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Axiomatic Characterizations of Grade for Commutative Rings

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Introduction

Let R be a commutative ring and I an ideal of R. Suppose that R is noetherian. Then, for every R-module M, we can define the grade of I on M in two different ways using M-regular sequences in I and $\mathrm{Ext}_R(R/I, M)$ (or $\mathrm{H}_I(M)$). In the case that R is not necessarily noetherian, there are two notions of grade which may be regarded as good generalizations of the one in the noetherian case. One is the homological grade (or 'Rees' grade) which is defined by using $\mathrm{Ext}_R(R/I, \ )$ or $\mathrm{H}_I(\ )$ in the same way as in the noetherian case. Another one is the polynomial grade which is a successful generalization of the notion of the longest M-regular sequence in I. As for the last one, a characterization was given by M. Hochster by making use of the Koszul complex in [4].

In this paper, we shall give axiomatic characterizations of the above two different notions of grade, which show a relationship between these two notions from another point of view.

In § 1 we shall see that the concept of localizing subcategories plays an essential role in the theory of homological grade in an abelian category. In § 2 we are mainly concerned with the homological grade in the category of R-modules. In this case we use Gabriel topologies on R instead of localizing subcategories. We shall also give a proof of the Auslander-Buchsbaum theorem on finite free resolutions in terms of homological grade. In § 3 we shall study the polynomial grade originally introduced by M. Hochster.

Throughout this paper all rings and algebras are commutative with identity and modules are unitary.

1. Homological grade in an abelian category

In this section, we shall discuss a homological theory of grade in an abelian category. Throughout this section $\mathscr{A}$ is a locally small abelian category with injective envelopes and products.

Let $\mathscr{C}$ be a localizing subcategory of $\mathscr{A}$ (cf. [8]). Then for each object M in $\mathscr{A}$ we can assign a non-negative integer or ∞ as follows:

$hgr(\mathscr{C}, M)=$ the least integer n such that $\mathrm{R}^n\mathrm{L}(M)\neq 0$ if there is such an integer and ∞ if there is not, where $L=L_{\mathscr{C}}$ is a functor $\mathscr{A}\to\mathscr{A}$ such that $L(M)$ is a largest $\mathscr{C}$-subobject of M and R^nL is the n-th right derived functor of L (cf. [5]). Prop. 2.13 of [5] (or Cor. to Prop. 2 in this section) shows that $hgr(\mathscr{C}, M)$ is also the least integer n such that $\mathrm{Ext}^n(N, M)\neq 0$ for some N in $\mathscr{C}$ if there is such an integer and ∞ if there is not. We say that $hgr(\mathscr{C}, M)$ is the homological grade of $\mathscr{C}$ on M.

Now we put $\psi(M)=hgr(\mathscr{C}, M)$; then ψ has the following properties:

(0) *$\psi(0)=\infty$, and if $M\cong N$, then $\psi(M)=\psi(N)$.*

(1) *If M is a subobject of N and $\psi(N)>0$, then $\psi(M)>0$.*

(2) *If $\psi(M)>0$, then $\psi(E(M)>0$, where $E(M)$ is an injective envelope of M.*

(3) *If $\{M_\nu\}$ is a family of objects with $\psi(M_\nu)>0$, then $\psi(\prod_\nu M_\nu)>0$.*

(4) *Let $0\to M\to E\to N\to 0$ be a short exact sequence. If E is an injective object with $\psi(E)>0$, then $\psi(N)=\psi(M)-1$.*

To prove the above properties, we first note that $hgr(\mathscr{C}, M)>0$ if and only if $\mathrm{Hom}(N, M)=0$ for every N in $\mathscr{C}$. Therefore the properties (0), (1), (2) and (3) are clearly satisfied. Next we note that $R^nL(E)=0$ for every $n>0$ if E is an injective object. Therefore we can easily show the property (4) by examining the long exact sequence in $R^{\cdot}L$.

It is the aim of this section to show that the above properties (0), (1), (2), (3) and (4) characterize $hgr(\mathscr{C},\ )$.

We say that a correspondence from $\mathrm{Obj}(\mathscr{A})$ to {non-negative integers and ∞} is a homological g-function on $\mathscr{A}$ if it satisfies the above conditions (0), (1), (2), (3) and (4). Let ψ be a homological g-function on $\mathscr{A}$. We put

$$\mathscr{M} = \{\text{object } M \text{ in } \mathscr{A} \text{ such that } \psi(M) > 0\}$$

and

$$\mathscr{C} = \{\text{object } N \text{ in } \mathscr{A} \text{ such that } \mathrm{Hom}(N, M) = 0 \text{ if } M \text{ belongs to } \mathscr{M}\}.$$

LEMMA 1. *$\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$, and $\mathscr{M}=\{$object M in $\mathscr{A}$ such that* $\mathrm{Hom}(N, M)=0$ *if N belongs to $\mathscr{C}\}$.*

PROOF. The conditions on ψ imply that $\mathscr{M}$ is closed under subobjects, essential extentions and products. Therefore the assertions follow from [8], Prop. 1.16, Part 1.

