$, then

$$<\alpha\delta, \phi>_{\mathscr{D}',\mathscr{D}} = <\alpha, \phi(x', 0) \otimes \delta> = <\alpha, w_\phi>.$$

Since $\mathscr{D}(R^N)$ is dense in $B_{p,\mu}(R^N)$, $\phi(x', 0)\otimes\delta = w_\phi \in B_{p',1/\mu}(R^N)$, which means that the trace mapping is defined.

(5)⇒(1). Let $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R_t^m)$, be a fixed restricted δ-sequence. Then the map

$$B_{p,\mu}(R^N) \ni u \longrightarrow \rho_j u \in \mathscr{D}'(R^N)$$

is continuous. By the Banach-Steinhaus theorem the map

$$B_{p,\mu}(R^N) \ni u \longrightarrow \lim_{j\to\infty} \rho_j u \in \mathscr{D}'(R^N)$$

is continuous, and $\lim_{j\to\infty} \rho_j u = u(x', 0)\otimes\delta$ for any $u \in \mathscr{D}(R^N)$. Thus the trace mapping is defined.

We suppose $1/\mu(0, \tau) \in L^{p'}(\Xi^m)$ and put $\nu_{p'}(\xi') = \left\{\int_{\Xi^m} \frac{1}{\mu^{p'}(\xi)} d\tau\right\}^{-1/p'}$. Let $t_0 \in R^m$ and $u \in \mathscr{D}(R^N)$. Then $\tau_{-t_0} u \in \mathscr{D}(R^N)$ and $(\tau_{-t_0} u)^\wedge = e^{i<t_0, \tau>} u$. In the proof of Proposition 1 we have shown

$$\|u(\cdot, t_0)\|_{p, \nu_{p'}} \leqq \left(\frac{1}{2\pi}\right)^{m/p'} \|u\|_{p, \mu}.$$

Thus the trace $u(\cdot, t_0)$ on $t = t_0$ belongs to the space $B_{p, \nu_{p'}}(R^n)$ for any $u \in B_{p, \mu}(R^N)$. For any $u \in B_{p, \mu}(R^N)$ there exists a sequence $\{u_j\}$, $u_j \in \mathscr{D}(R^N)$ such that $u = \lim_{j \to \infty} u_j$ in $B_{p, \mu}(R^N)$ and we have

$$\|u_j(\cdot, t_0) - u(\cdot, t_0)\|_{p, \nu_{p'}} \leqq \left(\frac{1}{2\pi}\right)^{m/p'} \|u_j - u\|_{p, \mu}.$$

Thus $u_j(\cdot, t_0)$ converges to $u(\cdot, t_0)$ in $B_{p, \nu_{p'}}(R^n)$ uniformly with respect to t_0. Since $t \to u_j(\cdot, t)$ are $B_{p, \nu_{p'}}(R^n)$-valued continuous functions, $u(\cdot, t)$ may be considered as a $B_{p, \nu_{p'}}(R^n)$-valued continuous function $\mathbf{u}(t)$ of t. Thus we have the following

PROPOSITION 4. *Suppose $1/\mu(0, \tau) \in L^{p'}(\Xi^m)$. Then every $u \in B_{p, \mu}(R^N)$ is identified with the $B_{p, \nu_{p'}}(R^n)$-valued continuous function $\mathbf{u}(t)$, where $\nu_{p'}(\xi') = \left\{\int_{\Xi^m} \frac{1}{\mu^{p'}(\xi', \tau)} d\tau\right\}^{-1/p'}$*

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