LEMMA 2. *$\psi(M)=hgr(\mathscr{C}, M)$ for every object M.*

PROOF. We use induction on $n=\min\{\psi(M), hgr(\mathscr{C}, M)\}$. The case $n=\infty$ is trivial. Therefore we may assume that $n<\infty$. Suppose that $n=0$. It is obvious that $\psi(M)=0$ if and only if $M\notin\mathscr{M}$. By Lemma 1, $M\notin\mathscr{M}$ if and only if $\mathrm{Hom}(N, M)\neq 0$ for some N in $\mathscr{C}$, i.e. $hgr(\mathscr{C}, M)=0$. This settles the case

$n=0$. We now suppose that $n>0$. Let E be an injective envelope of M. The property (2) on ψ and $h\,gr(\mathscr{C},\ )$ implies $\psi(E)>0$ and $h\,gr(\mathscr{C}, E)>0$; hence $\psi(E/M)=\psi(M)-1$ and $h\,gr(\mathscr{C}, E/M)=h\,gr(\mathscr{C}, M)-1$. Since $n-1=\min\{\psi(E/M), h\,gr(\mathscr{C}, E/M)\}$, our inductive hypothesis shows that $\psi(E/M)=h\,gr(\mathscr{C}, E/M)$. Therefore $\psi(M)=h\,gr(\mathscr{C}, M)$. This completes the proof.

In view of Lemma 1 and 2, we have

THEOREM 1. *If ψ is a homological g-function on $\mathscr{A}$, then there exists a unique localizing subcategory $\mathscr{C}$ of $\mathscr{A}$ such that $\psi=h\,gr(\mathscr{C},\ )$.*

Let now ψ be a homological g-function on $\mathscr{A}$, then ψ satisfies the following conditions:

(5) Let $0\to M'\to M\to M''\to 0$ be an exact sequence. Then the following statements hold:

(a) if $\psi(M')>\psi(M)$, then $\psi(M'')=\psi(M)$;

(b) if $\psi(M')<\psi(M)$, then $\psi(M'')=\psi(M')-1$;

(c) if $\psi(M')=\psi(M)$, then $\psi(M'')\geqq\psi(M)-1$.

(6) If E is an injective object in $\mathscr{A}$, then $\psi(E)=0$ or ∞.

In fact as to (6), consider the exact sequence $0\to E\to E\to 0\to 0$. If E is an injective object with $\psi(E)>0$, then $\psi(0)=\psi(E)-1$ by (4). Therefore $\psi(E)=\infty$ since $\psi(0)=\infty$. On the other hand, $\psi=h\,gr(\mathscr{C},\ )$ for some localizing subcategory $\mathscr{C}$ of $\mathscr{A}$. Therefore by examining the long exact sequence in $R^{\cdot}L$ we know that ψ satisfies (5).

Conversely, the conditions (0), (1), (2), (3), (5) and (6) imply (4). In fact, let $0\to M\to E\to N\to 0$ be an exact sequence and E an injective object such that $\psi(E)>0$. Then $\psi(E)=\infty$ by (6). Therefore if $\psi(M)<\infty$, then $\psi(N)=\psi(M)-1$ by (5)–(a); if $\psi(M)=\infty$, then $\psi(N)\geqq\psi(E)-1=\infty$ by (5)–(c). In any case we have $\psi(N)=\psi(M)-1$.

The following variation of the above theorem may be useful when we compare the homological grade with another notions of grade.

PROPOSITION 1. *Let $\mathscr{C}$ be a localizing subcategory of $\mathscr{A}$. A correspondence ψ from* Obj $(\mathscr{A})$ *to {non-negative integers and ∞} is a homological g-function and $\psi=h\,gr(\mathscr{C},\ )$ if ψ satisfies the following conditions:*

(i) *$\psi(0)=\infty$, and if $M\cong N$, then $\psi(M)=\psi(N)$.*

(ii) *$\psi(M)>0$ if and only if M is $\mathscr{C}$-pure i.e. M has no non-trivial $\mathscr{C}$-subobjects.*

(iii) *Let $0\to M\to E\to N\to 0$ be an exact sequence. If E is an injective object with $\psi(E)>0$, then $\psi(N)=\psi(M)-1$.*

For the rest of this section, we shall show some properties of $R^nL_{\mathscr{C}}$. Assume that $\mathscr{A}$ is an abelian category with enough injectives. Let $\mathscr{C}$ be a localizing subcategory of $\mathscr{A}$. For the definition of $\mathscr{C}$-divisorial objects and its properties, we shall refer to [5]. We denote $\mathrm{L}_{\mathscr{C}}$ by L for simplicity. The following is a generalization of [3], Prop. 5.1 § 5.

PROPOSITION 2. *Let M be an object in $\mathscr{A}$ and n a positive integer. If* $\mathrm{R}^p\mathrm{L}(M)=0$ *for* $p<n$, *then we have a functorial isomorphism* $\mathrm{Ext}^n(N, M) \cong \mathrm{Hom}(N, \mathrm{R}^n\mathrm{L}(M))$ *for every N in $\mathscr{C}$.*

PROOF. Let $M \to E_0 \to E_1 \to \cdots$ be an injective resolution of M. Our assumption implies that the complex $0 \to \mathrm{L}(E_0) \to \mathrm{L}(E_1) \to \cdots \to \mathrm{L}(E_n)$ is exact. Since each $\mathrm{L}(E_i)$ is $\mathscr{C}$-divisorial, $\mathrm{Im}(\mathrm{L}(E_{i-1}) \to \mathrm{L}(E_i))$ is a direct summand of $\mathrm{L}(E_i)$. Now let N be an object in $\mathscr{C}$. Then we have

$$\begin{aligned}
&\mathrm{Ker}(\mathrm{Hom}(N, E_n) \longrightarrow \mathrm{Hom}(N, E_{n+1})) \\
&\qquad = \mathrm{Ker}(\mathrm{Hom}(N, \mathrm{L}(E_n)) \longrightarrow \mathrm{Hom}(N, \mathrm{L}(E_{n+1}))) \\
&\qquad = \mathrm{Hom}(N, \mathrm{Ker}(\mathrm{L}(E_n) \longrightarrow \mathrm{L}(E_{n+1})))
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}(\mathrm{Hom}(N, E_{n-1}) \longrightarrow \mathrm{Hom}(N, E_n)) \\
&\qquad = \mathrm{Im}(\mathrm{Hom}(N, \mathrm{L}(E_{n-1})) \longrightarrow \mathrm{Hom}(N, \mathrm{L}(E_n))) \\
&\qquad = \mathrm{Hom}(N, \mathrm{Im}(\mathrm{L}(E_{n-1}) \longrightarrow \mathrm{L}(E_n))).
\end{aligned}$$

Therefore we have $\mathrm{Ext}^n(N, M) \cong \mathrm{Hom}(N, \mathrm{R}^n\mathrm{L}(M))$.

We already have proved the following corollary in [5]. But Prop. 2 enables us to give another proof of it.

COROLLARY. ([5], Prop. 2.13 § 2) *The following statements concerning an object M and positive integer n are equivalent:*

(i) $\mathrm{Ext}^p(N, M)=0$ *for every object N in $\mathscr{C}$ and $p<n$.*

(ii) $\mathrm{R}^p\mathrm{L}(M) = 0$ *for* $p < n$.

PROOF. We use induction on n. The case $n=0$ is trivial. Now assume that $n>0$ and that the assertion has been established in the case of smaller values of the inductive variable. First suppose that $\mathrm{R}^p\mathrm{L}(M)=0$ for $p<n$. Then by Prop. 2, $\mathrm{Ext}^{n-1}(N, M)=\mathrm{Hom}(N, \mathrm{R}^{n-1}\mathrm{L}(M))=0$ for every N in $\mathscr{C}$. Conversely suppose that $\mathrm{Ext}^p(N, M)=0$ for every N in $\mathscr{C}$ and $p<n$. Then also by Prop. 2 $\mathrm{Hom}(\mathrm{R}^{n-1}\mathrm{L}(M), \mathrm{R}^{n-1}\mathrm{L}(M)=\mathrm{Ext}^{n-1}(\mathrm{R}^{n-1}\mathrm{L}(M), M)=0$. Therefore $\mathrm{R}^{n-1}\mathrm{L}(M)=0$. This completes the proof.

Let f be an endomorphism of the functor $1_{\mathscr{A}}$. Assume that f commutes with the canonical morphism $\mathrm{L}\to 1_{\mathscr{A}}$ (note that L is a subfunctor of $1_{\mathscr{A}}$). Then the induced morphism $\mathrm{R}^n\mathrm{L}(M)\to\mathrm{R}^n\mathrm{L}(M)$ from $f(M)\colon M\to M$ is canonically equal to $f(\mathrm{R}^n\mathrm{L}(M))$.

PROPOSITION 3. *Let f be an endomorphism of the functor $1_{\mathscr{A}}$ which commutes with $\mathrm{L}\to 1_{\mathscr{A}}$. Assume that $f(N)$ is not a monomorphism for every N in $\mathscr{C}$. If M is an object such that $f(M)$ is a monomorphism, then $hgr(\mathscr{C}, M)>0$ and $hgr(\mathscr{C}, \mathrm{Coker} f(M))=hgr(\mathscr{C}, M)-1$.*

PROOF. Suppose that $hgr(\mathscr{C}, M)=0$ i.e. $\mathrm{L}(M)\neq 0$. Since $f(\mathrm{L}(M))\colon \mathrm{L}(M)\to\mathrm{L}(M)$ is not a monomorphism, so is $f(M)$, and this leads a contradiction. Therefore $hgr(\mathscr{C}, M)>0$. Now let $n=hgr(\mathscr{C}, M)$. Consider the exact sequence $0\to M\xrightarrow{f(M)} M\to N=\mathrm{Coker} f(M)\to 0$. Since $\mathrm{R}^p\mathrm{L}(M)=0$ for $p<n$, we have $\mathrm{R}^p\mathrm{L}(N)=0$ for $p<n-1$ by examining the long exact sequence in $\mathrm{R}^{\cdot}\mathrm{L}$. There also exists an exact sequence $0\to\mathrm{R}^{n-1}\mathrm{L}(N)\to\mathrm{R}^n\mathrm{L}(M)\xrightarrow{g}\mathrm{R}^n\mathrm{L}(M)$. Since $g=f(\mathrm{R}^n\mathrm{L}(M))$ and $\mathrm{R}^n\mathrm{L}(M)\ (\neq 0)\in\mathscr{C}$, g is not a monomorphism. Therefore $\mathrm{R}^{n-1}\mathrm{L}(N)\neq 0$, which completes the proof.

2. Homological grade of a Gabriel topology

Let R be a ring and $\mathrm{Mod}(R)$ the category of R-modules. By Theorem 1 § 1, for every homological g-function ψ on $\mathrm{Mod}(R)$, there exists a unique localizing subcategory $\mathscr{C}$ of $\mathrm{Mod}(R)$ such that $\psi=hgr(\mathscr{C},\ )$. Since $\mathscr{C}$ corresponds to a Gabriel topology F on R (see [5], § 3), there must be a relation between ψ and F. In fact we have

PROPOSITION 1. *Under the same notation and assumptions as above, $\psi(M)=$ the least integer n such that $\mathrm{Ext}^n_R(R/\mathfrak{a}, M)\neq 0$ for some $\mathfrak{a}$ in F if there is such an integer and ∞ if there is not.*

To prove the above proposition, it is sufficient to show the following lemma.

LEMMA 1. *Let F be a Gabriel topology on R and let $\mathscr{C}$ be the localizing subcategory of $\mathrm{Mod}(R)$ associated to F. Then the following statements, concerning an R-module M and a positive integer n, are equivalent:*

(i) $\mathrm{Ext}^p_R(R/\mathfrak{a}, M)=0$ *for all* $p<n$ *and* $\mathfrak{a}\in\mathrm{F}$.

(ii) $\mathrm{R}^p\mathrm{L}(M)=0$ *for all* $p<n$, *where* $\mathrm{L}=\mathrm{L}_{\mathscr{C}}$.

PROOF. Since $R/\mathfrak{a}\in\mathscr{C}$ for all $\mathfrak{a}\in\mathrm{F}$, the implication (ii)⇒(i) follows from Cor. to Prop. 2 § 1. (i)⇒(ii): Since $\mathrm{Hom}_R(R/\mathfrak{a}, M)=0$ for all $\mathfrak{a}$ in F, M is $\mathscr{C}$-pure i.e. $\mathrm{R}^0\mathrm{L}(M)=\mathrm{L}(M)=0$. Let $M\to E_0\xrightarrow{d_0}E_1\xrightarrow{d_1}\cdots\xrightarrow{d_{n-2}}E_{n-1}\xrightarrow{d_{n-1}}E_n$ be an injective resolution of M. Now choose an element x in $\mathrm{Ker}(\mathrm{L}(d_j))$ $(j=1, 2,\ldots,$

$n-1$). Since $\mathrm{Ann}_R(x) \in \mathrm{F}$, the morphism u: $R/\mathrm{Ann}_R(x) \to E_j$, defined by $u(1)=x$, can be extended to a morphism v: $R/\mathrm{Ann}_R(x) \to E_{j-1}$ by our assumption. Then $v(1) \in \mathrm{L}(E_{j-1})$ and the morphism $\mathrm{L}(d_{j-1})$ sends $v(1)$ into $u(1)$. Therefore $\mathrm{L}(E_{j-1}) \to \mathrm{L}(E_j) \to \mathrm{L}(E_{j-1})$ is exact; hence $\mathrm{R}^j\mathrm{L}(M)=0$ for $1 \leqq j < n$. This completes the proof.

For a Gabriel topology F on R, we denote the homological g-function associated to F by $hgr(\mathrm{F},\ )$, i.e. for every R-module M, $hgr(\mathrm{F}, M)$ = the least integer n such that $\mathrm{Ext}_R^n(R/\mathfrak{a}, M) \neq 0$ for some $\mathfrak{a}$ in F if there is such an integer and ∞ if there is not. Every homological g-function on $\mathrm{Mod}(R)$ is of this type. We say that $hgr(\mathrm{F}, M)$ is the homological grade of F on M. Now, let I be an ideal of R. Then I defines a homological g-function, which we denote by $hgr(I,\ )$, on $\mathrm{Mod}(R)$, i.e. for every R-module M, $hgr(I, M)$ = the least integer n such that $\mathrm{Ext}_R^n(R/I, M) \neq 0$ if there is such an integer and ∞ if there is not. Therefore there exists a unique Gabriel topology (we denote it by $\mathrm{F}(I)$) on R such that $hgr(I,\ ) = hgr(\mathrm{F}(I),\ )$. We say that $hgr(I, M)$ is the homological grade (or 'Rees' grade) of I on M. By the definition of $\mathrm{F}(I)$, we have $\mathrm{F}(I)=\{\text{ideal } \mathfrak{a} \text{ of } R \mid \mathrm{Hom}_R(R/\mathfrak{a}, M)=0$ for every R-module M such that $\mathrm{Hom}_R(R/I, M)=0\}$. In particular, $I \in \mathrm{F}(I)$.

Lemma 2. *If* $\mathfrak{a} \in \mathrm{F}(I)$, *then* $\mathrm{rad}(\mathfrak{a}) \supseteqq I$.

Proof. Let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a}$. Then $\mathfrak{p} \in \mathrm{F}(I)$; hence $hgr(I, R/\mathfrak{p})=0$, i.e. $\mathrm{Hom}_R(R/I, R/\mathfrak{p}) \neq 0$. Therefore we can choose an element a in $R-\mathfrak{p}$ such that $aI \subseteqq \mathfrak{p}$; hence $I \subseteqq \mathfrak{p}$. This shows that $I \subseteqq \mathrm{rad}(\mathfrak{a})$.

The converse of Lemma 2 is not true in general. For example, let $R = k[X_1, X_2,\dots, X_n,\dots]/(X_1, X_2^2,\dots, X_n^n,\dots)$ and $I=(x_1, x_2,\dots, x_n,\dots)$ where k is a field and $X_1, X_2,\dots$ are indeterminates (this example is given in [1]). It is easy to see that $hgr(I, R)>0$, i.e. $\mathrm{Hom}_R(R/I, R)=0$. Let now $\mathfrak{a}=\mathrm{Ann}_R(x_j)$ ($j \neq 1$). Since $\mathrm{Hom}_R(R/\mathfrak{a}, R) \neq 0$, we have $\mathfrak{a} \notin \mathrm{F}(I)$ but $\mathrm{rad}(\mathfrak{a})=I$. From the property of Gabriel topology (cf. [5], § 3), it is clear that the converse of Lemma 2 is true if I is finitely generated.

We shall now restate Prop. 1 § 1 in the following form.

Proposition 2. *Let* F *be a Gabriel topology on* R. *A correspondence* ψ *from* $\mathrm{Mod}(R)$ *to* {*non-negative integers and* ∞} *is a homological g-function* $hgr(\mathrm{F},\ )$ *if* ψ *satisfies the following conditions*:

(i) $\psi(0)=\infty$, *and if* $M \cong N$, *then* $\psi(M)=\psi(N)$.

(ii) $\psi(M)>0$ *if and only if* $\mathrm{Hom}_R(R/\mathfrak{a}, M)=0$ *for all* $\mathfrak{a}$ *in* F.

(iii) *Let* $0 \to M \to E \to N \to 0$ *be an exact sequence. If* E *is an injective module with* $\psi(E)>0$, *then* $\psi(N)=\psi(M)-1$.

Further, Prop. 3 § 1 takes the form:

PROPOSITION 3. *Let* F *be a Gabriel topology on* R, *and let* x *be an element of* R *such that* $x \in \operatorname{rad}(\mathfrak{a})$ *for all* $\mathfrak{a}$ *in* F. *If* M *is an* R*-module such that* x *is* M*-regular, then* $h\,gr(\mathrm{F}, M)>0$ *and* $h\,gr(\mathrm{F}, M/xM)=h\,gr(\mathrm{F}, M)-1$.

We shall conclude this section by establishing the Auslander-Buchsbaum theorem in terms of $h\,gr(\mathrm{F},\ )$.

LEMMA 3. ([6], Th. 8 Chap. 3) *Let* M *be an* R*-module, and let* $A=(a_{ij})$ *be a* $p \times q$ *matrix with entries in* R. *Then the equations*

$$\textstyle\sum_{j=1}^{q} a_{ij} e_j = 0, \qquad 1 \leq i \leq p,$$

have no non-trivial solutions on M *if and only if* $0:{}_M I=0$, *where* I *is the ideal generated by all the* $q \times q$ *minors of* A.

The following lemma is a generalization of Prop. 3 § 1.

LEMMA 4. *Let* $\phi\colon F \to G$ *be an* R*-module homomorphism where* $F \neq 0$ *and* $G \neq 0$ *are finite free* R*-modules of rank* q *and* p *respectively. Further let* F *be a Gabriel topology on* R *such that* $\mathrm{F}(I) \supseteqq \mathrm{F}$, *where* I *is the ideal generated by all the* $q \times q$ *minors of the matrix of* ϕ. *If* M *is an* R*-module such that* $\phi \otimes M$: $F \otimes M \to G \otimes M$ *is injective, then* $h\,gr(\mathrm{F}, M)>0$ *and* $h\,gr(\mathrm{F}, \operatorname{Coker}\phi \otimes M)= h\,gr(\mathrm{F}, M)-1$.

PROOF. If $h\,gr(\mathrm{F}, M)=0$, then $0:{}_M\mathfrak{a} \neq 0$ for some $\mathfrak{a}$ in F. Since $\mathfrak{a} \in \mathrm{F}(I)$ and I is finitely generated, $I^s \subseteqq \mathfrak{a}$ for some s; hence $0:{}_M I \neq 0$. On the other hand, by Lemma 3, the injectivity of $\phi \otimes M$ implies $0:{}_M I=0$, which is a contradiction. Therefore $h\,gr(\mathrm{F}, M)>0$. Now we put $n=h\,gr(\mathrm{F}, M)$ and consider the exact sequence $0 \to F \otimes M \to G \otimes M \to N=\operatorname{Coker}(\phi \otimes M) \to 0$. By examining the long exact sequence in $\mathrm{R}^{\cdot}\mathrm{L}$ (where L is the subfunctor of $1_{\mathrm{Mod}(R)}$ associated to F (cf. [5])), $\mathrm{RL}^p(N)=0$ for $p<n-1$. We also have an exact sequence $0 \to \mathrm{R}^{n-1}\mathrm{L}(N) \to \mathrm{R}^n\mathrm{L}(M) \xrightarrow{\alpha} \mathrm{R}^n\mathrm{L}(M)$. It is sufficient to show that α is not injective. Note that $\alpha=\phi \otimes \mathrm{R}^n\mathrm{L}(M)$. Therefore α is not injective if and only if $\operatorname{Hom}_R(R/I, \mathrm{R}^n\mathrm{L}(M)) \neq 0$. Since $h\,gr(\mathrm{F}, \mathrm{R}^n\mathrm{L}(M))=0$, $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{R}^n\mathrm{L}(M)) \neq 0$ for some $\mathfrak{a}$ in F. By our assumption, $I^s \subseteqq \mathfrak{a}$ for some s; hence $\operatorname{Hom}_R(R/I, \mathrm{R}^n\mathrm{L}(M)) \neq 0$. This completes the proof.

THEOREM 1. *Let* $C\colon 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$ $(n \geqq 1)$ *be a complex of finite free* R*-modules, and let* F *be a Gabriel topology on* R *such that* $\mathrm{F} \subseteqq \mathrm{F}(I)$, *where* I *is the ideal generated by all the* $q \times q$ *minors of the matrix of* ϕ_n *and* $q=\operatorname{rank}(F_n)$. *If* M *is an* R*-module such that* $C \otimes M$ *is exact, then* $h\,gr(\mathrm{F}, M) =n+h\,gr(\mathrm{F}, \operatorname{Coker}(\phi_1 \otimes M))$.

PROOF. By Lemma 4, we have $hgr(\mathrm{F}, M)=1+hgr(\mathrm{F}. \operatorname{Coker}(\phi_n\otimes M))$. Now let m be an integer such that $m\leqq n-1$ and assume that $hgr(\mathrm{F}, M)=n-m+hgr(\mathrm{F}, \operatorname{Coker}(\phi_{m+1}\otimes M))$. Since the sequence $0\to\operatorname{Coker}(\phi_{m+1}\otimes M)\to F_{m-1}\otimes M\to\operatorname{Coker}(\phi_m\otimes M)\to 0$ is exact and $hgr(\mathrm{F}, \operatorname{Coker}(\phi_{m+1}\otimes M))<hgr(\mathrm{F}, M)=hgr(\mathrm{F}, F_{m-1}\otimes M)$, we have $hgr(\mathrm{F}, \operatorname{Coker}(\phi_{m+1}\otimes M))=hgr(\mathrm{F}, \operatorname{Coker}(\phi_m\otimes M)+1$; hence, by induction, $hgr(\mathrm{F}, M)=n+hgr(\mathrm{F}, \operatorname{Coker}(\phi_1\otimes M))$.

By definition, an R-module M has the restricted projective dimension n, denoted by $Pd_R^*(M)$, if M possesses a finite free resolution and n is the length of the shortest such resolution. The following is the Auslander-Buchsbaum theorem on finite free resolutions in terms of homological grade.

COROLLARY. *Let R be a quasi local ring and let $M\neq 0$ be an R-module which admits a finite free resolution of finite length. For any ideal J such that* $\operatorname{rad}(J)$ *is the maximal ideal of R, we have $hgr(J, M)+Pd_R^*(M)=hgr(J, R)$.*

PROOF. Let $n=Pd_R^*(M)$, and let $0\to F_n\xrightarrow{\phi}F_{n-1}\to\cdots\to F_0\to M\to 0$ be a finite free resolution of M. The case $n=0$ is trivial. Assume now that $n\geqq 1$. Let I be the ideal generated by all the $q\times q$ minors of the matrix of ϕ and $q=\operatorname{rank}(F_n)$. Since $n=Pd_R^*(M)$, we must have $I\neq R$ (see [2], Lemma 1 §2). Since I is finitely generated, $\mathrm{F}(I)=\{\mathfrak{a}|\operatorname{rad}(\mathfrak{a})\supseteqq I\}$. On the other hand, $I\subseteqq\operatorname{rad}(\mathfrak{a})$ for all $\mathfrak{a}$ in $F(J)$. Therefore $\mathrm{F}(I)\supseteqq\mathrm{F}(J)$. Now the assertion follows from Th. 1.

3. Polynomial grade

Let R be a ring and $\operatorname{Mod}(R)$ the category of R-modules. Let I be an ideal of R and M an R-module. The classical grade of I on M is the length of the longest maximal R-sequence on M contained in I, and we denote it by $cgr(I, M)$. The polynomial grade of I on M, denoted by $\mathrm{Gr}_R(I, M)$, is defined by

$$\lim_{m\to\infty} cgr_{R[X_1,\dots,X_m]}(I[X_1,\dots, X_m], M[X_1,\dots, X_m]),$$

where $X_1, X_2,\dots, X_m,\dots$ are distinct indeterminates (cf. [6], Chap. 5). We assume some properties of polynomial grade whose proofs are all given in [6], Chap. 5:

(a) $\mathrm{Gr}_R(I, M)>0$ if and only if there exists a non-zero finitely generated ideal $J\subseteqq I$ such that $0:_M J\ (=\operatorname{Hom}_R(R/J, M))=0$.

(b) Let $\beta_1,\dots, \beta_s$ be an R-sequence on M contained in I. Then $\mathrm{Gr}_R(I, M)=\mathrm{Gr}_R(I, M/(\beta_1 M+\cdots+\beta_s M))+s$.

(c) Let S be an R-algebra and M an S-module. Then $\mathrm{Gr}_R(I, M)=\mathrm{Gr}_S(IS, M)$.

Moreover, by the definition of polynomial grade, we have:

(d) If $S=R[X_1,\dots, X_m]$ and M is an R-module, then $\mathrm{Gr}_R(I, M)=\mathrm{Gr}_S(I$

$\otimes_R S, M\otimes_R S)$.

Further, M. Hochster showed the following:

(d′) ([4], Cor. 1 to Prop. 2, §1) Let S be a faithfully flat R-algebra and M an R-module. Then $\mathrm{Gr}_R(I, M)=\mathrm{Gr}_S(I\otimes_R S, M\otimes_R S)$.

REMARK. As for (d′), we can prove it directly using the above properties (a), (b), (c) and (d).

Summarizing the properties of $\psi=\mathrm{Gr}_R(I,\ )$, we have

(i) $\psi(0)=\infty$, *and if* $M\cong N$, *then* $\psi(M)=\psi(N)$.

(ii) $\psi(M)>0$ *if and only if* $0:_M J=0$ *for some non-zero finitely generated ideal* $J\subseteqq I$.

(iii) *If* x *is an* M-*regular element contained in* I, *then* $\psi(M)=\psi(M/xM)+1$.

(iv) *Let* S *be a faithfully flat* R-*algebra, and let* $\psi|_S$ *be the restriction of* ψ *on* $\mathrm{Mod}(S)$. *Then*

(a) $\psi|_S$ *satisfies* (i), (ii) *and* (iii) *for* S-*modules and the ideal* IS *instead of* R-*modules and* I *respectively*.

(b) $\psi(M)=\psi|_S(M\otimes_R S)$ *for every* R-*module* M.

We shall show that the above properties characterize $\mathrm{Gr}_R(I,\ )$.

THEOREM 1. *Let* I *be an ideal of* R *and let* ψ *be a correspondence from* $\mathrm{Mod}(R)$ *to* {*non-negative integers and* ∞}. *If* ψ *satisfies the above conditions* (i), (ii), (iii) *and* (iv), *then* $\psi=\mathrm{Gr}_R(I,\ )$.

PROOF. First assume that $n=\mathrm{Gr}_R(I, M)$ is finite. Then $n=c\,gr_{R[X_1,\dots,X_m]}(I[X_1,\dots,X_m], M[X_1,\dots,X_m])$ for some m. Let $f_1,\dots,f_n$ be an $M[X_1,\dots,X_m]$-regular sequence in $I[X_1,\dots,X_m]$. Since ψ satisfies the conditions (iii) and (iv), we have

$$(*)\qquad \psi(M)=\psi|_{R[X_1,\dots,X_m]}(M[X_1,\dots,X_m])$$
$$=\psi|_{R[X_1,\dots,X_m]}(M[X_1,\dots,X_m]/(f_1,\dots,f_n)M[X_1,\dots,X_m])+n.$$

Note that $\psi|_{R[X_1,\dots,X_m]}(N)=0$ if and only if $\mathrm{Gr}_{R[X_1,\dots,X_m]}(I[X_1,\dots,X_m], N)=0$ for all $R[X_1,\dots,X_m]$-module N. Since $\mathrm{Gr}_{R[X_1,\dots,X_m]}(I[X_1,\dots,X_m], M[X_1,\dots,X_m]/(f_1,\dots,f_n)M[X_1,\dots,X_m])=0$, $\psi|_{R[X_1,\dots,X_m]}(M[X_1,\dots,X_m]/(f_1,\dots,f_n)M[X_1,\dots,X_m])=0$. Therefore $\psi(M)=n$. Finally assume that $\mathrm{Gr}_R(I, M)=\infty$. Then the equation $(*)$ shows that $\psi(M)\geqq n$ for all n. Therefore $\psi(M)=\infty$. This completes the proof.

In his paper [1], S. F. Barger has proved that $\mathrm{Gr}_R(I, M)$ is the least integer n such that $\mathrm{Ext}^n_R(R/I, M\infty)\neq 0$ if I is a finitely generated ideal, where $M\infty$ is the direct sum of countably many copies of M. We shall here give another proof of this statement as a corollary to Th. 1.

COROLLARY. *Suppose that I is a finitely generated ideal of R. Then* $Gr_R(I, M) = h\,gr(I, M\infty)$.

PROOF. For an R-module M, we define $\psi(M) = h\,gr(I, M\infty)$. By the above theorem, it is sufficient to prove that ψ satisfies the conditions (i), (ii), (iii) and (iv). Obviously ψ satisfies the conditions (i) and (ii). As for (iii), see Prop. 3 § 2. Therefore all that remains to be proved is that ψ satisfies the condition (iv). Let $S = R[X_1,\dots, X_m]$. Note that every injective S-module is also an injective R-module and $\mathrm{Hom}_R(R/I, M) = \mathrm{Hom}_S(S/IS, M)$ for all S-module M. Therefore $\mathrm{Ext}^n_R(R/I, M\infty) = \mathrm{Ext}^n_R(S/IS, M\infty)$ for all S-module M; hence $\psi|_S$ satisfies (i), (ii) and (iii). Finally for an R-module M, $\psi|_S(M \otimes_R S) = \psi(M[X_1,\dots, X_m]) = \psi(M\infty) = \psi(M)$. This completes the proof.

References

[1] S. F. Barger, A theory of grade for commutative rings, Proc. Amer. Math. Soc. **36** (1972), 365–368.

[2] D. A. Buchsbaum and D. Eisenbud, What makes a complex exact?, Jour. of Algebra **25** (1973), 259–268.

[3] A. Grothendieck, Local cohomology, Lecture Notes in Math. **41**, Springer-Verlarg, 1967.

[4] M. Hochster, Grade-sensitive modules and perfect modules, Proc. London Math. Soc. (3) **29** (1974), 55–76.

[5] S. Itoh, Divisorial objects in abelian categories, Hiroshima Math. J. **6** (1976), 539–554.

[6] D. G. Northcott, Finite free resolutions, Cambridge Univ. Press, 1976.

[7] D. Rees, The grade of an ideal or module, Proc. Cambridge Phil. Soc. **53** (1957), 28–42.

[8] R. G. Swan, Algebraic K-theory, Lecture Notes in Nath. **76**, Springer-Verlarg, 1968.

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